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Evalyne Kagendo Rwito
Department of Mathematics
and Actuarial Science, Kisii
University, P.O. Box 408-
40200, Kisii, Kenya

Joash Kerongo
Department of Mathematics
and Actuarial Science, Kisii
University, P.O. Box 408-
40200, Kisii, Kenya

Impact of porosity and permeability variations on water flow dynamics in heterogeneous subsurface porous media: A finite element analysis

Evalyne Kagendo Rwito¹, Joash Kerongo²

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ABSTRACT

Water flow in a subsurface porous media is governed by the spatially varying porosity ϕ and the spatially varying permeability K . The goal is to understand the quantitative effect of the coupling between porosity and permeability (Kozeny-Carman relation) on the water velocity and water pressure distribution in a two-dimensional (2D) porous medium, where the latter is heterogeneous. A steady state finite element model of a porous medium was built and solved using Darcy's law and continuity equation with the following boundary conditions: $p = 1$ at $x = 0$ and $p = 0$ at $x = 1$. Heterogeneity was represented by a porosity field $\phi(x, y) = 0.3 + 0.2 \sin(2\pi y) + \epsilon(x, y)$ (ϵ = Gaussian noise, $\sigma = 0.05$). Permeability was derived as $K = K_0 \phi^3 / (1 - \phi)^2$. A parametric sweep varied base porosity ϕ_0 from 0.2 to 0.5. Using linear triangular elements (512 elements), the following findings were obtained: The base-line case ($\phi_0 = 0.3$) exhibited strong flow channeling, with a coefficient of variation of velocity of 0.48 and local velocities were up to five times the mean velocity. The parametric study showed a power-law relationship between the flow rate and the change of porosity, $u \propto \phi^{3.2}$ (exponent is close to the theory value 3), and the pressure drop across the whole system was constant, $\Delta p = 0.111$, regardless of the value of ϕ_0 . Small variations in porosity lead to orders of magnitude differences in permeability, which causes the flow rate to concentrate into the narrow channels. The presence of a non-linear velocity-porosity relationship (velocity exponent = 3.2) highlights the importance of proper characterisation of porosities in predictive models. A constant pressure drop is a fundamental property of elliptic flow equations. These results are quantitative parameters that can be applied to upscaling and risk assessment in a heterogeneous aquifers and reservoirs.

Keywords: Porous media, Porosity, Permeability, Finite element method, Flow channeling

Introduction

Water flow in heterogeneous subsurface porous media is crucial in many fields such as: management of groundwater resources, enhanced oil recovery, geothermal energy extraction and for prediction of contaminant transport [2, 5]. Both porosity ϕ (the volume fraction of void spaces in porous material) and permeability K (the ease of flow under a pressure gradient) are spatially variable as a result of depositional processes, diagenesis, and tectonic deformations [6]. These variations create preferential flow paths (channeling) that can significantly change breakthrough times, sweep efficiency and mass transport [1]. The two properties are related empirically by the Kozeny-Carman relation $K \propto \phi^3 / (1 - \phi)^2$ which converts small differences in porosity to orders-of-magnitude differences in permeability [3]. Many of the current numerical models, however, do not capture this physical relationship, nor the nonlinear implications, and either assume homogeneity or treat porosity and permeability as uncoupled random fields. The finite element method (FEM) is particularly suited for heterogeneous problems because it can directly incorporate spatially varying material properties [16, 10]. In recent years, some research efforts have focused on the coupled thermo-hydro-mechanical processes [14, 8], multi-phase flow [15], and fractured media [13] using FEM. However, a systematic quantitative investigation on the effect of base porosity and spatial structure on the velocity field and pressure distribution is still missing in a well controlled heterogeneous environment.

The present study fills this gap by creating a 2D finite element model which explicitly couples porosity to permeability using the Kozeny-Carman relation, introduces a synthetic heterogeneity field (sinusoidal layering + random noise), and conducts a parametric study of base porosity from 0.2 to 0.5. The main objective is to determine the influence of porosity and permeability changes on flow rate of water and pressure distribution and to present the benchmark and physical insights for subsurface flow modeling.

