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FN Mburu
School of Pure and Applied
Sciences, Kirinyaga University,
Kirinyaga County, Kenya

PW Njori
School of Pure and Applied
Sciences, Kirinyaga University,
Kirinyaga County, Kenya

Jeremiah Osida Onunga
Department of Renewable
Energy and Technology, School
of Science and Technology,
Turkana University College,
Lodwar, Turkana County,
Kenya

Corresponding Author:
FN Mburu
School of Pure and Applied
Sciences, Kirinyaga University,
Kirinyaga County, Kenya

The zamkovoy canonical paracontact connection on LP-Kenmotsu manifolds

FN Mburu, PW Njori and Jeremiah Osida Onunga

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Abstract

This paper investigates the differential geometry of an n -dimensional Lorentzian Para-Kenmotsu (briefly, LP-Kenmotsu) manifold equipped with the Zamkovoy canonical Paracontact connection. Formulation of the explicit expression for this connection, establish its metric compatibility under specific structural conditions, and analyze the parallelism of the fundamental structure tensors. Additionally, computation of the torsion tensor and derive the major curvature properties, which includes the Riemannian curvature tensor, Ricci tensor, and scalar curvature, with mathematical proofs. Finally, discussion on the algebraic symmetry properties of the structure endomorphism ϕ , resolving a common discrepancy in the literature regarding its skew-symmetry versus symmetry on Lorentzian Paracontact metric structures. Beyond its intrinsic geometric value, this work also positions the formalism within emerging Responsible AI scholarship, proposing metric-compatible, torsion-governed connections such as the Zamkovoy connection as a candidate mathematical vocabulary for describing structurally consistent, auditable transformations in AI risk-classification and governance systems.

Keywords: Para-contact metric manifold, lorentzian almost paracontact manifold, lorentzian para-kenmotsu manifold, zamkovoy canonical paracontact connection, responsible AI, geometric formalism for AI governance

Introduction

The differential geometry of pseudo-Riemannian manifolds equipped with paracontact metric structures plays an important role in mathematical physics, in the study of general relativity, cosmological models, and non-symmetric gravitational theories. S. Zamkovoy^[1] introduced a canonical linear connection on Paracontact metric manifolds, referred to as the Zamkovoy canonical Paracontact connection. This connection serves as the Paracontact of the Tanaka-Webster connection on contact metric manifolds.

When dealing with a Lorentzian Para-Kenmotsu (LP-Kenmotsu) manifold, this connection provides profound structural insights that differ significantly from the standard Levi-Civita connection. This paper presents a rigorous exposition of the Zamkovoy Canonical Paracontact connection on LP-Kenmotsu manifolds, the propositions, theorems, and their proofs. It also address a fundamental algebraic property of the structure tensor ϕ specifically, its symmetry with respect to the Lorentzian metric which is crucial for establishing correct geometric identities and resolving mathematical discrepancies present in earlier works.

Furthermore, it connect these results with recent developments on the curvature properties of Lorentzian Para-Kenmotsu manifolds. Particularly, it highlights the extensive research carried out by F. N. Mburu and collaborators^[6-9] on several generalized curvature tensors, such as the W_3 , W_4 , W_5 , and W_6 curvature tensors, on LP-Kenmotsu manifolds. Their findings provide valuable geometric context and inspire deeper inquiries into how the choice of linear connections, like the Zamkovoy connection, affects the behavior of such curvature tensors.

Alongside these purely geometric aims, this paper situates its results within the broader, emerging conversation on Responsible AI. Responsible AI research increasingly calls for formal, mathematically grounded vocabularies for describing when a transformation applied to a system, a dataset, or a risk classification is well-behaved: consistent with an underlying

reference structure, free of unaccounted-for distortion, and traceable back to first principles. The Zamkovoy canonical Paracontact connection, precisely because its metric compatibility, torsion, and parallelism properties can be characterized exactly, offers a candidate formal analogue for these desiderata. Section 7 returns to this theme once the requisite geometric machinery has been established, using the connection's structural behavior as a lens on auditable, structure-preserving transformations relevant to AI governance frameworks.

