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Numerical range decomposition via primitive ideals and irreducible representations

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Abstract

The spatial numerical range of a bounded linear operator on a complex Banach space is not necessary closed or convex. The closed convex hull, on the other hand, is a compact convex set that contains all of the information about the numerical ranges. In this paper, a primitive ideal framework is developed from families of quotients and fibres numerical ranges. They are connected with prime and primitive ideals of the operator algebra to construct the closed convex hull. The Jacobson topology is used on the primitive ideal space and it is shown that primitive ideals are assigned to the quotient numerical ranges of the operator in an order-reversing manner. The key findings are that the full family of primitive ideals achieve complete numerical range envelope. However, finer decompositions are achieved by not including the zero ideal and Limiting to separating subfamilies of primitive ideal. All the findings are well illustrated.

Keywords: Irreducible representation, Separation family, Jacobson topology, Numerical Range, Primitive ideal and Banach algebra

Introduction

One of the main tools of operator theory has been the numerical range of a bounded linear operator, which is an essentially single subset of the complex plane, carrying at the same time spectral, geometric and algebraic information about the operator ^[5, 8]. In a Hilbert space, the classical numerical range of a quadratic form is the set of values that it assumes on the unit vectors and by the Toeplitz–Hausdorff theorem this set is always convex ^[9]. For inner product-free underlying space, Gunter Lumer's original idea was taken up by Frank F. Bonsall and John Duncan to develop the spatial numerical range, normed by norming pairs of vectors which was described in their two monographs entitled Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras ^[1, 2] respectively and was documented in great detail. In this generality convexity is not automatic. On the Banach space of square-summable sequences, the spatial numerical range of the unilateral shift is the open unit disc but this set can be neither closed nor convex on a general Banach space ^[5].

Due to the possible failure of convexity, it is common to turn to the topological closure or the closed convex hull of the spatial numerical range. This closed convex hull is exactly that of the algebra numerical range ^[6] which is generated by taking the operator at all states of the enveloping operator algebra ^[1, 2] and Bonsall and Duncan proved that the natural goal of any decomposition theory is to recover this convex envelope from smaller local pieces of the algebra ^[7]. One initial strategy is to divide by a closed two-sided ideal and to take the algebra range of the quotient. The quotient numerical range thus generated is always nonempty, compact and convex and for prime ideals it contains always the original spatial numerical range in the closed convex hull. The problem is that the zero ideal is prime so the entire family of prime ideals reduces to the closed convex hull without providing any finer structure and the set of prime ideals has no natural topology that links one quotient to another ^[3]. In recent years the interest in the numerical range and its convexity properties has not decreased but the questions raised by the numerical range remain open. In the non-unital setting, the convexity of the numerical range is sensitive whilst it is known that the numerical range of a convex

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Algebra is convex, the same is not known to be true of the spatial numerical range of a non-unital normed algebra. The question being open even for the Beurling algebra $L^1(G, \omega)$ [13]. The question of when a numerical range provides a spectral set in the sense of von Neumann's inequality has also been extended from operators on Hilbert spaces to general Banach algebras with positive answers for matrix algebras and for the Calkin algebra and a family of counterexamples for other Banach algebras by Blazhko et al. [16]. A related issue is the instability of the behaviour of the spectrum when considered as a function of quotient-construction. Brits constructed a general one demonstrating that for any given Banach algebra adding an appropriate inessential ideal in an algebra can make the spectrum function discontinuous [12]. On the representation-theoretic side; Farenick, Li, and Singla [14] have been actively developing boundary representations and matrix or operator-system ranges for finite-dimensional operator systems and the development of the extreme-point and boundary-representation machinery continues for which representations detect the boundary of a convex envelope as developed by Davidson and Kennedy [15]. Furthermore, the primitive ideal space and its Jacobson topology are also of current interest and recently Chen and Li have explicitly computed this space along with its Jacobson topology for the semigroup crossed product C^* -algebra of an integral domain [17]. None of these recent papers however focuses on the primitive ideal space through the Jacobson topology to aid in the reconstruction of the numerical range itself and this is exactly what the present paper fills. The prime ideal framework retains only the closed convex hull of the spatial numerical range in a trivial manner and without any topological structure in the arrangement of the various quotients relative to each other. Restricting attention to primitive ideals both flaws are corrected simultaneously. The kernels of the algebraically irreducible representations are always prime and the quotienting by them controls the numerical range and the Jacobson (or hull-kernel) topology is a quasi-compact and T_0 topology on the totality of primitive ideals recording how one primitive ideal is contained in the closure of another [10, 11]. In this paper, we then make a first attempt at numerical range decomposition in a primitive ideal setting. It is shown that the assignment of a primitive ideal to its associated quotient numerical range is order-reversing with respect to the specialization order on the primitive ideal space that the closed convex hull of the spatial numerical range is exactly recovered by the full primitive family and that the zero ideal being itself primitive is precisely why no finer reconstruction is possible from that family alone. That restricting to a separating subfamily of prime ideals one whose associated quotient-factoring states are weak*-dense in the state space and reconstructs the same closed convex hull while genuinely excluding the zero ideal thereby producing a nontrivial decomposition and that boundary representations realized through fibre numerical ranges attached to irreducible representations identify precisely the extreme points of the convex envelope. In each of these findings explicit corollaries and examples are provided and the paper ends with consolidating the main claims into one unifying theorem.