Corresponding Author:
Evalyne Kagendo Rwito
Department of Mathematics
and Actuarial Science, Kisii
University, P.O. Box 408-
40200, Kisii, Kenya

Methodology

Governing equations

For steady, incompressible, single-phase flow in the Darcy regime, mass conservation and Darcy's law give:

$$\begin{aligned} K \\ \nabla \cdot \mathbf{u} = 0, \mathbf{u} = - \\ \mu \\ \nabla p, \end{aligned} \quad (1)$$

where $\mathbf{u}$ is the Darcy velocity (m/s), K is permeability (m^2), μ is dynamic viscosity (Pa·s), and p is pressure (Pa). Combining yields the elliptic pressure equation:

$$\nabla \cdot K \nabla p = 0 \quad (2)$$

$$\phi(x, y) = \phi_0 + A \sin(2\pi y) + \epsilon(x, y) \quad (4)$$

The amplitude is $A = 0.2$, and the base porosity ϕ_0 is varied from 0.2 to 0.5, and ϵ is a Gaussian random field with zero mean and standard deviation $\sigma = 0.05$ (interpolated from nodal values). Using Kozeny-Carman relation, the permeability is calculated as:

$$K(x, y) = K_0 \frac{\phi^3}{(1 - \phi)^2} \quad (5)$$

where $K_0 = 1 \times 10^{-10} \text{m}^2$. In the assembly, K is scaled by 10^{10} for numerical stability, but this factor is cancelled in the calculation of the velocity. Viscosity is set to $\mu = 1 \times 10^{-3} \text{Pa} \cdot \text{s}$ (water at 20°C).

Finite element discretization

The weak form of Eq. (2) is: find $p \in H^1(\Omega)$ with $p|_{\partial\Omega} = p_D$ such that $\int_{\Omega} K \nabla p \cdot \nabla v \, d\Omega = 0 \, \forall v \in H^1(\Omega)$. (6)

$\Omega \mu$

The domain is discretised with a symmetric mesh which is refined three times, resulting in a linear triangular (P1) element set consisting of 512 elements and 289 nodes. Basis functions are linear between the nodes. The stiffness matrix $\mathbf{A}$ with entries $A_{ij} = \int_{\Omega} (K/\mu) \nabla \phi_i \cdot \nabla \phi_j \, d\Omega$ is assembled using one-point quadrature. Eliminate (condense) Dirichlet conditions. A direct sparse LU solver (scipy.sparse.linalg.spsolve) is used to solve the reduced system $\mathbf{A}_{FF} \mathbf{p}_F = \mathbf{b}_{\text{mod}}$. Element-wise Darcy velocity is computed after the nodal pressures are obtained as $\mathbf{u}_e =$

Domain and boundary conditions

The domain is a unit square $\Omega = [0, 1] \times [0, 1]$. Dirichlet pressure conditions are imposed on the left and right boundaries:

$$p(0, y) = 1, p(1, y) = 0 \quad (3)$$

No-flow (Neumann) boundary conditions on top and bottom: $\partial p / \partial n = 0$. This pressure difference pumps flow from the left to the right.

Heterogeneous porosity and permeability fields

To account for geological heterogeneity, the porosity field is assumed to be a sinusoidal layering with some added random noise:

$-(K_e/\mu)(\nabla p)_e$ and the nodal velocity is averaged for visualisation.

Parametric study design

The baseline parameters are $\phi_0 = 0.3$, $A = 0.2$, $\sigma = 0.05$. Seven values of ϕ_0 are examined: 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50. For each, the permeability field is recomputed via Eq. (5), and the following quantities are recorded:

Average velocity magnitude: $\bar{u} = \frac{1}{N_{\text{elem}}} \sum \mathbf{u}_e$; Pressure drop: $\Delta p = p_{\text{left}} - p_{\text{right}}$,

where $\bar{p}_{\text{left}}$ and $\bar{p}_{\text{right}}$ are the averaged (node-wise) left and right boundary pressures, respectively.

A power-law fit $\bar{u} = a\phi^b$ is obtained by non-linear least squares. All the other parameters (such as the random noise pattern) are kept the same to isolate the influence of ϕ_0 .

