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On fuzzy sublattice of a lattice

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Abstract

The definition of fuzzy sublattice of a lattice and its properties are discussed. Characterization is given through level sublattice of that lattice. The concepts of fuzzy sublattice and level sublattice are studied and their properties are discussed. The characterization Theorem of a fuzzy sublattice is presented through the level sublattice. The concept of level sublattice of a lattice is studied. Then the necessary and sufficient condition of a fuzzy subset of a lattice to be a fuzzy sublattice of that lattice is proved. Further, the necessary and sufficient condition of equality of two level sublattices in a family of level sublattices is obtained. The union and intersection of two level sublattices are defined. Also, the complement of a level sublattice is defined. Then a chain of level sublattices of a fuzzy sublattice is obtained. The relation between a non-empty subset of a lattice and the fuzzy sublattice of the lattice, defined in terms of that subset, is established. We made an attempt to study the algebraic nature of fuzzy subset of lattice especially fuzzy sublattice, by using the ideas of fuzzy theory and lattice theory.

Keywords: Fuzzy sublattice, level sublattice

Introduction

The notion of fuzzy sets was introduced by Lofti. A. Zadeh in 1965. Zadeh had initiated fuzzy set theory as a modification of the ordinary set theory. The concept of lattice was first defined by Dedekind in 1897, and then developed by Birkhoff G. in 1933.

Here we made an attempt to study the algebraic nature of fuzzy subset of lattice especially fuzzy sublattice, by using the ideas of fuzzy theory and lattice theory.

1. Preliminaries

Some well - known definitions and preliminary results are recalled

Definition 1.1

A relation defined on a set S which is reflexive, antisymmetric and transitive is called a partial ordering on S . A set S with a partial ordering ρ defined on it is called a partially ordered set or a poset and is denoted by (S, ρ) .

Definition 1.2

Let $(P, \leq)$ be a poset. Let A be a non-empty subset of P . An element $u \in P$ is called an upper bound of A if $a \leq u$ for all $a \in A$. An element $u \in P$ is called the least upper bound (l.u.b) of A if

1. u is an upper bound of A .
2. if v is any other upper bound of A , then $u \leq v$.

An element $l \in P$ is called the lower bound of A if $l \leq a$ for all $a \in A$. An element $l \in P$ is called the greatest lower bound (g.l.b) of A if

1. l is a lower bound of A .
2. if m is any other lower bound of A , then $m \leq l$.

Definition 1.3

A **lattice** is a poset in which any two elements have a g.l.b and a l.u.b. We denote the l.u.b of $a \vee b$ and g.l.b by $a \wedge b$.

Definition 1.4

Let $(L, \vee, \wedge)$ be a lattice. A non-empty subset S of L is called anfilter of L if it satisfies the following conditions:

1. $x, y \in S \Rightarrow x \wedge y \in S$.
2. $x \in S$ and $r \in L$ with $r \geq x \Rightarrow r \in S$.

Definition 1.5

Let X be a non-empty set. A mapping $\mu: X \rightarrow [0, 1]$ is called a fuzzy subset of X .

Definition 1.6

Let μ be any fuzzy subset of a set X and $\mu = \{(x_i, t_i) / i = 1 \text{ to } n \text{ and } t_i \in [0, 1]\}$. Then, $\{t_i / i = 1 \text{ to } n\}$ is called the image set of μ and is denoted by $\text{Im } \mu$.

Definition 1.7

Let μ be any fuzzy subset of a set X and $t \in \text{Im } \mu$. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the level subset of μ .

Clearly, $\mu_t \subseteq \mu_s$ whenever $t > s$.

Fuzzy sublattice of a lattice

Here, the concepts of fuzzy sublattice and level sublattice are studied and their properties are discussed. The characterization Theorem of a fuzzy sublattice is presented through the level sublattice.

Definition: 2.1

A fuzzy subset μ of a lattice L , is called a fuzzy sublattice of L , if the following conditions are satisfied:

For all $x, y \in L$,

1. $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$
2. $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$.