2. Preliminaries

Let M be an n -dimensional smooth manifold with an almost paracontact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1,1)$ -tensor field, ξ is the Reeb vector field, η is a 1-form, and g is a Lorentzian metric.

Definition 2.1. An n -dimensional almost Paracontact metric manifold (M, ϕ, ξ, η, g) is known as a Lorentzian Para-Kenmotsu (LP-Kenmotsu) manifold if the following conditions hold for all vector fields $X, Y \in \chi(M)$ [2, 5]:

$$\eta\eta(\xi) = -1, \phi^2 X = X + \eta(X)\xi, \phi\xi = 0, \eta \cdot \phi = 0 \quad (2.1)$$

$$g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + g(X)g(Y) \quad (2.2)$$

$$\nabla_X \phi = -g(\phi X, Y)\xi - \eta(Y)\phi X \quad (2.3)$$

where ∇ denotes the Levi-Civita connection of g .

On an LP-Kenmotsu manifold, the covariant differentiation of the Reeb vector field ξ and the 1-form η satisfies the following standard relations:

$$\nabla_X \xi = -X - \eta(X)\xi \quad (2.4)$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y) \quad (2.5)$$

Some authors adopt the opposite sign convention for the Reeb vector field derivative:

$$\nabla_X \xi = X + \eta(X)\xi \quad (2.6)$$

$$(\nabla_X \eta)Y = g(X, Y) + \eta(X)\eta(Y) \quad (2.7)$$

Proposition 2.2. On Lorentzian para-Kenmotsu manifold (M, ϕ, ξ, η, g) , the structure tensor ϕ is symmetric with respect to the Lorentzian metric g

$$g(\phi(X, Y) = g(X, \phi Y) \quad (2.8)$$

Proof.

By the metric compatibility of ϕ in Eq. (2.2),

$$g(\phi^2 X, \phi Y) = g(\phi X, Y) + \eta(\phi X)\eta(Y) \quad (2.9)$$

$\eta(\phi X) = 0$ by Eq.(2.1), (2.9) simplifies to:

$$g(\phi^2 X, \phi Y) = g(\phi X, Y) \quad (2.10)$$

On other hand, substituting the definition of $\phi^2 X$ from (2.1) into the left-hand side of (2.10), to get:

$$\begin{aligned} g(\phi^2 X, \phi Y) &= g(X + \eta(X)\xi, \phi Y) \\ &= g(X, \phi Y) + \eta(X)g(\xi, \phi Y) \end{aligned} \quad (2.11)$$

By (2.2) and (2.1), this results to $g(\xi, \phi Y) = \eta(\phi Y) = 0$. Hence

$$g(\phi^2 X, \phi Y) = g(X, \phi Y) \quad (2.12)$$

Comparing this with (2.10) results to this conclusion:

$$g(\phi X, Y) = g(X, \phi Y) \quad (2.13)$$

Hence symmetric.

For almost contact geometry where $\eta(\xi) = 0$, the structure tensor ϕ is skew-symmetric that is $g(\phi X, Y) = -g(X, \phi Y)$. Yet, on Lorentzian Paracontact manifolds where $\eta(\xi) = -1$, the compatibility relation (2.2) mathematically forces ϕ to be symmetric. This distinction is pivotal for avoiding sign errors in subsequent derivations.

3. The Zamkovoy Connection

The Zamkovoy connection ∇^Z on an almost Paracontact metric manifold is defined by ^[1, 2]

$$\nabla_X^Z Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi \quad (3.1)$$

$\forall$ vector fields $X, Y \in \chi(M)$. Consider this connection to LP-Kenmotsu manifolds under both conventions.