2. Notations, Definitions and Preliminaries

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{R}$ the field of real numbers. Let X denote a complex Banach space with $\dim X \geq 2$, X^* its continuous dual space, and $B(X)$ the unital Banach algebra of all bounded linear operators on X with identity operator I_X (or simply I) [4]. The standard dual pairing between $x \in X$ and $f \in X^*$ is written as (x, f) or $f(x)$. For any subset $S \subseteq \mathbb{C}$ or $S \subseteq B(X)$, $\bar{S}$ denotes its topological closure, $co(S)$ its convex hull, and $\overline{co}(S) = co(\bar{S})$ its closed convex hull. The boundary of $S \subseteq \mathbb{C}$ is denoted ∂S , and $Ext(K)$ denotes the set of extreme points of a compact convex set $K \subset \mathbb{C}$. The spectrum of an operator $T \in B(X)$ is denoted by $\sigma(T)$. The global numerical radius of an operator $T \in B(X)$ is defined as $v(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. The algebra numerical range of T in a unital Banach algebra A is $V(T, A)$ with state space $S(A)$, while $W(T)$ and $\overline{W}(T)$ denote the spatial numerical range of T and its topological closure, respectively. For a closed two-sided ideal $I \subseteq B(X)$, $[T]_I = T + I$ denotes the coset of T in the quotient Banach algebra $B(X)/I$, and $W[T]_I$ is the corresponding quotient numerical range. $ClPrime(A)$ and $Prim(A)$ denote the spaces of closed prime ideals and primitive ideals of A , respectively. For $P \in Prim(B(X))$ with algebraically irreducible representation $\pi_P: B(X) \rightarrow B(Y_P)$, $V_P(T) = W(\pi_P(T))$ denotes the fibre numerical range.

Definition 2.1 (Spatial and Algebra Numerical Range; Bonsall & Duncan 1971). Let $T \in B(X)$.

The spatial numerical range of T is

$$W(T) = \{(Tx, f) : x \in X, f \in X^*, \|x\| = \|f\| = 1, (x, f) = 1\}.$$

Its topological closure in $\mathbb{C}$ is denoted $\overline{W}(T)$.

A linear functional $E \in B(X)^*$ is a state on $B(X)$ if $\|E\| = E(I) = 1$. The set of all states is denoted $S(B(X))$.

The algebra numerical range of $T \in B(X)$ is

$$V(T, B(X)) = \{E(T) : E \in S(B(X))\}$$

Lemma 2.2 (Bonsall & Duncan 1971) for every $T \in B(X)$, the algebra numerical range $V(T, B(X))$ is a nonempty, compact, and convex subset of $\mathbb{C}$.

Proof. Consider the one-dimensional subspace $\mathbb{C}I \subset B(X)$ and define $\psi: \mathbb{C}I \rightarrow \mathbb{C}$ by $\psi(\lambda I) = \lambda$. Clearly, $|\psi(\lambda I)| = |\lambda| = \|\lambda I\|$, so $\|\psi\| = 1$ and $\psi(I) = 1$. By the Hahn–Banach extension theorem, there exists a bounded linear functional $E \in B(X)^*$ such that $E|_{\mathbb{C}I} = \psi$ and $\|E\| = \|\psi\| = 1$. Since $E(I) = 1, E \in S(B(X))$, proving $S(B(X)) \neq \emptyset$ and hence $V(T, B(X)) \neq \emptyset$.

To establish convexity of $S(B(X))$, let $E_1, E_2 \in S(B(X))$ and $t \in [0, 1]$. Define $E_t = tE_1 + (1-t)E_2$. Then $E_t(I) = tE_1(I) + (1-t)E_2(I) = t + 1 - t = 1$. By the triangle inequality, $\|E_t\| \leq t\|E_1\| + (1-t)\|E_2\| = t + 1 - t = 1$. Since $|E_t(I)| = 1 \leq \|E_t\| \leq 1$, we conclude $\|E_t\| = 1$, so $E_t \in S(B(X))$. The state space $S(B(X))$ is the intersection of the weak-* closed unit ball of

$B(X)^*$ with the weak-* closed hyperplane $\{E \in B(X)^*: E(I) = 1\}$. By the Banach–Alaoglu theorem, the closed unit ball of $B(X)^*$ is weak-* compact. Thus $S(B(X))$ is weak-* compact.

The evaluation map $ev_T: B(X)^* \rightarrow \mathbb{C}$ defined by $ev_T(E) = E(T)$ is linear and continuous in the weak-* topology. Therefore, $V(T, B(X)) = ev_T(S(B(X)))$, being the continuous image of a compact convex set under a linear map, is compact and convex in $\mathbb{C}$. $\square$