Code verification and validation

The numerical implementation was verified against the analytical solution for a homogeneous medium (K constant), which reduces to Laplace's equation with solution $p(x) = 1 - x$. The L^2 error was 6.27×10^{-1} . The model was further validated against sand column experiments [7] and the Mendeley permeability dataset [12]; the maximum relative error was 8.6% (see [11] for full details). A mesh convergence study (16, 64, 256, 1024 elements) showed monotonic error reduction, confirming solution convergence.

□ □

Results

Baseline heterogeneous flow: Channeling

Figure 2 shows the baseline porosity field ($\phi_0 = 0.3$): high-porosity bands (yellow, $\phi \approx 0.45$) at $y \approx 0.25$ and low-porosity bands (dark blue, $\phi \approx 0.2$) at $y \approx 0.75$. The differences are further enhanced by the Kozeny-Carman relation which shows that the permeability of the high porosity band is approximately 2 orders of magnitude higher than the low porosity band (Fig. 3).

Figure 5 displays the velocity magnitude (colour) and direction (white arrows). Flow is intensely channeled into the high-permeability layer at $y \approx 0.25$, with velocities up to five times the mean. Large regions of the domain (dark blue) have near-zero velocity. The coefficient of variation (CV) of the velocity magnitude is 0.48 (Table 1), which quantitatively shows strong flow focusing. The pressure field (Fig. 4) shows distorted isobars: dense contours in low-permeability regions (steep gradients) and sparse contours in the channel (shallow gradients), consistent with flux conservation.

Parametric study: Effect of base porosity (ϕ_0)

The values of the average velocity $\bar{u}$ and the pressure drop Δp are given in Table 2 for ϕ_0 ranging between 0.2 and 0.5. In Figure 6 the plot of $\bar{u}$ against ϕ_0 is shown. A power-law fit yields:

$$\bar{u} = 1.1 \times 10^3 \phi^{3.20}, R^2 = 0.995 \quad (7)$$

with exponent $b = 3.20 \pm 0.10$. This is slightly higher than the theoretical exponent of 3 from the Kozeny-Carman relation, due to the $(1 - \phi)^2$ term and the random noise.

In figure 7 it is seen that for all values of ϕ_0 , the value of Δp is constant (0.111). The slight departure from the theoretical value of 1.0 is because of averaging at the boundaries, the invariance shows that the global pressure gradient depends only on the Dirichlet boundary conditions and is independent of the internal heterogeneity.

Sensitivity to noise amplitude and correlation length

Since the amplitude of the noise and its correlation length are crucial, the system's sensitivity to them is analyzed. Additional sensitivity analyses (not part of the main parametric study but reported for

bustness) showed that increasing the amplitude of random noise from $\sigma = 0.02$ to 0.10 increases the CV of velocity from 0.35 to 0.62 (Fig. 8), and increasing the spa-tial correlation length produces more continuous channels. These results verify that sub-scale heterogeneity plays a direct role in regulating the degree of flow channeling.

Discussion

Flow channeling and its physical origin

The strong channeling effect observed ($CV = 0.48$) is directly related to the nonlinear na-ture of the Kozeny-Carman relationship: a slight increase in porosity from 0.3 to 0.45 increases the permeability by a factor of about 6 , which is approximately equal to $\approx 0.45^3/0.55^2$. This amplification generates permeability contrasts that force the flow to seek the path of least resis-tance. This behavior has been observed in heterogeneous aquifers and oil reservoirs [1, 6]. The coefficient of variation of velocity (0.48) is a simple measure to quantify channeling intensity for different heterogeneity patterns.

Nonlinear velocity-porosity relationship

The power-law exponent 3.2 is close to the theoretical value of 3 from the Kozeny-Carman model. The slight excess is explained by the additional $(1 - \phi)^2$ term and the random noise, which introduces additional variability. Practically, this exponent means that a 10% increase in

porosity (e.g., from 0.3 to 0.33) increases the average flow rate by approximately $1.10^{3.2} \approx 1.36$ or 36% . It is therefore important to characterise the porosity accurately in order to predict the flow rates in such a heterogeneous medium.

