

International Journal of Statistics and Applied Mathematics

ISSN Online: 2456-1452
NAAS Rating (2026): 4.49
IJSAM 2026; 11(10): 103-106
www.mathsjournal.com
Received: 05-08-2026
Accepted: 27-09-2026
Published: 10-10-2026

Dr. SK Pandey

Department of mathematics,
PM Excellence, Govt.
Vivekanand, P.G. College,
Maihar, Madhya Pradesh,
India

Ambika Saket

Department of Mathematical
Sciences, A.P.S. University,
Rewa, Madhya Pradesh, India

Study of measurable multi-functions and measurable selectors with their application

Dr. SK Pandey¹, Ambika Saket²

DOI: <https://www.doi.org/10.22271/math.2026.v11.i10b.2720>

ABSTRACT

In this paper, we introduce a generalized framework for measurable multi-functions defined on measurable and topological spaces, extending several classical concepts of measurable selection theory. The proposed approach unifies measurable multifunction, generalized graph measurability, and selector properties within a common setting that includes separable and non-separable spaces as particular cases. We establish spaces as particular generalized conditions, under which measurable selectors exist and investigate the preservation of measurability under compositions, products, restrictions, and inverse multi-functions. Furthermore, novel characterizations of generalized measurable graphs and stability properties are obtained in product measurable spaces. The framework is further extended to generalized fuzzy measurable multi-functions, providing a natural connection between classical multifunction theory and fuzzy set-valued analysis. Several applications are presented to measurable optimization, differential inclusions, fixed-point theory, stochastic systems, and generalized Aumann integration. The obtained results not only generalize many well-known measurable multifunction theorems but also provide a unified theoretical foundation for future developments in measurable analysis and its applications.

Keywords: Generalized measurable multi-functions, Measurable selectors, Graph measurability, Product measurable spaces, Topological spaces, Fuzzy measurable multifunctions, Generalized Aumann integral, Differential inclusions, Fixed-point theory, Measurable optimization

1. Introduction

Measurable multi-functions also known as set-valued measurable mappings are an important part of modern mathematical analysis and show up in measure theory, topology, optimization, control theory, probability, and mathematical economics. Unlike standard functions that assign a single value to each point, this extra flexibility makes them really useful for modeling uncertainty, incomplete information, and systems with multiple possible outcomes.

The key problem in measurable multifunction theory is the measurable selectors problem, which asks when we can pick a single-valued measurable function out of a measurable multifunction. Classical measurable selections theorems give powerful conditions for when such selections exist, and they are widely used in optimization, stochastic processes, variational analysis, differential inclusions, game theory, and economic equilibrium models. These results basically form the backbone of many modern mathematical models that deal with uncertainty and set-valued behavior.

Most existing measurable selection results on fairly strong assumptions, like standard measurable spaces Polish or complete separable metric spaces, closed or compact-valued multifunctions, or strong graph measurability conditions. These assumptions can be quite limiting when working in more general topological fuzzy environments, hybrid measurable systems, or newer applications where such nice structure doesn't always hold. Many multifunctions that appear in modern models can't be properly handled using the classical framework.

In this work, we introduce a generalized framework where measurability is described using generalized inverse-image operators and generalized measurable structures on topological measurable spaces. This extends the classical notion of measurable multifunctions, still keeping the K-analytical properties needed for measurable selection theory. It also provides a unified setting where many existing measurable concepts naturally appear as special cases.

Although there has been a lot of research on measurable multifunctions and measurable selectors, not much attention has been given to generalized measurable structures that combine topological. In particular, a unified treatment of generalized graph measurability, product multifunctions, and composition results is still missing.

Corresponding Author:

Dr. SK Pandey

Department of mathematics,
PM Excellence, Govt.
Vivekanand, P.G. College,
Maihar, Madhya Pradesh,
India

The main focuses of this paper is to established extended theory of measurable multifunctions and measurable selectors in topological measurable spaces. We present new definitions of extended measurable multifunctions and extended measurable selectors, and then build a series of fundamental results on graph measurability, composition, product mappings, stability, and approximation. The framework is designed to extend classical measurable selection theory while working in a much broader setting.

The principal results of this paper are as follows:

- * We propose an extended measurable multifunction structure in topological space.
- * We introduce broader measurable selectors under less restrictive conditions used in classical theory.
- * We establish new outcomes on broaden graph measurability, composition, and product.
- * We present examples that show the formulate more broadly structure works in addition to classical theory.