Theorem 2.3 (Bonsall–Duncan Theorem). For every $T \in B(X)$,

$$V(T, B(X)) = \overline{co} W(T).$$

Proof. Step 1: Inclusion $\overline{co} W(T) \subseteq V(T, B(X))$. Let $\lambda \in W(T)$. Then $\lambda = f(Tx)$ for some $x \in X$, $f \in X^*$ with $\|x\| = \|f\| = f(x) = 1$. Define the functional $E_{x,f}: B(X) \rightarrow \mathbb{C}$ by $E_{x,f}(S) = f(Sx)$ for all $S \in B(X)$. Then $|E_{x,f}(S)| \leq \|S\|$, so $\|E_{x,f}\| \leq 1$. Since $E_{x,f}(I) = 1$, we have $\|E_{x,f}\| = 1$, which implies $E_{x,f} \in S(B(X))$. Thus $\lambda = E_{x,f}(T) \in V(T, B(X))$. This shows $W(T) \subseteq V(T, B(X))$. Since $V(T, B(X))$ is closed and convex by Lemma 2.2, taking the closed convex hull gives $\overline{co} W(T) \subseteq V(T, B(X))$.
Step 2: Inclusion $V(T, B(X)) \subseteq \overline{co} W(T)$. Both $V(T, B(X))$ and $\overline{co} W(T)$ are compact convex subsets of $\mathbb{C} \cong \mathbb{R}^2$. By the Hahn–Banach separation theorem, a compact convex set in $\mathbb{R}^2$ is uniquely determined by its support functions. For any $\theta \in [0, 2\pi)$, define the support function $h_A(\theta) = \sup\{Re(e^{-i\theta}z) : z \in A\}$. By standard numerical radius estimations (cf. ^[1], Theorem 9.4),

$$\sup\{Re(e^{-i\theta}E(T)) : E \in S(B(X))\} = \sup\{Re(e^{-i\theta}\lambda) : \lambda \in W(T)\}.$$

Linearity of the real-part functional implies $\sup_{\lambda \in W(T)} Re(e^{-i\theta}\lambda) = \sup_{z \in \overline{co} W(T)} Re(e^{-i\theta}z)$. Thus, $h_{V(T, B(X))}(\theta) = h_{\overline{co} W(T)}(\theta)$ for all $\theta \in [0, 2\pi)$. Consequently, $V(T, B(X)) = \overline{co} W(T)$.

Definition 2.4 (Quotient Numerical Range). For a closed two-sided ideal $I \subseteq B(X)$ and $T \in B(X)$, the quotient numerical range of T with respect to I is

$$W[T]_I = V([T]_I, B(X)/I) = \{E([T]_I) : E \in S(B(X)/I)\}.$$

Lemma 2.5 (Pullback of States). Let I be a closed two-sided ideal of $B(X)$ and $\pi_I: B(X) \rightarrow B(X)/I$ be the canonical quotient homomorphism defined by $\pi_I(S) = [S]_I$. If $E \in S(B(X)/I)$, then the composition $E \circ \pi_I$ belongs to $S(B(X))$.

Proof. Let $E \in S(B(X)/I)$. Then $\|E\| = 1$ and $E([I]_I) = 1$. For any $S \in B(X)$,

$$|(E \circ \pi_I)(S)| = |E([S]_I)| \leq \|E\| \| [S]_I \| = \| [S]_I \| = \inf_{R \in I} \|S + R\| \leq \|S\|.$$

Hence $\|E \circ \pi_I\| \leq 1$. Furthermore $(E \circ \pi_I)(I) = E([I]_I) = 1$. Therefore, $\|E \circ \pi_I\| = 1$ and $(E \circ \pi_I)(I) = 1$, which proves $E \circ \pi_I \in S(B(X))$. $\square$

Theorem 2.6 (Universal Inclusion for Quotients). For every closed two-sided ideal I of $B(X)$ and every $T \in B(X)$,

$$W[T]_I \subseteq \overline{co} W(T).$$

Proof. Let $\mu \in W[T]_I$. By Definition 2.4, there exists a state $E \in S(B(X)/I)$ such that $\mu = E([T]_I)$. By Lemma 2.5, $\psi = E \circ \pi_I \in S(B(X))$. Thus,

$$\mu = E([T]_I) = E(\pi_I(T)) = \psi(T) \in V(T, B(X)).$$

Since μ was arbitrary, $W[T]_I \subseteq V(T, B(X))$. Applying Theorem 2.3 yields $W[T]_I \subseteq \overline{co} W(T)$. $\square$

Theorem 2.7 (Spectral Inclusion for Quotients). For every closed two-sided ideal I of $B(X)$ and every $T \in B(X)$,

$$\sigma([T]_I) \subseteq \sigma(T).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma(T)$. Then $(T - \lambda I)$ is invertible in $B(X)$, so there exists $R \in B(X)$ satisfying $(T - \lambda I)R = R(T - \lambda I) = I$. Applying the quotient homomorphism π_I to both sides gives:

$$\pi_I(T - \lambda I)\pi_I(R) = \pi_I(R)\pi_I(T - \lambda I) = \pi_I(I),$$

which simplifies to $([T]_I - \lambda[I]_I)[R]_I = [R]_I([T]_I - \lambda[I]_I) = [I]_I$. Thus $[T]_I - \lambda[I]_I$ is invertible in $B(X)/I$, meaning $\lambda \notin \sigma([T]_I)$. By contraposition, $\sigma([T]_I) \subseteq \sigma(T)$. $\square$

Theorem 2.8 (Monotonicity under Inclusion of Ideals). Let I and J be closed two-sided ideals of $B(X)$ with $I \subseteq J$. Then for every $T \in B(X)$,

$$\sigma([T]_J) \subseteq \sigma([T]_I) \subseteq \sigma(T) \quad \text{and} \quad W[T]_J \subseteq W[T]_I \subseteq \overline{\text{co}} W(T).$$

Proof. Since $I \subseteq J$, the canonical map $\rho_{I,J}: B(X)/I \rightarrow B(X)/J$ given by $\rho_{I,J}(T + I) = T + J$ is a well-defined bounded algebra homomorphism with norm $\|\rho_{I,J}\| \leq 1$. If $\lambda \notin \sigma([T]_I)$, there exists $[R]_I \in B(X)/I$ such that $([T]_I - \lambda[I]_I)[R]_I = [I]_I$. Applying $\rho_{I,J}$ yields $([T]_J - \lambda[I]_J)[R]_J = [I]_J$, so $\lambda \notin \sigma([T]_J)$. Combined with Theorem 2.7, $\sigma([T]_J) \subseteq \sigma([T]_I) \subseteq \sigma(T)$.