Example: 2.2

Consider the fuzzy subset μ_1 of the lattice $(Z, \vee, \wedge)$.

$$\mu_1(x) = \begin{cases} .6 & \text{if } x \in \langle 2 \rangle \\ .3 & \text{if } x \notin \langle 2 \rangle \end{cases}$$

Then μ_1 is a fuzzy sublattice of Z .

Example: 2.3

Consider the fuzzy subset μ_2 of the lattice in Example: 1.6.

$\mu_2(x) = \{(\emptyset, .9), (\{1\}, .8), (\{2\}, .3), (\{3\}, .6), (\{1, 2\}, .5), (\{1, 3\}, .7), (\{2, 3\}, .2), (\{1, 2, 3\}, .4)\}$. Then μ_2 is not a fuzzy sublattice of Z .

For example, let $x = \{2\}$ and $y = \{3\}$.

- Then $x \vee y = \{2, 3\}$.
- Here $\mu_2(x) = .3$ and $\mu_2(y) = .6$.
- Therefore, $\min \{\mu_2(x), \mu_2(y)\} = \min \{.3, .6\} = .3$.
- But $\mu_2(x \vee y) = .2$.
- Hence $\mu_2(x \vee y) \not\geq \min \{\mu_2(x), \mu_2(y)\}$. Thus μ_2 is not a fuzzy sublattice.

Proposition: 2.4

Let μ be a fuzzy sublattice of L . Then for all $x \in L$, $\mu(x) \leq \mu(0)$, where 0 is the least element of L .

Now, the relation between a non-empty subset of a lattice and the fuzzy sublattice of the lattice, defined in terms of that subset, is established. Then it is also proved that the converse relation is also true.

Theorem: 2.5

Let H be any non-empty subset of a lattice L , $H \neq L$. If μ is a fuzzy sublattice of L , defined by

$$\mu(x) = \begin{cases} s & \text{if } x \in H \\ t & \text{if } x \in L \sim H \end{cases}$$

Where $s, t \in [0, 1]$, $s > t$, then H is a sublattice of L .

Proof

Let H be any non-empty subset of a lattice L , $H \neq L$.

Let $x, y \in L$ be arbitrary.

Consider the function, $\mu(x) = \begin{cases} s & \text{if } x \in H \\ t & \text{if } x \in L \sim H \end{cases}$ where $s, t \in [0, 1], s > t$.

Assume that μ is a fuzzy sublattice of L . Then,

1. $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$
 2. $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$
- To prove H is a sublattice of L .
 - Let $a, b \in H$ be arbitrary.
 - Then, $\mu(a) = s$ and $\mu(b) = s$.
 - Here $\min \{\mu(a), \mu(b)\} = \min\{s, s\} = s$
 - Hence all the values of $\mu(a \vee b)$ and $\mu(a \wedge b)$ are greater than or equal to s .
 - But μ has only two values s and t with $s > t$.
 - Therefore, all the values of $\mu(a \vee b)$ and $\mu(a \wedge b)$ are equal to s .
 - This implies $(a \vee b)$ and $(a \wedge b) \in H$.
 - This proves that H is a sublattice of L .

Theorem: 2.6

If H is any sublattice of a lattice L , $H \neq L$, then the fuzzy subset μ of L defined by

$$\mu(x) = \begin{cases} s & \text{if } x \in H \\ t & \text{if } x \in L \sim H \end{cases}$$

Where $s, t \in [0, 1], s > t$ is a fuzzy sublattice of L .

Proof

Let H be any sublattice of a lattice L , $H \neq L$.
Consider the fuzzy subset μ of L , defined by

$$\mu(x) = \begin{cases} s & \text{if } x \in H \\ t & \text{if } x \in L \sim H \end{cases}$$

Where $s, t \in [0, 1], s > t$

- Let $x, y \in L$ be arbitrary.
- To prove μ is a fuzzy sublattice of L .

