

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2026): 4.49
Maths 2026; 11(9): 09-17
© 2026 Stats & Maths
<https://www.mathsjournal.com>
Received: 03-07-2026
Accepted: 06-08-2026

Pranjal Kaser
Research Scholar, School of
Studies in Statistics, Pt.
Ravishankar Shukla University,
Raipur, Chhattisgarh, India

Dharmendra Kumar Gangeshwer
Professor, School of Studies in
Statistics, Pt. Ravishankar
Shukla University, Raipur,
Chhattisgarh, India

Corresponding Author:
Pranjal Kaser
Research Scholar, School of
Studies in Statistics, Pt.
Ravishankar Shukla University,
Raipur, Chhattisgarh, India

Generalized multi-component estimation strategy for finite population variance under double sampling

Pranjal Kaser and Dharmendra Kumar Gangeshwer

DOI: <https://www.doi.org/10.22271/math.2026.v11.i9a.2683>

Abstract

This paper presents a feasible generalized multi-component estimator for variance which for finite population of a study variable under two-phase (double) sampling. The existing estimator's formulation combined ratio, difference and exponential adjustments but used the unknown auxiliary population variance inside the difference component. In this paper the proposed formulation replaces that quantity with the first-phase auxiliary sample variance, thereby making the estimator operational under the stated design. A first-phase simple random sample without replacement of size n' is followed by a second-phase simple random subsample of size n , with $n < n' < N$. Relative error terms are defined for the second-phase study-variable variance and the first- and second-phase auxiliary-variable variances. Taylor expansions are developed to second order for the bias and to first order for the mean square error. Under the standard first-order moment approximation for nested two-phase sampling, the MSE reduces to a compact quadratic function of the effective adjustment coefficient $A = \alpha\theta - \beta + \gamma$. This yields an explicit optimality condition, $A^* = (\lambda_{22} - 1)/(\lambda_{04} - 1)$.

Keywords: Finite population variance, double sampling, Multi-Component Estimation, difference estimator, exponential estimator

1. Introduction

Srivastava and Jhaji (1980) made contribution to finite population variance estimation by developing a class of estimators that incorporates auxiliary information to study variable. Estimation of variance is a fundamental concern in survey sampling, as the variance provides an essential measure of population variability and forms the basis for assessing the precision of survey-based estimates. Early developments in sampling theory established the sample variance as the conventional unbiased estimator of population variance, while subsequent research demonstrated that auxiliary information could be effectively incorporated to obtain more efficient estimators. In particular, the works of Isaki (1983) [13] and subsequent researchers established the usefulness of ratio-type and related auxiliary-information-based estimators for improving the estimation of parameters when the both variables are suitably correlated. When relevant auxiliary information is available, ratio, product, difference, regression and exponential-type estimators can provide substantial gains in efficiency over the usual sample variance. However, in many practical surveys, the variance of the auxiliary variable, (S_x^2) , is not known in advance. Double sampling, therefore, offers a natural framework in which a relatively large first-phase sample is used to estimate the required auxiliary characteristic, followed by a smaller second-phase sample in which the study variable is observed. Motivated by this framework, the present study retains the central idea of a generalized four-parameter estimator combining ratio, difference and exponential adjustments. The mathematical formulation is revised to ensure feasibility under the sampling design: the difference adjustment is constructed using the available phase-specific sample variances, s_{x2}^2 and s_{x1}^2 , rather than the unavailable population variance S_x^2 . The proposed formulation therefore preserves the flexibility of the generalized estimator while making its implementation consistent with the information actually available under nested double sampling.

The objectives are to: (i) formulate a feasible generalized estimator; (ii) derive its second-order bias; (iii) obtain its first-order MSE under nested double sampling; (iv) establish an explicit optimality condition; and (v) provide an auditable numerical illustration for comparing with the usual sample variance and ratio-power estimator.