Now let $\mu \in W[T]_J$. Then $\mu = \psi([T]_J)$ for some $\psi \in S(B(X)/J)$. Define $E = \psi \circ \rho_{I,J}: B(X)/I \rightarrow \mathbb{C}$. Then $\|E\| \leq \|\psi\| \|\rho_{I,J}\| \leq 1$ and $E([I]_I) = \psi(\rho_{I,J}([I]_I)) = \psi([I]_J) = 1$. Hence $E \in S(B(X)/I)$. Thus $\mu = \psi([T]_J) = \psi(\rho_{I,J}([T]_I)) = E([T]_I) \in W[T]_I$. Combined with Theorem 2.6, $W[T]_J \subseteq W[T]_I \subseteq \overline{\text{co}} W(T)$. $\square$

Definition 2.9 (Prime and Primitive Ideal; Dales 2000). Let A be a Banach algebra of bounded linear operators, then:

- A proper closed two-sided ideal $P \subset A$ is prime if for all closed two-sided ideals $I, J \subseteq A$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. The set of all closed prime ideals of A is denoted $ClPrime(A)$.
- A proper closed two-sided ideal $P \subset A$ is primitive if $P = \ker \pi$ for some algebraically irreducible representation $\pi: A \rightarrow B(Y)$ on a complex Banach space Y . The set of all primitive ideals of A is denoted $Prim(A)$.

Lemma 2.10. Every primitive ideal of a Banach algebra A is a closed prime ideal. That is, $Prim(A) \subseteq ClPrime(A)$.

Proof. Let $P = \ker \pi \in Prim(A)$, where $\pi: A \rightarrow B(Y)$ an algebraically irreducible representation is. Since π is continuous, $P = \pi^{-1}(0)$ is closed. To show P is prime, let I, J be closed two-sided ideals of A such that $IJ \subseteq P$. Then $\pi(I)\pi(J) = \pi(IJ) = 0$. Consider the subspace $V = \overline{\text{span}} \pi(J)Y \subseteq Y$. Since J is an ideal, V is $\pi(A)$ -invariant. By algebraic irreducibility of π , either $V = 0$ or $V = Y$. Case 1: If $V = 0$, then $\pi(J)y = 0$ for all $y \in Y$, so $\pi(J) = 0$. Thus $J \subseteq \ker \pi = P$.

Case 2: If $V = Y$, then $\pi(I)Y = \pi(I)(\pi(J)Y) = (\pi(I)\pi(J))Y = 0Y = 0$. Thus $\pi(I) = 0$, so $I \subseteq P$. In either case, $I \subseteq P$ or $J \subseteq P$, so P is prime. $\square$

Lemma 2.11. If A is a simple unital Banach algebra, then $0 \in Prim(A)$.

Proof. Let M be a maximal left ideal of A . By Zorn's lemma and unitality, maximal left ideals exist and are closed. The quotient space $Y = A/M$ is a complex Banach space, and the map $\pi: A \rightarrow B(Y)$ defined by $\pi(a)(b + M) = ab + M$ is an algebraically irreducible representation. The kernel $\ker \pi = \{a \in A: aA \subseteq M\}$ is a two-sided ideal contained in $M \neq A$. Since A is simple, its only proper two-sided ideal is 0. Thus $\ker \pi = 0$, so $0 \in Prim(A)$. $\square$

Corollary 2.12. Every maximal proper closed two-sided ideal of $B(X)$ is primitive.

Proof. If M is a maximal proper closed two-sided ideal of $B(X)$, then the quotient algebra $B(X)/M$ is a simple unital Banach algebra. By Lemma 2.11, $0 \in Prim(B(X)/M)$. Pulling back through the canonical quotient map shows $M \in Prim(B(X))$. $\square$

Corollary 2.13. If X is a non-zero complex Banach space, then $0 \in Prim(B(X))$.

Proof. The identity map $id: B(X) \rightarrow B(X)$ is an algebraically irreducible representation of $B(X)$ on X . Its kernel is $\ker(id) = 0$, hence $0 \in Prim(B(X))$. $\square$

Corollary 2.14. (Calkin, 1941) If H is an infinite-dimensional Hilbert space, then both 0 and the compact operator ideal $K(H)$ belong to $Prim(B(H))$.

Proof. $0 \in Prim(B(H))$ By Corollary 2.13. The ideal of compact operators $K(H)$ is the unique maximal closed two-sided ideal of $B(H)$ when H is separable (and maximal in general). By Corollary 2.12, $K(H) \in Prim(B(H))$. $\square$

Definition 2.15. (Jacobson Topology; Jacobson, 1945). The Jacobson topology (or hull-kernel topology) on $Prim(B(X))$ is defined by specifying the closed sets to be of the form

$$\text{hull}(I) = \{P \in Prim(B(X)): I \subseteq P\},$$

Where I ranges over all closed two-sided ideals of $B(X)$.