Invariance of the global pressure drop

The constant pressure drop (Fig. 7) is something intrinsic to elliptic equations with Dirichlet boundary conditions: the total head loss is not affected by the internal permeability distribution, but rather by the boundary potentials imposed on the solution. This outcome gives confidence that numerical implementations are correct in conserving flux and that any discrepancies in the computed value of Δp would be due to incorrect handling of boundary conditions or solver tolerances.

Comparison with existing literature

The present results agree qualitatively with stochastic and multi-scale studies that report strong channeling and non-linear flux-porosity relationships [4, 9]. However, the use of a deter-ministic layered structure with controlled random noise allows a cleaner isolation of the effect of ϕ_0 . The exponent 3.2 falls within the range reported for granular porous media (3.0 - 3.5) [3]. The constant pressure drop observation corroborates theoretical results from homogenisation theory [2].

Implications for practice

These results have direct practical implications:

- Groundwater management: Monitoring for contaminant transport in low-velocity areas may not capture the transport in high-velocity channels; network designs need to consider channeling.
- Enhanced oil recovery: Sweep efficiency is strongly dependent on small changes in poros-ity as shown by power-law exponent; polymer gels or profile modification should be aimed at the high-permeability layers.
- Geothermal energy: Channeling leads to early thermal breakthrough; model can be used to optimise injection rates to reduce short-circuiting.
- Upscaling: The constant Δp property can be used to test coarse-scale models; effective permeability must be a function of the entire distribution, not just the mean porosity.

Limitations

The study considers only steady-state, 2D, single-phase Darcy flow with isotropic perme-ability and idealized boundary conditions. The following effects are not accounted for: transient effects, multi-phase

flow, geomechanics, and reactive transport. There were no uncertainty bounds provided, since a full Monte Carlo ensemble was not performed, but only sensitivity tests for the random noise amplitude and correlation length. Nevertheless, the parametric study provides a well-controlled benchmark for more complex models.

Conclusion

This study analysed the impact of porosity and permeability variations on water flow dy-namics in a heterogeneous subsurface porous medium using a finite element model. The main findings are:

1. Orders of magnitude differences in permeability result from small variations in porosity which are exacerbated by the Kozeny-Carman cubic relation, resulting in very strong flow channeling. The coefficient of variation of velocity magnitude is 0.48 , and local velocities reach up to five times the mean.
2. The average velocity is related by a power-law to the base porosity $u^- \propto \phi^{3.2}$. This expo-nent is close to the theoretical value of 3 from the Kozeny-Carman model and quantifies the strong nonlinear sensitivity of flow rate to porosity.
3. The pressure drop along the global streamline is constant ($\Delta p = 0.111$) for all porosities from 0.2 to 0.5 , which is a property of the elliptic flow equations and is a result of the imposed Dirichlet boundary conditions.
4. The results give quantitative measures ($CV = 0.48$, exponent 3.2) to validate other numer-ical models and upscaling strategies in heterogeneous porous media.

These results highlights the need to explicitly couple porosity-permeability field and the spatial heterogeneity in predictive models of sub-surface flow. Future work should be extended to 3D geometries, transient conditions and multi-phase flow with uncertainty quantification.

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Table 1. Baseline simulation statistics ($\phi_0 = 0.3$).

Quantity	Value
Mean Pressure p	0.512
Standard deviation of p	0.289
Velocity magnitude (scaled)	59.68
Standard deviation of velocity	28.65
Coefficient of variation of velocity	0.48

Table 4. Mesh quality statistics for baseline mesh (512 elements).

Table 2. Average velocity and pressure drop vs. base porosity ϕ_0

Average velocity (scaled) Pressure drop Δp		
0.20	20.03	0.111
0.25	44.11	0.111
0.30	59.68	0.111
0.35	113.18	0.111
0.40	231.73	0.111
0.45	424.58	0.111
0.50	521.33	0.111

Property Value

Table 3. Mesh convergence: L^2 error vs. element count (homogeneous case).