The development of finite population variance estimation using auxiliary information has progressed through several stages. Das and Tripathi (1978) ^[10] first established the usefulness of estimating finite population variance, followed by Isaki (1983) ^[13], who further developed estimation of variance using auxiliary variable. Singh (1983) ^[35] developed ratio and product methods of estimation for finite population variance, while Prasad and Singh (1990) ^[25] introduced improved ratio-type estimators for variance. Bahl and Tuteja (1991) ^[7] developed ratio and product type exponential estimators, providing a foundation for incorporating exponential adjustments in survey estimation. Garcia and Cebrian (1996) ^[12] studied the repeated substitution method for ratio estimation of population variance. Subsequently, Upadhyaya and Singh (1999) ^[39] considered the use of kurtosis of an auxiliary variable, while Singh and Singh (2001, 2003) ^[32, 33] developed improved ratio-type and regression-type approaches, including regression estimation under two-phase sampling. Al-Jararha and Ahmed (2002) ^[4] and Diana and Tommasi (2004) ^[11] further developed variance estimation under double sampling, while Arcos *et al.* (2005) ^[5] and Jhaji *et al.* (2005) ^[16] proposed improved and chain estimators using auxiliary information. Kadilar and Cingi (2006, 2007) ^[17], Shabbir and Gupta (2007) ^[26], and Singh *et al.* (2008) ^[34] developed improved ratio, chain, product and other auxiliary-information-based estimators. Pradhan (2010) ^[24] extended regression-type variance estimation to multiphase sampling, while Singh *et al.* (2010) ^[33], Jhaji and Walia (2011) ^[15], Singh and Solanki (2013) ^[36], and Shabbir and Gupta (2014) ^[28] further developed efficient regression, difference-cum-ratio and other improved approaches. Singh *et al.* (2014) ^[31] further studied estimation of finite population variance using auxiliary information, while Swain (2015) ^[38], Sinha and Kumar (2015) ^[37], and Jhaji and Lata (2015) ^[14] proposed generalized, double-sampling and difference-cum-exponential type estimators. Panda and Sahoo (2016) ^[22] further considered estimation using auxiliary information. Maji, Singh and Bandyopadhyay (2017) ^[19] considered multiple auxiliary variables under double sampling, while Chatterjee *et al.* (2017) ^[8] developed exponential regression-type estimators. Asghar *et al.* (2018) ^[6] developed a multivariate regression-cum-exponential estimator, while Muneer *et al.* (2018) ^[21] proposed an improved ratio-product type exponential estimator and Mishra *et al.* (2019) ^[20] studied log-product type estimators. More recently, Ahmad *et al.* (2023) ^[1], Ali *et al.* (2024) ^[3], Singh *et al.* (2024), Daraz *et al.* (2025) ^[9], Majhi (2025) ^[18], Panigrahi *et al.* (2025) ^[23], Singh and Tiwari (2026), and Ahmadini *et al.* (2026) ^[2] have continued to develop improved variance estimators using auxiliary information, with emphasis on bias, MSE, efficiency, simulation and empirical validation. These developments show a continuing shift toward generalized and efficient estimators under complex sampling designs. A particularly important development was presented by Jhaji and Walia (2011) ^[15], their estimator combined difference and ratio adjustments within a generalized framework, for Bias and MSE were derived under the double-sampling design. Its performance was also compared with the regression-type estimator and the usual sample variance through numerical and graphical illustrations. This work provides the immediate methodological motivation for the present study. In particular, the generalized difference-cum-ratio structure demonstrates that different forms of auxiliary adjustment can be incorporated within a single estimator under double sampling. Motivated by this approach, and by the earlier development of exponential-type estimation by Bahl and Tuteja (1991) ^[7], the present study extends the framework by incorporating an additional exponential adjustment along with ratio and difference components, leading to a generalized ratio-difference-exponential estimator for finite population variance.