It is enough to prove that

1. $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$
2. $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$

We prove this in three cases

Case (i) Let $x, y \in H$

Then, $\mu(x) = s, \mu(y) = s$

$$\Rightarrow \min \{\mu(x), \mu(y)\} = \min \{s, s\} = s.$$

Here, $x, y \in H$

$$\Rightarrow x \vee y, x \wedge y \in H, \text{ since } H \text{ is a sublattice of } L.$$

$$\Rightarrow \mu(x \vee y) = s, \mu(x \wedge y) = s.$$

$$\text{Now, } \mu(x \vee y) = s = \min \{\mu(x), \mu(y)\}$$

$$\Rightarrow \mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$$

$$\text{And } \mu(x \wedge y) = s = \min \{\mu(x), \mu(y)\}$$

$$\Rightarrow \mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}$$

Then all the inequalities are satisfied in this case.

Case (ii) Let $x, y \in L \sim H$

Then, $\mu(x) = t, \mu(y) = t$

And $\min \{ \mu(x), \mu(y) \} = \min \{ t, t \} = t$.

Here $x \vee y, x \wedge y \in L$.

Then all the values of $\mu(x \vee y)$ and $\mu(x \wedge y)$ are either t or s .

If $\mu(x \vee y) = s$, then $\mu(x \vee y) = s > t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \vee y) = t$, then $\mu(x \vee y) = t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \wedge y) = s$, then $\mu(x \wedge y) = s > t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \wedge y) = t$, then $\mu(x \wedge y) = t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}.$$

Then all the inequalities are satisfied in this case also.

Case (iii) Let $x \in H, y \in L \sim H$

Here, $\mu(x) = s, \mu(y) = t$

And $\min \{ \mu(x), \mu(y) \} = \min \{ s, t \} = t$.

Clearly $x \vee y, x \wedge y \in L$

Then all the values of $\mu(x \vee y)$ and $\mu(x \wedge y)$ are either t or s .

If $\mu(x \vee y) = s$, then $\mu(x \vee y) = s > t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \vee y) = t$, then $\mu(x \vee y) = t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \wedge y) = s$, then $\mu(x \wedge y) = s > t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}.$$

If $\mu(x \wedge y) = t$, then $\mu(x \wedge y) = t = \min \{ \mu(x), \mu(y) \}$

$$\Rightarrow \mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}.$$

Then all the inequalities are satisfied in this case also.

Thus μ is a fuzzy sublattice of L .

Remark: 2.7

χ_H is the characteristic function of the subset H of L .

$$\text{That is, } \mathbb{Y}_H = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \in L \sim H \end{cases}$$

Corollary: 2.8

If a non - empty subset H of a lattice L , is a sublattice of L , then $\mathbb{Y}_H$ is a fuzzy sublattice of L .

From theorem 2.5 and theorem 2.6, we have,

Theorem: 2.9

Let H be any non - empty subset of a lattice L , $H \neq L$. Let μ be any fuzzy subset of L defined by

$$\mu(x) = \begin{cases} s & \text{if } x \in H \\ t & \text{if } x \in L \sim H \end{cases}$$

Where $s, t \in [0, 1]$, $s > t$. Then μ is a fuzzy sublattice of L if and only if, H is a sublattice of L .

Here, the concept of level sublattice of a lattice is studied. Then the necessary and sufficient condition of a fuzzy subset of a lattice to be a fuzzy sublattice of that lattice is proved. Further, the necessary and sufficient condition of equality of two level sublattices in a family of level sublattices is obtained.

Proposition: 2.10

Let μ be any fuzzy sublattice of L . Then the level subsets μ_t , $t \in \text{Im } \mu$ are sublattices of L .

Theorem: 2.11 [Characterization Theorem]

A fuzzy subset μ of a lattice L , is a fuzzy sublattice of L if and only if, the level subsets μ_t , $t \in \text{Im } \mu$ are sublattices of L .

Proof

- First part is shown Proposition 2.10.
- Conversely, assume that the level subsets μ_t , $t \in \text{Im } \mu$ are sublattices of L .
- To prove: μ is a fuzzy sublattice of L .
- Let $x, y \in L$ be arbitrary.