Theorem 2.16 (Jacobson 1945 & S. Willard 1970). The space $Prim(B(X))$ equipped with the Jacobson topology is a T_0 , quasi-compact topological space.

Proof. Step 1: Axioms of Topology. (i) $\text{hull}(0) = Prim(B(X))$ and $\text{hull}(B(X)) = \emptyset$. (ii) For any family of ideals I_α ,

$$\bigcap_{\alpha} \text{hull}(I_\alpha) = \{P: I_\alpha \subseteq P \text{ for all } \alpha\} = \left\{P: \sum_{\alpha} I_\alpha \subseteq P\right\} = \text{hull}\left(\sum_{\alpha} I_\alpha\right).$$

For any two ideals I, J , $\text{hull}(I) \cup \text{hull}(J) = \text{hull}(IJ)$. If $P \in \text{hull}(I) \cup \text{hull}(J)$, then $I \subseteq P$ or $J \subseteq P$, so $IJ \subseteq I \cap J \subseteq P$, meaning $P \in \text{hull}(IJ)$. Conversely, if $P \in \text{hull}(IJ)$, then $IJ \subseteq P$. By Lemma 2.10, P is prime, so $I \subseteq P$ or $J \subseteq P$, meaning $P \in \text{hull}(I) \cup \text{hull}(J)$.

Step 2: T_0 Property. Let $P, Q \in \text{Prim}(B(X))$ with $P \neq Q$. Without loss of generality, assume $P \not\subseteq Q$. Then $Q \notin \text{hull}(P)$, while $P \in \text{hull}(P)$. Thus $U = \text{Prim}(B(X)) \setminus \text{hull}(P)$ is an open set containing Q but not P . Hence the space is T_0 .

Step 3: Quasi-compactness. Let $\{U_\alpha\}$ be an open cover of $\text{Prim}(B(X))$, where $U_\alpha = \text{Prim}(B(X)) \setminus \text{hull}(I_\alpha)$ for closed ideals I_α . Then $\bigcap_\alpha \text{hull}(I_\alpha) = \emptyset$, which implies $\text{hull}(\sum_\alpha I_\alpha) = \emptyset$. Since every proper ideal is contained in a primitive ideal (Corollary 2.12), we must have $\sum_\alpha I_\alpha = B(X)$. Because $I \in B(X)$, there exist finitely many elements $a_1 \in I_{\alpha_1}, \dots, a_n \in I_{\alpha_n}$ such that $\|I - \sum_{j=1}^n a_j\| < 1$. By the Neumann series theorem, $S = \sum_{j=1}^n a_j$ is invertible in $B(X)$. Thus the ideal $\sum_{j=1}^n I_{\alpha_j}$ contains I and equals $B(X)$. Consequently, $\bigcap_{j=1}^n \text{hull}(I_{\alpha_j}) = \text{hull}(B(X)) = \emptyset$, so $\{U_{\alpha_j}\}_{j=1}^n$ covers $\text{Prim}(B(X))$. $\square$

3. Main Results

Definition 3.1 (Fibre Numerical Ranges). Let $P \in \text{Prim}(B(X))$ and let $\pi_P: B(X) \rightarrow B(Y_P)$ be an algebraically irreducible representation on a complex Banach space Y_P such that $\ker \pi_P = P$. The fibre numerical range of $T \in B(X)$ at P is

$$V_P(T) = W(\pi_P(T)) = \{(\pi_P(T)y, g) : y \in Y_P, g \in Y_P^*, \|y\| = \|g\| = 1, (y, g) = 1\}.$$

The localized numerical radius at P is defined as $v(\pi_P(T)) = \sup\{|\lambda| : \lambda \in V_P(T)\}$.

Theorem 3.2. (Refined Chain of Inclusions). For every $P \in \text{Prim}(B(X))$ and $T \in B(X)$,

$$\sigma(\pi_P(T)) \subseteq \sigma([T]_P) \subseteq \sigma(T) \quad \text{and} \quad V_P(T) \subseteq W[T]_P \subseteq \overline{\text{co}} W(T).$$

Proof. Since $\ker \pi_P = P$, the representation $\pi_P: B(X) \rightarrow B(Y_P)$ factors uniquely as $\pi_P = \tilde{\pi}_P \circ \pi_P^{\text{quot}}$, where $\pi_P^{\text{quot}}: B(X) \rightarrow B(X)/P$ is the canonical quotient map and $\tilde{\pi}_P: B(X)/P \rightarrow B(Y_P)$ is an injective continuous homomorphism. If $\lambda \notin \sigma([T]_P)$, then $[T]_P - \lambda[I]_P$ is invertible in $B(X)/P$. Applying $\tilde{\pi}_P$ maps this inverse to an inverse for $\pi_P(T) - \lambda I_{Y_P}$ in $B(Y_P)$. Thus $\lambda \notin \sigma(\pi_P(T))$, proving $\sigma(\pi_P(T)) \subseteq \sigma([T]_P)$. The second spectral inclusion is Theorem 2.7. For numerical ranges, let $\lambda \in V_P(T)$. Then $\lambda = g(\pi_P(T)y)$ for some $y \in Y_P$, $g \in Y_P^*$ with $\|y\| = \|g\| = 1$ and $(y, g) = 1$. Define $E: B(X)/P \rightarrow \mathbb{C}$ by $E([S]_P) = g(\tilde{\pi}_P([S]_P)y)$. Then $\|E\| \leq \|g\| \|\tilde{\pi}_P\| \|y\| \leq 1$ and $E([I]_P) = g(y) = 1$, so $E \in S(B(X)/P)$. Thus $\lambda = E([T]_P) \in W[T]_P$. Combining this with Theorem 2.6 establishes $V_P(T) \subseteq W[T]_P \subseteq \overline{\text{co}} W(T)$.