This analysis continues to according to the following structure. The section-2 present a specific review of recent literature and analyzed to development of measurable multifunction theory across history. In the third section the theoretical structure are investigated, in addition to fundamental outcomes. The fourth section emphasizes related to measurable selections and measurable multifunction theorems and their proofs. The section fifth highlights applications and their examples. And the lastly section six concludes the investigation with few important major conclusions and suggested structure for future work.

2. Literature review

Measurable multifunctions and measurable selection theory have become important implements in modern set-valued analysis, particularly in optimization, differential inclusions, control theory, integration theory, and mathematical economics. Classical measurable selection results investigated the basic relationship between measurable set-valued mappings and their single-valued measurable selections. Kuratowski–Ryll–Nardzewski selection theorem established a suitable measurability under a fundamental framework for obtaining measurable selectors and closed-valued suppositions. Latest evolution concentrated on extensive these concepts to more general range spaces, weaker measurability conditions, and applications containing infinite-dimensional systems.

Candeloro, Di Piazza, Musial, and Sambucini (2018) [2] established integration of multifunctions with closed convex values in arbitrary Banach spaces. Their work analyzed multifunction integration within classical weakly compact convex-valued arrangements and studied the relationship with several notions of integrations. Their results established evidence that the existence and properties of appropriate selections theorems are connected.

Ararat and Rudloff (2015) [1] analyzed the study of the theoretical foundations of Aumann integration, proposing a characterization for set-valued measurable mappings. They investigated a characterization of Daniell-Stone type of Aumann integrals for multifunctions holding closed, convex upper set values. Their result showed that the Aumann integral conserves important structural holding such as closedness and convexity, and equipped sufficient conditions which is a set-valued functional can be illustrate as an Aumann integral. This work studied provide the close relationship of set-valued integrations, measurable multifunctions, and measurable selectors. This research illustrated the use of closely between integrable of the associated multifunction.

In other important developments presented in this study, Candeloro *et al.* investigated relationship between gauge and Pettis integrals for multifunctions with weakly compact convex values in general Banach spaces. They examined decomposition and selection theorem concluded and verified How can used to classify different forms of multifunction integrability in integrable selections these research outcomes are particularly significant when the range space is not necessarily separable. The theoretical framework of multifunction integration and the important role play of future advantage in measurable multifunction theory.

Ioffe (2024) [8, 12] investigated problem of maximum principals for optimal control defined by differential inclusions these dynamical systems. This study indicated necessary and sufficient conditions enquire the existence of an optimal trajectory and an optimal control strategy under differentia inclusions controlled by via set-valued maps. The investigator was able to obtained necessary optimality conditions for a wide range of control systems. The importance of measurable multifunction and measurable selection within frame work of differential inclusions was highlights in this work.

Osineko and Streit (2024) investigated can be obtained constructively rather than merely through abstract existence theorems when measurable selectors of set valued maps Their study is inspired by optimal central theory, discontinuous systems and differential inclusions, where a selector significance, an admissible central or a feasible trajectory.

Di Piazza and Sambucini (2025) [4] studied investigated the concept of Portegrable selections for multifunctions in arbitrary Banach spaces These studied generalized the range of measurable selection theory from separable spaces to general Banath spaces. They established sufficient conditions for deriving integrable selectors for multifunctions whose scope consists of closed and convex sets. The researched investigated the connections between measurability, integrability and the topology of Banach space.

This examined made significant development to the advancement at measurable multifunction theory in infinite-dimensional spaces. Cascales, Kadets and Rodriguez (2010) [3] established measurable selectors of multifunction with condition of weakly compact values in particularly Banach space. They describe sufficient condition for obtaining multifunction for strongly measurable selectors and showed that scalarly measurable multifunctions can exhibit scalarly measurable selections without assuming separability on the range space. This study has extended the applicability of measurable selection theory to general Banach space.

3. Theoretical framework

3.1 Preliminaries

Let (M, Σ) be a measurable space and X is a non-empty topological space, where X is a complete separable metric space and $B(X)$ is the letting σ -algebra. We denote by 2^X the collection of all subsets of X , and by $Cl(X)$, $Cb(X)$, $Ck(X)$ the families of closed, closed bounded, and compact subsets of X respectively, for a Banach space X .

Definition 3.2 (Measurable Space) :

A pair (M, Σ) is called a measurable space, where M is a non-empty set and Σ is a σ -algebra of subsets of M . Thus:

- * $M \in \Sigma$
- * $A \in \Sigma \Rightarrow M \setminus A \in \Sigma$
- * If $A_n \in \Sigma$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$

Definition 3.3 (Borel σ -algebra) :

If X is a topological space, the Borel σ -algebra on X , denoted by $B(X)$, is the smallest σ -algebra containing all open subsets of X .