It is enough to prove that,

1. $\mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}$
2. $\mu(x \wedge y) \geq \min \{ \mu(x), \mu(y) \}$

Let $\min \{ \mu(x), \mu(y) \} = r$

$\Rightarrow$ Either $\mu(x) = r$ and $\mu(y) \geq \mu(x) = r$ or $\mu(y) = r$ and $\mu(x) \geq \mu(y) = r$

$\Rightarrow \mu(x) \geq r$ and $\mu(y) \geq r$

$\Rightarrow x, y \in \mu_r$.

$\Rightarrow x \vee y, x \wedge y \in \mu_r$, since μ_r is a sublattice of L .

$\Rightarrow \mu(x \vee y) \geq r$; $\mu(x \wedge y) \geq r \rightarrow (1)$

For (i)

Let $\mu(x \vee y) = s_1 \rightarrow (2)$

To prove $\mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}$.

That is to prove $s_1 \geq r$

Suppose $s_1 < r$. $\rightarrow (3)$

From (2) and (3), we have, $\mu(x \vee y) = s_1 < r$.

$\Rightarrow \mu(x \vee y) < r$, which is a contradiction to (1).

Hence $s_1 \geq r$.

Thus $\mu(x \vee y) \geq \min \{ \mu(x), \mu(y) \}$.

Similarly, we can prove others.

Thus μ is a fuzzy sublattice of L .

Definition: 2.12

Let μ be any fuzzy sublattice of L ; $t \in [0, 1]$; and $t \leq \mu(0)$. Then the sublattice μ_t of L is called a level sublattice of μ .

Remark: 2.13

Let μ be any fuzzy sublattice of a lattice L . From the above theorem, μ_t , $t \in \text{Im } \mu$ are level sublattices of L . Then $F_\mu = \{ \mu_t / t \in \text{Im } \mu \}$ is called as the family of level sublattices of μ .

Theorem: 2.14

Two level sublattices μ_s and μ_t (with $s < t$) of a fuzzy sublattice μ of L are equal if and only if, there is no x in L such that $s \leq \mu(x) < t$.

Proof

Let μ_s and μ_t be two level sublattices of a fuzzy sublattices μ of L .

Here $s < t$.

Assume that μ_s and μ_t are equal.

To prove there is no x in L such that $s \leq \mu(x) < t$.

On the contrary, assume that $s \leq \mu(x) < t$ for some x in L .

$$\Rightarrow \mu(x) \geq s \text{ and } \mu(x) < t$$

$$\Rightarrow x \in \mu_s \text{ and } x \notin \mu_t$$

$$\Rightarrow \mu_s \neq \mu_t.$$

This is a contradiction to our assumption and hence there is no x in L such that $s \leq \mu(x) < t$.

Conversely, assume that no x in L such that $s \leq \mu(x) < t$. $\longrightarrow$ (1)

$$\mu_s = \{ x \in L / \mu(x) \geq s \} \text{ and } \mu_t = \{ x \in L / \mu(x) \geq t \} \text{ and } s < t.$$

$$\text{Then clearly } \mu_t \subseteq \mu_s. \longrightarrow (2)$$

It is enough to show that $\mu_s \subseteq \mu_t$

$$\text{Let } x \in \mu_s. \text{ Then } \mu(x) \geq s$$

$$\text{Suppose } \mu(x) < t. \text{ Then, } s \leq \mu(x) < t.$$

This is a contradiction to (1).

$$\text{Hence } \mu(x) \geq t.$$

$$\Rightarrow x \in \mu_t.$$

$$\Rightarrow \mu_s \subseteq \mu_t \longrightarrow (3)$$

From (2) and (3), we have $\mu_s = \mu_t$. Hence two level sublattices are equal.

Next, the union and intersection of two level sublattices are defined. Also, the complement of a level sublattice is defined. Then a chain of level sublattices of a fuzzy sublattice is obtained. Further it is shown that $(F_\mu, \vee, \wedge)$ is a completely distributive lattice.