Theorem 3.3 (Order-Reversing Property). Let $P, Q \in \text{Prim}(B(X))$ with $P \subseteq Q$. Then for every $T \in B(X)$,

$$\sigma([T]_Q) \subseteq \sigma([T]_P) \subseteq \sigma(T) \quad \text{and} \quad W[T]_Q \subseteq W[T]_P \subseteq \overline{\text{co}} W(T).$$

In particular, the assignment $P \mapsto W[T]_P$ is an order-reversing map with respect to ideal inclusion.

Proof. Since $P \subseteq Q$, Theorem 2.8 applies directly with $I = P$ and $J = Q$, yielding $\sigma([T]_Q) \subseteq \sigma([T]_P)$ and $W[T]_Q \subseteq W[T]_P$. $\square$

Theorem 3.4 (Collapse of Full Primitive Family). For every $T \in B(X)$,

$$\overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} W[T]_P = \overline{\text{co}} W(T).$$

Proof. By Theorem 2.6, $W[T]_P \subseteq \overline{\text{co}} W(T)$ for every $P \in \text{Prim}(B(X))$. Taking the union over all P and then the closed convex hull yields:

$$\overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} W[T]_P \subseteq \overline{\text{co}} W(T). \quad (1)$$

For the reverse inclusion, Corollary 2.13 establishes that $0 \in \text{Prim}(B(X))$. Setting $P = 0$, the quotient algebra $B(X)/0$ is canonically isometric to $B(X)$, so $W[T]_0 = V(T, B(X))$. By Theorem 2.3, $V(T, B(X)) = \overline{\text{co}} W(T)$. Thus:

$$\overline{\text{co}} W(T) = W[T]_0 \subseteq \bigcup_{P \in \text{Prim}(B(X))} W[T]_P \subseteq \overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} W[T]_P. \quad (2)$$

Combining both inclusions (1) and (2) proves equality. $\square$

Corollary 3.5 (Spectral Formula). For every $T \in B(X)$,

$$\bigcup_{P \in \text{Prim}(B(X))} \sigma([T]_P) = \sigma(T).$$

Proof. Theorem 2.7 shows $\sigma([T]_P) \subseteq \sigma(T)$ for all $P \in \text{Prim}(B(X))$, so $\bigcup_P \sigma([T]_P) \subseteq \sigma(T)$. Since $0 \in \text{Prim}(B(X))$ by Corollary 2.13, $\sigma([T]_0) = \sigma(T)$, which forces equality. $\square$

Remark 3.6 Theorem 3.4 and Corollary 3.5 prove that the full primitive ideal space trivially recovers the global numerical range envelope and spectrum because of the single element $P = 0$. To obtain non-trivial local-to-global decompositions, one must restrict to subfamilies that omit 0.

Definition 3.7 (Separating Family). A family of closed prime ideals $\mathcal{F} = \{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{ClPrime}(B(X))$ is called separating if the set of quotient-factoring states

$$\Phi_{\mathcal{F}} = \{E \circ \pi_{P_\alpha} : \alpha \in \Lambda, E \in S(B(X)/P_\alpha)\}$$

is weak-* dense in $S(B(X))$.

Theorem 3.8 (Reconstruction via Separating Families). Let $\mathcal{F} = \{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{ClPrime}(B(X))$ be a separating family of prime ideals. Then for every $T \in B(X)$,

$$\overline{\text{co}} \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha} = \overline{\text{co}} W(T).$$

Proof. Step 1: Inclusion ($\subseteq$). For each $\alpha \in \Lambda$, Theorem 2.6 gives $W[T]_{P_\alpha} \subseteq \overline{\text{co}} W(T)$. Taking the union and closed convex hull gives $\overline{\text{co}} \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha} \subseteq \overline{\text{co}} W(T)$.

Step 2: Inclusion ($\supseteq$). Let $K = \overline{\text{co}} \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha}$. The set K is a compact convex subset of $\mathbb{C}$. Suppose for contradiction that there exists $z_0 \in \overline{\text{co}} W(T) \setminus K$. By the strict Hahn–Banach separation theorem on $\mathbb{C} \cong \mathbb{R}^2$, there exist a direction angle $\theta \in [0, 2\pi)$ and a real constant $c \in \mathbb{R}$ such that:

$$\text{Re}(e^{-i\theta} z_0) > c \geq \sup_{z \in K} \text{Re}(e^{-i\theta} z).$$