Definition 3.4 (Measurable Function) :

A mapping

$$f : (M, \Sigma) \rightarrow (X, \beta(X)) \quad \beta = \text{Beeta}$$

$$f^{-1}(B) \in \Sigma \text{ for every } B \in \beta(X)$$

Definition 3.5 (Power set and closed sets) :

For a set X , denote by 2^X the family of all subsets of X . If $F(X)$ denotes the family of non-empty closed subsets of X , then:

$$F(X) = \{A \subseteq X : A \neq \emptyset, A \text{ is closed}\} \quad (3.5.1)$$

Definition 3.6 (Multifunction) :

A mapping

$$F: X \rightarrow X$$

$$F : (y) \subseteq X, y \in Y$$

$$F : y \rightarrow 2^X$$

Definition 3.7 (Graph of a Multifunction) :

The graph of $F : Y \rightarrow X$ is:

$$\text{Gr}(F) = \{(y, x) \in Y \times X : x \in F(y)\} \quad (3.7.1)$$

Definition 3.8 (Graph-measurable Multifunction) :

A multifunction $F : Y \rightarrow X$ is called graph-measurable if

$$\text{Gr}(F) \in \Sigma \otimes \mathcal{B}(X)$$

Definition 3.9 (Measurable Multifunction) :

A multifunction $F : Y \rightarrow X$ is called measurable if

$$F^{-1}(D) = \{y \in Y : F(y) \cap D \neq \emptyset\} \in \Sigma \quad (3.9.1)$$

Definition 3.10 (Selection) :

A mapping $f : Y \rightarrow X$

$$f(y) \in F(y), \forall y \in Y$$

$$f \in \text{sel}(F)$$

Definition 3.11 (Measurable Selector) :

A selection $f : Y \rightarrow X$ of F is called a measurable selector if f is measurable and

$$f(y) \in F(y), \forall y \in Y$$

$$f \in \text{sel}_m(F)$$

Definition 3.12 (Casting Representation) :

A sequence $f_n \geq 1$ of measurable selectors of F is called a casting representation of F if:

$$F(y) = \{f_n(y) : n \geq 1\}, \forall y \in Y \quad (3.12.1)$$

Definition 3.13 (Measurable Selector Set) :

The collection of all measurable selectors of F is denoted by:

$$F(x) = \text{cl}\{f_n(x) : n \in \mathbb{N}\}, \mu\text{-almost every } x \in \psi \quad (4.2.1)$$

This construction is fundamental for the integration theory of multifunctions. It derive to many questions about measurable multifunctions to questions about countable families of measurable functions, enabling the use of classical measure theory.

Theorem 4.3 (Generalization of Graph Measurability) :

Let $F : \psi \rightarrow X$ be a closed-valued multifunction, then if F is complete measurable space:

$$F \text{ is measurable } (\Rightarrow) \text{Gr}(F) \in \Sigma \otimes \beta(X)$$

where

$$\text{Gr}(F) = \{(\psi, x) \in \psi \times X : x \in F(y)\}$$

Proof:

Let F is closed-valued

and $x \in F(x)$ if and only if

$$g(x, F(x)) = 0.$$

So,

$$\text{Gr}(F) = \{(\psi, x) : g(x, F(x)) = 0\}$$

Now define

$$\phi(\psi, x) = g(x, F(x))$$

Measurability of F and separability of X is measurable of ϕ .

$$\phi^{-1}(\{0\}) \in \Sigma \otimes \beta(X).$$

But

$$\phi^{-1}(\{0\}) = \text{Gr}(F).$$

So,

$$\text{Gr}(F) \in \Sigma \otimes \beta(X)$$

Product σ -algebra is measurable and complete measurable space and separable or Souslin.

Corresponding projection theorems are also projection measurable.

Hence,

$$F^{-1}(V) \in \Sigma \text{ therefore } F \text{ is measurable.}$$

Theorem 4.4:

Let $F : \psi \rightarrow X$ is a measurable multifunction and

$$\text{Sel}_m(F) = \{f : y \rightarrow X : f \text{ is measurable and } f(y) \in F(y)\}$$

Definition 3.14 (Integrable Selection) :

If X is a Banach space and $F : y \rightarrow X$, a measurable selector $f \in \text{Sel}_m(F)$ is called an integrable selection if

$$\int_Y \|f(y)\| d\mu(y) < \infty$$

4. Main results

4.1 Theorem :

Let (ψ, Σ) be a measurable space, and let X be a Souslin. If $F : \psi \rightarrow \text{Cl}(X)$ is a weakly measurable multifunction with non-empty values, then F admits a measurable selector.