3.1 Connection under Convention (i) $\nabla_X \xi = -X - \eta(X)\xi$

Using relations in (2.4) and (2.5), substitute these into (3.1)

$$\begin{aligned} \nabla_X^Z Y &= \nabla_X Y + \eta(X)\phi Y - \eta(Y)(-X - \eta(X)\xi) - g(X, Y) - \eta(X)\eta(Y)\xi \\ &= \nabla_X Y - g(X, Y)\xi + \eta(Y)X + \eta(X)\phi Y \end{aligned} \quad (3.2)$$

Hence, connection reduced to

$$\nabla_X^Z Y = \nabla_X Y - g(X, Y)\xi + \eta(Y)X + \eta(X)\phi Y \quad (3.3)$$

3.2 Connection under Convention (ii) $\nabla_X \xi = X + \eta(X)\xi$

By alternative relations in (2.6) and (2.7), and substituting these into (3.1)

$$\begin{aligned} \nabla_X^Z Y &= \nabla_X Y + \eta(X)\phi Y - \eta(Y)(X + \eta(X)\xi) + g(X, Y) + \eta(X)\eta(Y)\xi \\ &= \nabla_X Y + \eta(X)\phi Y - \eta(Y)X - \eta(X)\eta(Y)\xi + g(X, Y)\xi + \eta(X)\eta(Y)\xi \\ &= \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y \end{aligned} \quad (3.4)$$

Then, the connection simplifies to

$$\nabla_X^Z Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y \quad (3.5)$$

It is clear the two simplified expressions (3.3) and (3.5) differ only in the signs of the second and the third terms.

3.3 Examining the metric compatibility of the Zamkovoy connection under both conventions.

Proposition 3.1. Let (M, ϕ, ξ, η, g) be an LP-Kenmotsu manifold.

By (3.3), the covariant derivative of g satisfies:

$$(\nabla_X^Z g)(Y, Z) = -2\eta(X)g(\phi Y, Z) \quad (3.6)$$

By (3.5), the covariant derivative of g satisfies:

$$(\nabla_X^Z g)(Y, Z) = -2\eta(X)g(\phi Y, Z) \quad (3.7)$$

Hence, the connection is metric compatible ($\nabla^Z g = 0$) distributed horizontally

Proof. (3.3).

By definition is given by

$$(\nabla_X^Z g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X^Z Y, Z - g(Y, \nabla_X^Z Z). \quad (3.8)$$

using the fact that ∇ is the Levi-Civita connection therefore,

$$X(g(Y, Z)) - g(\nabla_X, Z) - g(Y, \nabla_X Z) = 0$$

$$\begin{aligned} (\nabla_X^Z g)(Y, Z) &= -g(-g(X, Y)\xi + \eta(Y)X + \eta(X)\phi Y, Z) - g(X, Z)\xi + \eta(Z)X + \eta(X)\phi Z \\ &= g(X, Y)\eta(Z) - \eta(Y)g(X, Z) - \eta(X)g(\phi Y, Z) + g(X, Z)\eta(Y) - \eta(Z)g(Y, X) - \eta(X)g(Y, \phi Z) \end{aligned} \quad (3.9)$$

By symmetric and proposition (2.2) the remaining terms gives

$$\begin{aligned} (\nabla_X^Z g)(Y, Z) &= -\eta(X)g(\phi Y, Z) - \eta(X)g(\phi Y, Z) \\ (\nabla_X^Z g)(Y, Z) &= -2\eta(X)g(\phi Y, Z) \end{aligned} \quad (3.10)$$

Similarly (3.5) yields

$$\begin{aligned}
 (\nabla_X^Z g)(Y, Z) &= -g(g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y, Z) - g(Y, g(X, Z)\xi - \eta(Z)X + \eta(X)\phi Z) \\
 &= g(X, Y)\eta(Z) + \eta(Y)g(X, Z) - \eta(X)g(\phi Y, Z) - g(X, Z)\eta(Y) + \eta(Z)g(Y, X) - \eta(X)g(Y, \phi Z) \\
 &= -\eta(X)g(\phi Y, Z) - \eta(X)g(Y, \phi Z) \\
 &= -2\eta(X)g(\phi Y, Z)
 \end{aligned} \tag{3.11}$$

Thus proved

3.4 Analyzing the parallelism of the ξ (Reeb vector) and the structure tensor ϕ using these conventions.

Theorem 3.2. Let (M, ϕ, ξ, η, g) be an LP-Kenmotsu manifold.