Since $z_0 \in \overline{\text{co}} W(T) = V(T, B(X))$ (Theorem 2.3), there exists a state $\psi_0 \in S(B(X))$ such that $\psi_0(T) = z_0$. Consequently, $\text{Re}(e^{-i\theta} \psi_0(T)) > c$. Define the open half-space $U = \{E \in B(X)^* : \text{Re}(e^{-i\theta} E(T)) > c\}$. Since the evaluation functional $E \mapsto E(T)$ is weak-* continuous, U is a weak-* open neighborhood of ψ_0 in $B(X)^*$. Because $\mathcal{F}$ is separating, Definition 3.7 asserts that $\Phi_{\mathcal{F}}$ is weak-* dense in $S(B(X))$. Thus $\Phi_{\mathcal{F}} \cap U \neq \emptyset$. Let $\psi \in \Phi_{\mathcal{F}} \cap U$. Then $\psi = E_\beta \circ \pi_{P_\beta}$ for some $\beta \in \Lambda$ and $E_\beta \in S(B(X)/P_\beta)$. Calculating the value of $\psi(T)$:

$$\psi(T) = (E_\beta \circ \pi_{P_\beta})(T) = E_\beta([T]_{P_\beta}) \in W[T]_{P_\beta} \subseteq \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha} \subseteq K.$$

Since $\psi \in U$, we have $\text{Re}(e^{-i\theta} \psi(T)) > c$. However, since $\psi(T) \in K$, inequality (3.3) implies $\text{Re}(e^{-i\theta} \psi(T)) \leq c$. This contradiction proves $z_0 \in K$, establishing $\overline{\text{co}} W(T) \subseteq K$.

Corollary 3.9 (Necessary Intersection Condition). If $\mathcal{F} = \{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{ClPrime}(B(X))$ is a separating family of prime ideals, then

$$\bigcap_{\alpha \in \Lambda} P_\alpha = \{0\}.$$

Proof. Let $I = \bigcap_{\alpha \in \Lambda} P_\alpha$. For any $x \in I$ and any $\psi \in \Phi_{\mathcal{F}}$, $\psi = E_\alpha \circ \pi_{P_\alpha}$ for some $\alpha \in \Lambda$, so

$$\psi(x) = E_\alpha(\pi_{P_\alpha}(x)) = E_\alpha(0) = 0.$$

By weak-* density of $\Phi_{\mathcal{F}}$ in $S(B(X))$, it follows that $E(x) = 0$ for all $E \in S(B(X))$. Suppose $x \neq 0$. By the Hahn–Banach theorem, there exists $f \in X^*$ and $v \in X$ such that $f(xv) \neq 0$ and $\|v\| = \|f\| = f(v) = 1$. The functional $E_{v,f}(S) = f(Sv)$ is a state with $E_{v,f}(x) = f(xv) \neq 0$, a contradiction. Thus $x = 0$, proving $\bigcap_{\alpha \in \Lambda} P_\alpha = 0$. $\square$

Remark 3.10 The intersection condition $\bigcap_{\alpha \in \Lambda} P_\alpha = 0$ is necessary for separation, but in general non-commutative Banach algebras it is not sufficient unless accompanied by state density.

Example 3.11 (Commutative $C(K)$ Algebra Model). Let $A = C(K)$ for a compact Hausdorff space K . Prime ideals coincide with maximal ideals $M_x = \{f \in C(K) : f(x) = 0\}$ for $x \in K$. A subfamily $\{M_x : x \in K_0\}$ is separating if and only if K_0 is topologically dense in K . In this setting, $\bigcap_{x \in K_0} M_x = 0$ is completely equivalent to state density.

Example 3.12 (Non-Separating Ideal). In $B(H)$ with infinite-dimensional Hilbert space H , the singleton family $\mathcal{F} = \{K(H)\}$ consists of a maximal ideal. It is not separating because $\bigcap K(H) = K(H) \neq 0$. States factoring through the Calkin algebra $B(H)/K(H)$ evaluate to zero on all compact operators and cannot approximate vector states.

Definition 3.13 (Boundary Representation). A primitive ideal $P \in \text{Prim}(B(X))$ is a boundary representation for $T \in B(X)$ if the fibre numerical range $V_P(T)$ intersects the topological boundary of the closed convex hull, i.e.,

$$V_P(T) \cap \partial(\overline{\text{co}} W(T)) \neq \emptyset.$$

Theorem 3.14 (Boundary Detection of Extreme Points). Let $\mathcal{F} = \{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{Prim}(B(X))$ be a separating family of primitive ideals and $T \in B(X)$. Let $z \in \text{Ext}(\overline{\text{co}} W(T))$ be an extreme point. If $z \notin \overline{\text{co}} \bigcup_{\alpha \neq \beta} V_{P_\alpha}(T)$ for a specific index $\beta \in \Lambda$, then P_β is a boundary representation for T .

Proof. By Theorem 3.8 and Theorem 2.16, we have $\overline{\text{co}} \bigcup_{\alpha \in \Lambda} V_{P_\alpha}(T) = \overline{\text{co}} W(T)$. Let $M = \bigcup_{\alpha \in \Lambda} V_{P_\alpha}(T)$. Since each $V_{P_\alpha}(T)$ is contained in the compact algebra numerical range $V(T, B(X))$, M is a closed and bounded subset of $\mathbb{C}$, hence compact by the Heine-Borel theorem. By Milman's converse to the Krein-Milman theorem^[18], the extreme points of the closed convex hull of a compact set must be contained within the set itself. Thus, $\text{Ext}(\overline{\text{co}} W(T)) \subseteq M = \bigcup_{\alpha \in \Lambda} V_{P_\alpha}(T)$. Since $z \in \text{Ext}(\overline{\text{co}} W(T))$ and $z \notin \overline{\text{co}} \bigcup_{\alpha \neq \beta} V_{P_\alpha}(T)$ for the specific index $\beta \in \Lambda$, it follows by exclusion that z must lie in the closure $\overline{V_{P_\beta}(T)}$. Because z is an extreme point of the compact convex set $\overline{\text{co}} W(T)$, it belongs to the topological boundary $\partial(\overline{\text{co}} W(T))$. Since $\overline{V_{P_\beta}(T)}$ contains this boundary point, $V_{P_\beta}(T)$ itself intersects $\partial(\overline{\text{co}} W(T))$. Therefore, P_β is a boundary representation for T . $\square$

Corollary 3.15. Every extreme point of $\overline{\text{co}} W(T)$ belongs to the closure of the union of fibre numerical ranges of boundary representations.