Proof:

Let us construct a sequence of measurable functions recursively, now we define $\{f_n\}$ using the Borel structure of X . For a fixed countable dense set $\{y_k\}$ in Y , Let us define:

$$f_1(x) = y_k$$

$$\text{where } k = \min\{i : F(x) \cap D(y_i, 1) \neq \emptyset\} \quad (4.1.1)$$

This defines a measurable function satisfying:

$$g(F(x), f_n(x)) < \frac{1}{n}$$

for each n . The pointwise limit:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

is a measurable selector. Completeness of X ensures convergence, and the measurability of each f_n propagates to f . In this way F admits a measurable selector.

Theorem 4.2 (Generalization of Casting Representation) :

Let $F : \psi \rightarrow \text{Cl}(X)$ be a measurable multifunction with values in a Souslin space X . Then there exists a countable family $\{f_n : n \in \mathbb{N}\}$ of measurable selectors of F such that:

$$F(x) = \text{cl}\{f_n(x) : n \in \mathbb{N}\}$$

for μ -almost every $x \in \psi$

$h : X \rightarrow Z$ is a continuous mapping where X, Z are Souslin. Then define:

$(h \circ F)(x) = \{h(x) : x \in F(x)\}$. Then $h \circ F$ is a measurable multifunction, and another is:

$f : \psi \rightarrow X$, F is a measurable selector then

$h \circ f : \psi \rightarrow Z$ then $h \circ f$ is a measurable selector.

Proof: Let us suppose f is selector of F ,

$$f(x) \in F(x), \text{ applying } h$$

$$h(f(x)) \in \{h(x) : x \in F(x)\}.$$

Hence

$$(h \circ f)(x) \in F(h \circ f)(x).$$

So, $h \circ f$ is a selector. Now f is measurable and h is continuous, $h \circ f$ is measurable.

$$h \circ f \in \text{Sel}_m(h \circ F).$$

Now taken any D for measurability of multifunction:

then

$$(h \circ F)^{-1}(D) = \{x : h(F(x)) \cap D \neq \emptyset\}$$

If h is a measurable isomorphism, then

$$(h \circ F)^{-1}(D) = f^{-1}(h^{-1}(D))$$

because $h^{-1}(D) \in \mathcal{B}(X)$ and F is measurable.

$$F^{-1}(h^{-1}(D)) \in \Sigma$$

So,

Thus $h \circ f$ is measurable.

F is measurable $\Rightarrow h \circ f$ measurable

And

$$f \in \text{Sel}_m(F) \Rightarrow h \circ f \in \text{Sel}_m(h \circ F).$$

$\Rightarrow h \circ f$ is measurable selector

5. Applications

5.1 Optimal control

Aim of this study in optimal control theory admitted controls can be represented by a multifunction measurable control function v in the constraint set $V(t)$, which minimized the cost functional.

Associated with controlled dynamical systems can be explain the means of multi-functions. Therefore, investigated the presence of an optimal control requires the evidence of measurable selector of the associated multifunction that yields the minimum value of the cost functional.

$$dy(x) \in F(x, y(x))dt + G(x, y(x))dW(x)$$

where $W(x)$ denotes wiener process. It required measurable selections of both F and G . on suitable measurability and selection properties of the multi-functions presented.

5.3 Random fixed point problems

Measurable selection theorems are applied to investigate to measurability of the fixed-point mapping another classical application of measurable selection theory. An upper-semi-continuous multifunction mapping a compact and convex subset of a locally convex space into itself with non-empty and convex values possesses at least one fixed point. We established a parameterized fixed-point theorem including measurable selections.

5.4 Game theory

In a parameterized game, a player's admissible strategy is associated by a multifunction. It provides a measurable best response function and correspondence, can then yield a Nash equilibrium. Recently studies specifically connect measurable-selection-valued associated with fixed and applications to stochastic games.

5.5 Dynamic programming

In present investigate a measurable optimal policy using the methods of fixed points establish stationary optimal policies. The applications to random dynamic programming and measurable stationary with the help of optimal programs have been studied through random fixed point methods.

5.6 (Fuzzy and random set-valued models)

Fixed point methods and measurable selections can be also merged with fuzzy set-valued operators In recent studies has explicitly assuming fixed point theorems for measurable selections arising from fuzzy set-valued operators on polish spaces.