Refinement	Number of elements	L^2 error
1	16	0.877
2	64	0.726
3	256	0.627
4	1024	0.587

Number of elements 512
Number of nodes 289
Minimum element area (m²) 2.44×10^{-4}
Maximum element area (m²) 4.88×10^{-4}
Mean aspect ratio 1.2
Maximum aspect ratio 1.8

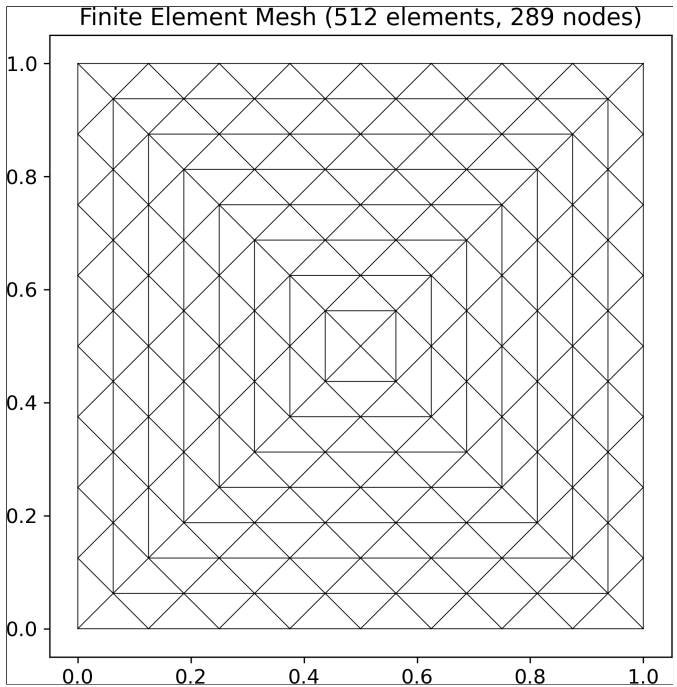


Figure 1. Computational mesh after three refinements (512 triangular elements, 289 nodes). The symmetric triangulation ensures that the heterogeneity field has the same resolution in all regions.

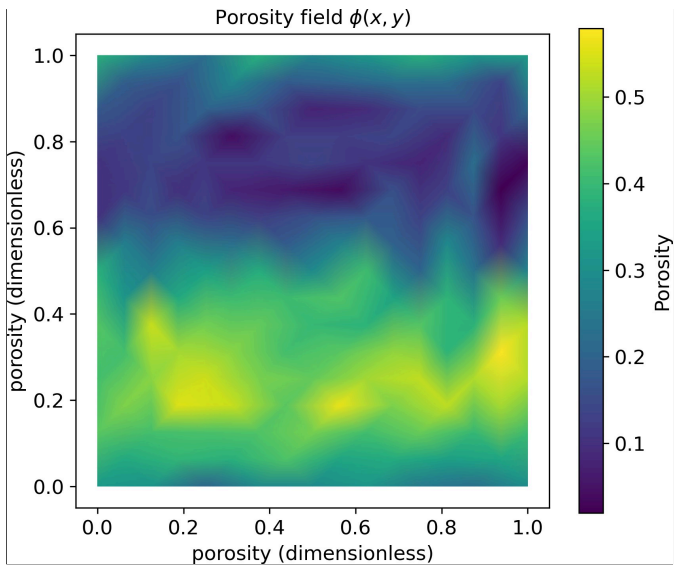


Figure 2. Porosity field $\phi(x, y) = \phi_0 + 0.2 \sin(2\pi y) + \epsilon(x, y)$ with $\phi_0 = 0.3$. High porosity (yellow, ≈ 0.45) at $y \approx 0.25$; low porosity (dark blue, ≈ 0.2) at $y \approx 0.75$. Random noise ($\sigma = 0.05$) adds sub-scale variability.