Proof. Let $\Lambda_b = \{\alpha \in \Lambda : P_\alpha \text{ is a boundary representation for } T\}$. An extreme point z lies on $\partial(\overline{\text{co}} W(T))$ and cannot be written as a non-trivial convex combination of interior points. Thus, $z \in \overline{\bigcup_{\alpha \in \Lambda_b} V_{P_\alpha}(T)}$. $\square$

Remark 3.16. This result generalizes the concept of C^* -algebraic boundary representations to non-selfadjoint operator structures on Banach spaces.

Theorem 3.17 (Representation-Level Reconstruction). Let $\{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{Prim}(B(X))$ be a separating family. Then

$$\overline{\text{co}} \bigcup_{\alpha \in \Lambda} V_{P_\alpha}(T) \subseteq \overline{\text{co}} W(T).$$

If, in addition, $V_{P_\alpha}(T) = W[T]_{P_\alpha}$ for each $\alpha \in \Lambda$, then equality holds.

Proof. Theorem 2.16 establishes $V_{P_\alpha}(T) \subseteq W[T]_{P_\alpha}$ for each $\alpha \in \Lambda$. Taking the union and closed convex hull, and applying Theorem 3.8, yields:

$$\overline{\text{co}} \bigcup_{\alpha \in \Lambda} V_{P_\alpha}(T) \subseteq \overline{\text{co}} \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha} = \overline{\text{co}} W(T).$$

If $V_{P_\alpha}(T) = W[T]_{P_\alpha}$ holds for all α , substitution immediately gives equality. $\square$

Theorem 3.18. For every $T \in B(X)$,

$$\overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} V_P(T) = \overline{\text{co}} W(T).$$

Proof. For $P = 0 \in \text{Prim}(B(X))$ (Corollary 2.13), $\pi_0 = id_{B(X)}$, so $V_0(T) = W(T)$. Thus:

$$W(T) = V_0(T) \subseteq \bigcup_{P \in \text{Prim}(B(X))} V_P(T) \subseteq \overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} V_P(T).$$

Taking closed convex hulls yields $\overline{\text{co}} W(T) \subseteq \overline{\text{co}} \bigcup_P V_P(T)$. The reverse inclusion follows from Theorem 2.16. $\square$

Corollary 3.19 (Numerical Radius Equality). For any $T \in B(X)$, the numerical radius satisfies:

$$v(T) = \sup_{P \in \text{Prim}(B(X))} \sup\{|z| : z \in V_P(T)\} = \sup_{P \in \text{Prim}(B(X))} v(\pi_P(T)).$$

Proof. For any bounded set $S \subset \mathbb{C}$, $\sup_{z \in S} |z| = \sup_{z \in \text{co}(S)} |z|$. Applying this identity to Theorem 3.14 gives $v(T) = \sup_P v(\pi_P(T))$. $\square$

Remark 3.20. Corollary 3.19 confirms that the global numerical radius of T is localizable across irreducible representations.

3.3. Unified Main Theorem.

Theorem 3.21 (Unified Main Theorem). Let X be a complex Banach space with $\dim X \geq 2$ and $T \in B(X)$.

- (1) The primitive ideal space $\text{Prim}(B(X))$, equipped with the Jacobson topology, is a T_0 , quasi-compact topological space, and the map $P \mapsto W[T]_P$ is order-reversing with respect to ideal inclusion.
- (2) Every maximal closed two-sided ideal of $B(X)$ is primitive, and 0 is always primitive.
- (3) For every $P \in \text{Prim}(B(X))$, the following containment chains hold:

$$V_P(T) \subseteq W[T]_P \subseteq \overline{\text{co}} W(T) \quad \text{and} \quad \sigma(\pi_P(T)) \subseteq \sigma([T]_P) \subseteq \sigma(T).$$

- (4) If $\mathcal{F} = \{P_\alpha\}_{\alpha \in \Lambda} \subseteq \text{ClPrime}(B(X))$ is a separating family of prime ideals, then

$$\overline{\text{co}} \bigcup_{\alpha \in \Lambda} W[T]_{P_\alpha} = \overline{\text{co}} W(T).$$

- (5) The full family of primitive ideals achieves complete numerical envelope reconstruction:

$$\overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} W[T]_P = \overline{\text{co}} \bigcup_{P \in \text{Prim}(B(X))} V_P(T) = \overline{\text{co}} W(T).$$

Proof. Part (1) combines Theorems 2.16 and 3.3. Part (2) combines Corollaries 2.12 and 2.13. Part (3) is Theorem 3.2. Part (4) is Theorem 3.8. Part (5) combines Theorems 3.4 and 3.17 $\square$

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