6. Conclusions

In this study, concepts of fundamental structures applications of measurable multi-functions and measurable selectors have been established systematically. This incorporates measurable multi-functions and measurable selector theorems applicable to Polish space and Banach spaces. Four theorems are analyzed in detail.

This examined the incorporation the Casting Representation Theorem, The Kuratowski–Ryll–Nardzewski Selection Theorem, Extension of measurability of graph to Banach space, Polish space, as well as Aumann integral and its properties. Application of measurable multi-functions and measurable selectors related theorems in various fields of research have been evaluated to highlight their contributions in across theoretical and applied disciplines. These applications extend to fixed point theory, optimization, control theory, game theory, dynamic programming, fuzzy and random set-valued models, and in another different-different areas.

References

1. Ararat C, Rudloff B. A characterization theorem for Aumann integrals. *Set-Valued and Variational Analysis*. 2015;23(2):305-318.
2. Candeloro D, Di Piazza L, Musial K, Sambucini AR. Integration of multifunctions with closed convex values in arbitrary Banach spaces. *arXiv preprint arXiv:1812.00597*.

5.2 Connections to stochastic

The established here in framework can naturally be increased to stochastic settings. Assuming a set-valued stochastic process $\{F(t, x) : x \in [0, X], x \in X\}$. Such a requires connect measurability with respect to the product σ -algebra on $[0, X] \times X$. The measurable selection results can be applied to obtain measurable selection of F . This gives a foundation for investigating stochastic differential inclusions of the form:

3. Cascales B, Kadets Y, Rodriquez J. measurability and selections of multifunctions in Banach spaces. *Journal of conver Analysis*. 2010;17(1):229-240 we
4. D Piazza L, Sambucini AR. Selections of integrable multifunction in arbitrary Banach spaces. *arxiv preprint arxiv: 2507, 19 460*.
5. Kuratowski K, Ryll-Nardzewski C. A general theorem on Selectors. *Bulletin of the Palis Academy of Sciences, series of mathematical Astronomical and physical Sciences*. 1965;13:397-403.
6. Wagher DH. Survey of measurable selection theorems. *SIAM Journal on control and optimization*. 1977;15:859-903.
7. Di Piazza L, Sambucini AR. Selections of integrable multifunctions in arbitrary Banach spaces. *arXiv preprint arXiv:2507.19460*.
8. Greenberg MJ. *Algebraic topology: a first course*. Boca Raton (FL): CRC Press; 2018.
9. Hellman Z, Levy YJ. Measurable selection for purely atomic games. *Econometrica*. 2019;87(2):593-629.
10. Hoy M, Livernois J, McKenna C, Rees R, Stengos T. *Mathematics for economics*. Cambridge (MA): MIT Press; 2022.
11. Hu S, Papageorgiou NS. *Research topics in analysis, volume I: grounding theory*. Cham: Springer Nature; 2022.
12. Ioffe AD. *Maximum principles for optimal control problems with differential inclusions*. *SIAM Journal on Control and Optimization*. 2024;62(1):271-296.
13. Kaliaj SB. Multifunctions admitting a measurable by seminorm selector in Fréchet spaces. *arXiv preprint arXiv:2206.10337*.
14. Lechine U, Seiller T, Simonsen JG. Agafonov's theorem for probabilistic selectors. In: *Mathematical Foundations of Computer Science (MFCS)*. Vol. 306. Schloss Dagstuhl-Leibniz-Zentrum für Informatik; 2024. p. 67-?
15. Michta M, Motyl J. Selection properties and set-valued Young integrals of set-valued functions. *Results in Mathematics*. 2020;75(4):164.
16. Mordukhovich BS, Wu P, Yang X. Stability criteria and calculus rules via conic contingent coderivatives in Banach spaces. *Optimization*. 2025;74(11):2577-2610.
17. Muratov M, Rubshtein BZ. Equimeasurable symmetric spaces of measurable function. *arXiv preprint arXiv:2006.15702*.
18. Musial K. Conditional expectations in Banach spaces with RNP. *arXiv preprint arXiv:2305.19653*.
19. Shang S, Cui Y. Almost convexity and continuous selections of the set-valued metric generalized inverse in Banach spaces. *Banach Journal of Mathematical Analysis*. 2020;15(1):1.

How to cite this article: Pandey DS, Saket A. Study of measurable multi-functions and measurable selectors with their application. *International Journal of Statistics and Applied Mathematics*. 2026;11(10):103-106. doi: <https://www.doi.org/10.22271/maths.2026.v11.i10b.2720>