1. By Convention (i) and the connection (3.3), the covariant derivative of the (ξ) Reeb vector field fulfills:

$$\nabla_X^Z \xi = -2(X + \eta(X)\xi) \tag{3.12}$$

2. By Convention (ii) and the connection (3.5) the Reeb vector field fulfills

$$\nabla_X^Z \xi = 2(X + \eta(X)\xi) \tag{3.13}$$

Proof

From (3.3), let $Y = \xi$ and substitute it to the definition:

$$\nabla_X^Z \xi = \nabla_X \xi - g(X, \xi)\xi + \eta(\xi)X + \eta(X)\phi\xi \tag{3.14}$$

But,

$$\nabla_X \xi = -X - \eta(X)\xi$$

$$g(X, \xi) = \eta(X)$$

$$\eta(\xi) = -1$$

$$\phi\xi = 0$$

Substitute in (3.14)

$$\begin{aligned}
 \nabla_X^Z \xi &= (-X - \eta(X)\xi) - \eta(X)\xi - X \\
 \nabla_X^Z \xi &= -2(X + \eta(X)\xi)
 \end{aligned} \tag{3.15}$$

Proved

By (3.5), let $Y = \xi$ and substitute into the definition:

$$\nabla_X^Z \xi = \nabla_X \xi + g(X, \xi)\xi - \eta(\xi)X + \eta(X)\phi\xi \tag{3.16}$$

Thus

$$\begin{aligned}
 \nabla_X^Z \xi &= (X + \eta(X)\xi) + \eta(X)\xi - (-1)X \\
 \nabla_X^Z \xi &= 2(X + \eta(X)\xi)
 \end{aligned} \tag{3.17}$$

Proved

By choosing the connection (3.3) under convection(ii)

$$\nabla_X^Z \xi = \nabla_X \xi - g(X, \xi)\xi + \eta(\xi)X = (X + \eta(X)\xi) - X = 0 \tag{3.18}$$

This indicates the Reeb vector ξ strictly parallel

4. Computing the torsion tensor $T^Z(X, Y) = \nabla_X^Z Y - \nabla_Y^Z X - [X, Y]$ of the connection using both simplified forms.**Theorem 4.1.** Let (M, ϕ, ξ, η, g) be an LP-Kenmotsu manifold.

$$\text{Connection (3.3)} \quad T^Z(X, Y) = \eta(X)(\theta Y - Y) - \eta(Y)(\phi X) + \eta(Y)X \quad (4.1)$$

$$\text{And connection (3.5)} \quad T^Z(X, Y) = \eta(X)(\theta Y + Y) - \eta(Y)(\phi X) - \eta(Y)X \quad (4.2)$$

*Proof.**By definition*

$$T^Z(X, Y) = \nabla_X^Z Y - \nabla_Y^Z X - [X, Y] \quad (4.3)$$

As the Levi-Civita connection ∇ is torsion-free, substituting the connection components

$$T^Z(X, Y) = (-g(X, Y)\xi + \eta(Y)X + \eta(X)\phi Y) - (-g(Y, X)\xi + \eta(Y)\phi X - \eta(X)Y)$$

$$T^Z(X, Y) = \eta(Y)X - \eta(X)Y - \eta(Y)\phi X + \eta(X)\phi Y$$

$$T^Z(X, Y) = -\eta(Y)(\phi X - X) + \eta(X)(\phi Y) \quad (4.4)$$

Using (3.5) gives

$$T^Z(X, Y) = (g(X, Y)\xi + \eta(X)\phi Y - \eta(Y)X) - (g(Y, X)\xi - \eta(Y)\phi X + \eta(X)Y)$$

$$T^Z(X, Y) = \eta(Y)X - \eta(X)Y + \eta(Y)\phi X - \eta(X)\phi Y$$

$$T^Z(X, Y) = \eta(X)(Y + \phi Y) - \eta(Y)(X + \phi X) \quad (4.5)$$

Thus proved

5. The Curvature PropertiesSuppose R and R^Z be the Riemannian curvature tensors of the Levi-Civita connection ∇ and the Zamkovoy connection ∇^Z , respectively is defined by

$$R^Z(X, Y)Z = \nabla_X^Z \nabla_Y^Z Z - \nabla_Y^Z \nabla_X^Z Z - \nabla_{[X, Y]}^Z Z. \quad (5.1)$$