Definition: 2.15

Let μ be any fuzzy sublattice of a lattice L . Let μ_t and μ_s be any two level sublattices of μ . Now, we define $\cup$ and $\cap$ on F_μ by,

$$\mu_t \cup \mu_s = \{ x \in L / \mu(x) \geq \min \{ t, s \} \}$$

$$\mu_t \cap \mu_s = \{ x \in L / \mu(x) \geq \max \{ t, s \} \}$$

Theorem: 2.16

Let μ be any fuzzy sublattice of a lattice L . Let μ_t and μ_s be any two level sublattices of μ . Then $\mu_t \cup \mu_s$ and $\mu_t \cap \mu_s$ are also level sublattices of μ .

Proof

Let μ be any fuzzy sublattices of a lattice L .

Let μ_t and μ_s be any two level sublattices of μ .

$\Rightarrow \mu_t$ and μ_s are sublattices of L .

To prove (i) $\mu_t \cup \mu_s$ and (ii) $\mu_t \cap \mu_s$ are level sublattices of μ .

That is to prove: $\mu_t \cup \mu_s$ and $\mu_t \cap \mu_s$ are sublattices of L .

For (i)

Let $x, y \in \mu_t \cup \mu_s$ be arbitrary.

$\Rightarrow x, y \in L$ such that $\mu(x) \geq \min \{t, s\}$ and $\mu(y) \geq \min \{t, s\}$ ----- $\rightarrow$ (1)

Case (1) Let $\min \{t, s\} = t$

$\Rightarrow x \in L$ such that $\mu(x) \geq t$ and $y \in L$ such that $\mu(y) \geq t$, by (1)

$\Rightarrow x, y \in \mu_t$.

$\Rightarrow x \vee y, x \wedge y \in \mu_t$, since μ_t is a sublattice of L .

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq t$; $\mu(x \wedge y) \geq t$.

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq \min \{t, s\}$; $\mu(x \wedge y) \geq \min \{t, s\}$, by the assumption in this case.

$\Rightarrow x \vee y \in \mu_t \cup \mu_s$; $x \wedge y \in \mu_t \cup \mu_s$.

Hence $\mu_t \cup \mu_s$ is a sublattice of L .

Thus, $\mu_t \cup \mu_s$ is a level sublattice of μ .

Case (2) Let $\min \{t, s\} = s$

$\Rightarrow x \in L$ such that $\mu(x) \geq s$ and $y \in L$ such that $\mu(y) \geq s$, by (1)

$\Rightarrow x, y \in \mu_s$.

$\Rightarrow x \vee y, x \wedge y \in \mu_s$, since μ_s is a sublattice of L .

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq s$; $\mu(x \wedge y) \geq s$.

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq \min \{t, s\}$; $\mu(x \wedge y) \geq \min \{t, s\}$, by the assumption in this case.

$\Rightarrow x \vee y \in \mu_t \cup \mu_s$; $x \wedge y \in \mu_t \cup \mu_s$.

Hence $\mu_t \cup \mu_s$ is a sublattice of L . Thus, $\mu_t \cup \mu_s$ is a level sublattice of μ .

For (ii)

Again, let $x, y \in \mu_t \cap \mu_s$ be arbitrary.

$\Rightarrow x \in L$ such that $\mu(x) \geq \max \{t, s\}$ and $y \in L$ such that $\mu(y) \geq \max \{t, s\}$ -- $\rightarrow$ (2)

Case (1) Let $\max \{t, s\} = t$

$\Rightarrow x \in L$ such that $\mu(x) \geq t$ and $y \in L$ such that $\mu(y) \geq t$, by (1)

$\Rightarrow x, y \in \mu_t$.

$\Rightarrow x \vee y, x \wedge y \in \mu_t$, since μ_t is a sublattice of L .

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq t$; $\mu(x \wedge y) \geq t$.

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq \max \{t, s\}$; $\mu(x \wedge y) \geq \max \{t, s\}$, by the assumption in this case.

$\Rightarrow x \vee y \in \mu_t \cap \mu_s$; $x \wedge y \in \mu_t \cap \mu_s$.

Hence $\mu_t \cap \mu_s$ is a sublattice of L .