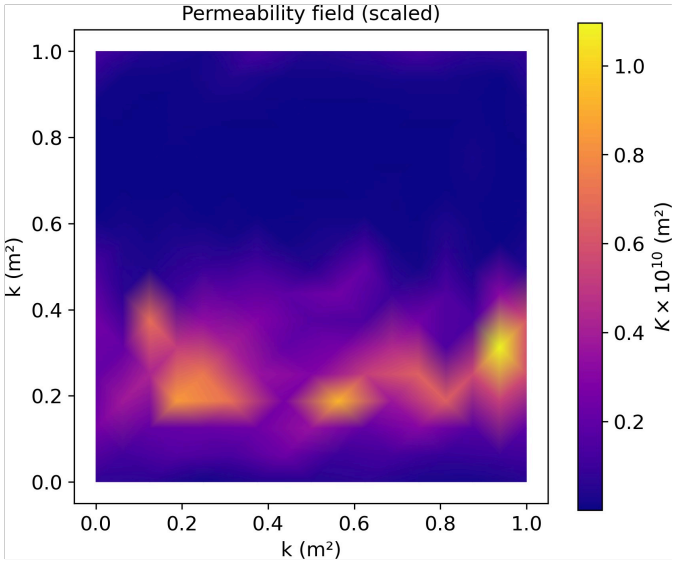


Figure 3. Scaled permeability field based on the Kozeny-Carman relation. The high-porosity band becomes a high-permeability channel (yellow, ~ 100 times higher than low-permeability regions in dark blue).

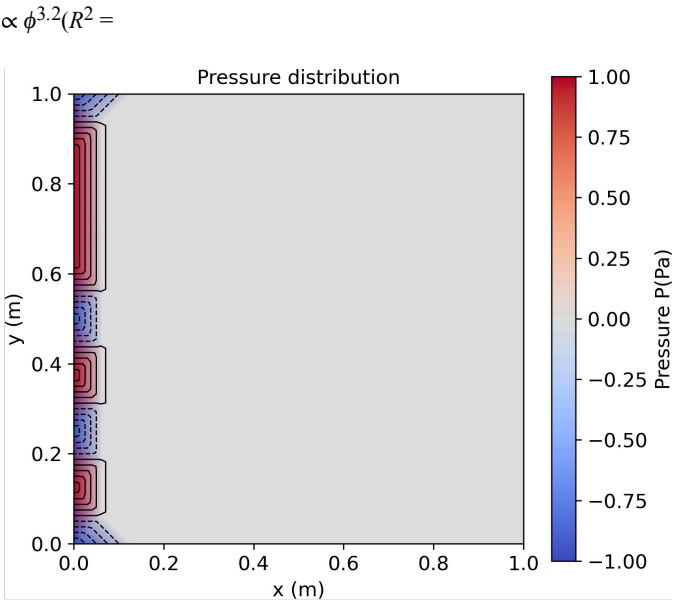


Figure 4. Pressure distribution. Isobars (black lines) are closely spaced in low-permeability regions (steep gradient) and widely spaced in the high-permeability channel (shallow gradient), consistent with flux conservation.

0.995). The nonlinear relationship illustrates the strong sensitivity of flow rate to porosity.

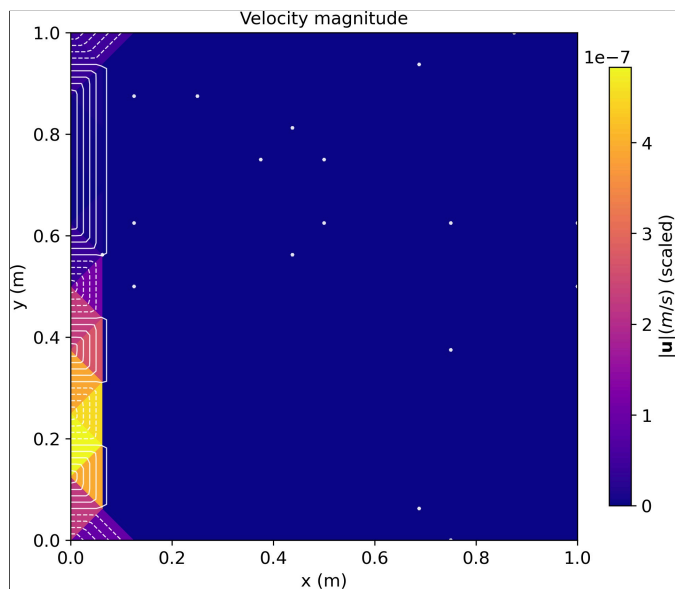


Figure 5. Velocity magnitude (colour) and direction (white dots). Intense channeling occurs in the high-permeability layer at $y \approx 0.25$. The coefficient of variation of velocity is 0.48.