Theorem 5.1. Riemannian curvature tensor R^Z of the Zamkovoy connection (3.3) on an LP-Kenmotsu manifold is related to the Levi-Civita curvature tensor R by:

$$\begin{aligned} R^Z(X, Y)Z &= R(X, Y)Z + 2g(Y, Z)\eta(X)\xi + 3g(Y, Z)X \\ &\quad - 3g(X, Z)Y - 2g(X, Z)\eta(Y)\xi + 2g(Y, \phi Z)\eta(X)\xi \\ &\quad - 2g(X, \phi Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y \\ &\quad + 2\eta(X)\eta(Z)\phi Y - 2\eta(Y)\eta(Z)\phi X \end{aligned} \quad (5.2)$$

Corollary 5.2. Curvature tensor R^Z fulfills the first Bianchi identity Defined by

$$R^Z(X, Y)Z + R^Z(Y, Z)X + R^Z(Z, X)Y = 0 \quad (5.3)$$

*Proof*Equation (5.3) is proofed direct computation of $\nabla_X^Z \nabla_Y^Z Z - \nabla_Y^Z \nabla_X^Z Z - \nabla_{[X, Y]}^Z Z$ using equation (3.3) and Lorentzian Para Kenmotsu structure equations.Adding up (5.2) over permutations (cyclic) of X, Y, Z and applying first Bianchi identity of Levi-Civita Curvature tensor $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ algebraic terms cancels.**Theorem 5.3.**Ricci tensor $S^Z(Y, Z) = \sum_{i=1}^n \epsilon_i g(R^Z(e_i, Y)Z, e_i)$ of Zamkovoy connection (3.3) on LP-Kenmotsu manifold is provided by:

$$S^Z(Y, Z) = S(Y, Z) + (3n - 5)g(Y, Z) + (2n - 4)\eta(Y)\eta(Z) - 2g(Y, \phi Z)$$

For S is the Ricci tensor.

Proof

Setting $\{e_i\}$ be orthonormal basis of M defined as $\epsilon_i = g(e_i, e_i)$

By contracting every term in (5.2) w.r.t $X = e_i$

Taking inner product (e_i)

$$\sum_i \epsilon_i g(R^Z(e_i, Y)Z, e_i) = S(Y, Z)$$

$$\sum_i \epsilon_i [3g(Y, Z)g(e_i, e_i) - 3g(e_i, Z)g(Y, e_i)] = 3ng(Y, Z) = 3(n-1)g(Y, Z)$$

$$\sum_i \epsilon_i [2g(Y, Z)\eta(e_i)g(\xi, e_i) - 2g(e_i, Z)\eta(Y)g(\xi, e_i)] = 2g(Y, Z)g(\xi, \xi) - 2\eta(Y)g(Z, \xi)$$

$$= -2g(Y, Z) - 2\eta(Y)\eta(Z)$$

$$\sum_i \epsilon_i [2g(Y, \phi Z)\eta(e_i)g(\xi, e_i) - 2g(e_i, \phi Z)\eta(Y)g(\xi, e_i)] = 2g(Y, \phi Z)g(\xi, \xi) - 2\eta(Y)g(\phi Z, \xi)$$

$$= -2g(Y, \phi Z)$$

$$\sum_i \epsilon_i [-2\eta(Y)\eta(Z)g(\phi e_i, e_i) + 2\eta(e_i)\eta(Z)g(\phi Y, e_i)] = -2\eta(Y)\eta(Z)\phi + 2\eta(Z)\eta(\phi Y) = 0$$

Adding up contracted terms

$$S^Z(Y, Z) = S(Y, Z) + (3n-3)g(Y, Z) - 2g(Y, Z) - 2\eta(Y)\eta(Z) - 2g(Y, \phi Z) + (2n-2)\eta(Y)\eta(Z)$$

$$S^Z(Y, Z) = S(Y, Z) + (3n-5)g(Y, Z) + (2n-4)\eta(Y)\eta(Z) - 2g(Y, \phi Z) \quad (5.5)$$

Thus proved.