Thus, $\mu_t \cap \mu_s$ is a level sublattice of μ .

Case (2) Let $\max \{t, s\} = s$

$\Rightarrow x \in L$ such that $\mu(x) \geq s$ and $y \in L$ such that $\mu(y) \geq s$, by (1)

$\Rightarrow x, y \in \mu_s \Rightarrow x \vee y, x \wedge y \in \mu_s$, since μ_s is a sublattice of L .

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq s$; $\mu(x \wedge y) \geq s$.

$\Rightarrow x \vee y, x \wedge y \in L$ such that $\mu(x \vee y) \geq \max \{t, s\}$; $\mu(x \wedge y) \geq \max \{t, s\}$, by the assumption in this case.

$\Rightarrow x \vee y \in \mu_t \cap \mu_s$; $x \wedge y \in \mu_t \cap \mu_s$.

Hence $\mu_t \cap \mu_s$ is a sublattice of L .

Thus, $\mu_t \cap \mu_s$ is a level sublattice of μ .

Theorem: 2.17

Let μ be any fuzzy sublattice of a lattice L . If $\text{Im } \mu = \{t_0, t_1, t_2, \dots, t_n\}$ with $t_0 > t_1 > t_2 > \dots > t_n$, then we have the following chain of

level sublattices of μ : $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \dots \subseteq \mu_{t_n} = L$.

Proof

Let μ be any fuzzy sublattice of a lattice L .

Let $\text{Im } \mu = \{t_0, t_1, t_2, \dots, t_n\}$ with $t_0 > t_1 > t_2 > \dots > t_n$. $\longrightarrow$ (1)

Let $x_1 \in \mu_{t_0} \Rightarrow x_1 \in L$ such that $\mu(x_1) \geq t_0$.

$\Rightarrow \mu(x_1) \geq t_1$, by (1).

Hence $x_1 \in L$ such that $\mu(x_1) \geq t_1$.

$\Rightarrow x_1 \in \mu_{t_1}$.

Thus, we have $\mu_{t_0} \subseteq \mu_{t_1}$.

Again, let $x_2 \in \mu_{t_1}$

$\Rightarrow x_2 \in L$ such that $\mu(x_2) \geq t_1$.

$\Rightarrow \mu(x_2) \geq t_2$, by (1).

Hence $x_2 \in L$ such that $\mu(x_2) \geq t_2 \Rightarrow x_2 \in \mu_{t_2}$.

Thus, we have $\mu_{t_1} \subseteq \mu_{t_2}$.

Hence we get $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2}$.

Proceeding like this, we get $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \dots \subseteq \mu_{t_n} = L$.

Remark: 2.18

Let μ be any fuzzy sublattice of a lattice L . Let $t, s \in \text{Im } \mu$ with $s \geq t$. Then, by previous Theorem, we have, μ_t and μ_s are two level sublattices of μ such that $\mu_s \subseteq \mu_t$. Here, $\min \{s, t\} = t$ and $\max \{s, t\} = s$.

$$\text{Now, } \mu_s \cup \mu_t = \{x \in L / \mu(x) \geq \min \{s, t\}\}$$

$$\Rightarrow \mu_s \cup \mu_t = \{x \in L / \mu(x) \geq t\} = \mu_t.$$

$$\text{Again, } \mu_s \cap \mu_t = \{x \in L / \mu(x) \geq \max \{s, t\}\}$$

$$\Rightarrow \mu_s \cap \mu_t = \{x \in L / \mu(x) \geq s\} = \mu_s.$$

Theorem: 2.19

Let μ be any fuzzy sublattice of a lattice L . Then $(F_\mu, \vee, \wedge)$ is a completely distributive lattice.

Proof

Define $\vee$ and $\wedge$ on F_μ by

$$\mu_t \vee \mu_s = \mu_t \cup \mu_s$$

$$\text{And } \mu_t \wedge \mu_s = \mu_t \cap \mu_s.$$

Where μ_t and $\mu_s \in F_\mu$. Then it is easy to verify that $(F_\mu, \vee, \wedge)$ is a completely distributive lattice.

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