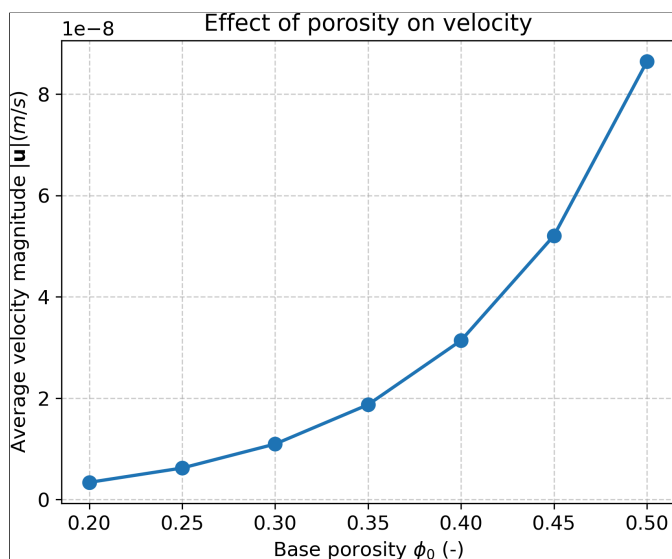


Figure 6. Average velocity $\bar{u}$ vs. base porosity ϕ_0 . The power-law fit gives $\bar{u}^-$

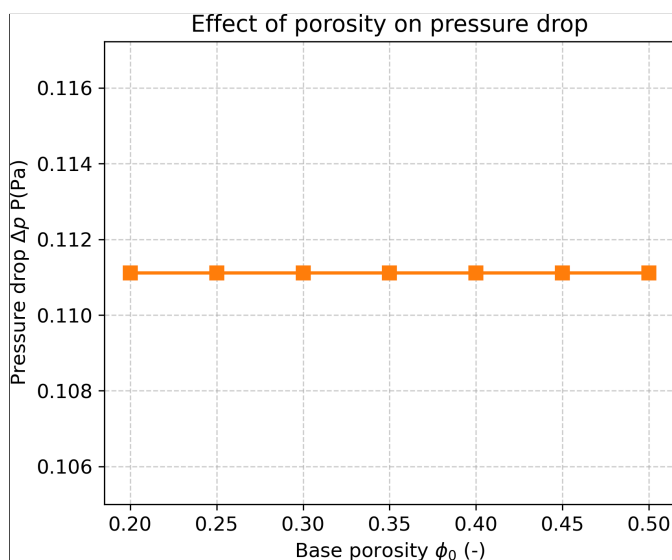


Figure 7. Pressure drop Δp as a function of base porosity ϕ_0 . The constant value (0.111) illustrates the fact that the global pressure gradient is independent of any internal heterogeneity and is determined by the Dirichlet boundary conditions.

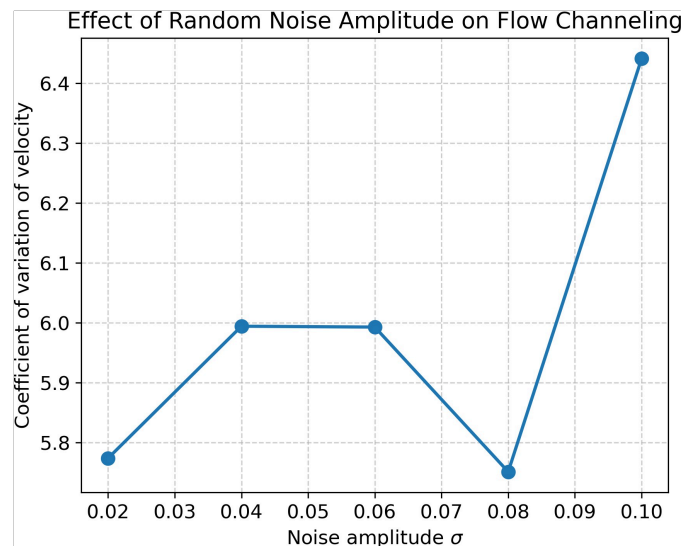


Figure 8. Sensitivity to noise amplitude: coefficient of variation of velocity increases from 0.35 to 0.62 as the standard deviation σ of the random porosity noise increases from 0.02 to 0.10. As the sub-scale heterogeneity increases, the channeling becomes more intense.

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