Corollary 5.4. Ricci tensor S^Z is symmetric if only $\phi = 0$, or any anti-invariant sub manifold of M for $g(Y, \phi Z) = 0$

Theorem 5.5. Scalar curvature r^Z of Zamkovoy connection (3.3) is given by

$$r^Z = r + (3n-4)(n-1) - 2tr(\phi) \quad (5.6)$$

r is the scalar curvature of Levi-Civita connection

Proof

contract $S^Z(Y, Z)$ w.r.t g

$$r^Z = \sum_j \epsilon_j S^Z(e_j, e_j)$$

$$= r + (3n-5)n + (2n-4)g(\xi, \xi) - 2tr(\phi)$$

$$= r + 3n^2 - 5n + 2n - 4(-1) - 2tr(\phi)$$

$$= r + (n-1)(3n-4) - 2tr(\phi) \quad (5.7)$$

For paracontact structure $tr(\phi) = 0$

Hence

$$r^Z = r + (n-1)(3n-4) \quad (5.8)$$

6. Submanifolds and Solitons

If M is an anti-invariant submanifold of an Lorentzian Para Kenmotsu manifold then, $g(Y, \phi Z) = 0$ condition holds. Thus, the Ricci tensor S^Z is completely symmetric:^[3]

$$S^Z(Y, Z) = S(Y, Z) + 3n-5)g(Y, Z) + (2n-4)\eta(Y)\eta(Z) \quad (6.1)$$

Allowing the study of Einstein solitons and η -Einstein solitons w.r.t Zamkovoy connection on such submanifolds.

An Einstein soliton (g, V, λ) is defined by^[4]

$$\mathcal{L}_V g + 2S + (2\lambda - r) = 0 \quad (6.2)$$

A manifold admits an Einstein soliton w.r.t the Zamkovoy connection, gives:

$$\mathcal{L}_v^z g + 2S^z + (2\lambda - r^z)g = 0 \quad (6.3)$$

From these findings, depending on the scalar, the soliton is: Expanding if $r > -(3n - 8)(n - 1)$, Steady if $r = -(3n - 8)(n - 1)$ and Shrinking if $r < -(3n - 8)(n - 1)$

These results are enriched by the curvature investigations of Mburu *et al.* [6-9], that describes specialized W-curvature tensors on these manifolds.

7. Responsible AI Perspective: Structural Geometry as a Vocabulary for Trustworthy Transformations

The results developed above are geometric in their own right, but they also carry a suggestive analogy for Responsible AI, understood here as the interdisciplinary effort to make AI systems' internal transformations traceable, consistent, and auditable. Three structural features of the Zamkovoy canonical Paracontact connection on an LP-Kenmotsu manifold are of particular interest from this vantage point.

First, metric compatibility (Section 3) means the connection preserves the geometry's own notion of length and angle as it transports structure across the manifold. In a Responsible AI setting, this is analogous to requiring that a model update, a re-training step, or a risk-reclassification procedure preserve the relationships a governance framework has already certified as meaningful, rather than silently distorting them. Second, the explicit torsion tensor computed in Section 4 gives a quantitative handle on exactly how, and by how much, a transformation departs from being symmetric; a vanishing or bounded torsion term is the geometric counterpart of an audit trail showing that a system update introduces no unaccounted-for asymmetry. Third, the classification of Einstein solitons in Section 6 into expanding, steady, and shrinking regimes offers a formal vocabulary for describing whether a system governed by such a connection is drifting away from, holding at, or converging toward a reference equilibrium over time, a question directly analogous to monitoring whether an AI system's risk profile is stabilizing under continued operation.

These are offered as a structural analogy rather than a claim that LP-Kenmotsu geometry directly models any deployed AI system; the connection between differential geometry and Responsible AI governance frameworks, such as those surveyed by Jobin *et al.* [10], remains at the level of shared formal concerns: compatibility with a reference structure, quantifiable and bounded distortion, and well-characterized long-run behavior. Making this analogy precise, for instance by formalizing a risk-classification manifold on which an explicit connection of this kind acts, is left as a direction for future interdisciplinary work between differential geometry and Responsible AI research.

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