In this paper, the proposed formulation replaces that quantity with the first-phase auxiliary sample variance, thereby making the estimator operational under the stated design. A first-phase simple random sample without replacement of size n' is followed by a second-phase simple random subsample of size n , with $n < n' < N$. Relative error terms are defined for the second-phase study-variable variance and the first- and second-phase auxiliary-variable variances. Taylor expansions are developed to second order for the bias and to first order for the mean square error. Under the standard first-order moment approximation for nested two-phase sampling, the MSE reduces to a compact quadratic function of the effective adjustment coefficient $A = \alpha\theta - \beta + \gamma$. This yields an explicit optimality condition, $A^* = (\lambda_{22} - 1)/(\lambda_{04} - 1)$.

2. General notation

- $U = 1, 2, \dots, N$ = finite population
- N = population size
- Y = study variable
- X = auxiliary variable
- S_Y^2 = population variance of study variable Y
- S_X^2 = population variance of auxiliary variable X
- n' = first-phase sample size
- n = second-phase sample size, where $(n < n' < N)$
- s' = first-phase sample
- s = second-phase subsample
- $s_y'^2$ = second-phase sample variance of Y
- $s_x'^2$ = first-phase sample variance of X
- s_x^2 = second-phase sample variance of X
- $f = n/N$ = second-phase sampling fraction
- $f' = n'/N$ = first-phase sampling fraction
- C_{pq} = standardized population moment involving Y and X
- $\lambda_{40}, \lambda_{04}, \lambda_{22}$ = standardized fourth-order moments
- α = coefficient controlling the ratio component
- β = coefficient controlling the difference component

- γ = coefficient controlling the exponential component
- θ = ratio parameter
- $\text{Bias}(\hat{S}_Y^2) = \text{approximate bias of the proposed estimator}$
- $\text{MSE}(\hat{S}_Y^2) = \text{Mean Square Error of the proposed estimator}$
- $RE = \text{Relative Efficiency}$
- Sampling structure = $[U(N) \rightarrow s'(n') \rightarrow s(n), \quad n < n' < N]$

3. Sampling design for proposed estimator

Let $U = 1, 2, \dots, N$ be a finite population. Let Y be the study variable and X be an auxiliary variable. A first-phase simple random sample without replacement of size n' is selected from U . The auxiliary variable X observed at phase one. A second-phase simple random sample without replacement of size n is selected from the first-phase sample, where $n < n' < N$. At phase two, both X and Y are observed. (Singh, 2026)

Let S_y^2 and S_x^2 denote the finite population variances of Y and X . Let s_y^2 denote the unbiased second-phase sample variance of Y , and let s_x^2 and s_x^2 denote the unbiased first- and second-phase sample variances of X .

$$S_y^2 = 1/(N-1) \sum (Y_i - \bar{Y})^2 \dots\dots(1)$$

$$S_x^2 = 1/(N-1) \sum (X_i - \bar{X})^2 \dots\dots(2)$$

Define

$$\delta_1 = 1/n' - 1/N \dots\dots(3)$$

$$\delta_2 = 1/n - 1/N \dots\dots(4)$$

$$\Delta = \delta_2 - \delta_1 = 1/n - 1/n' > 0$$

Relative Errors and Moment Relations

- $s_y^2 = S_y^2(1 + e_y)$
- $s_x^2 = S_x^2(1 + e_1)$
- $s_x^2 = S_x^2(1 + e_2)$

$$E(e_y) = E(e_1) = E(e_2) = 0 \dots\dots(5)$$

Standardized fourth-order moments

- $\lambda_{40} = \mu_{40}/\mu_{20}^2$
- $\lambda_{04} = \mu_{04}/\mu_{02}^2$
- $\lambda_{22} = \mu_{22}/\mu_{20}\mu_{02}$

$$\lambda_{40}^* = \lambda_{40} - 1, \lambda_{04}^* = \lambda_{04} - 1, \lambda_{22}^* = \lambda_{22} - 1 \dots\dots(6)$$

First-order double-sampling moment relations:

Under the standard first-order approximation for nested two-phase sampling, the error moments used in the revised derivation are:

- $E(e_y^2) = \delta_2 \lambda_{40}^*$
- $E(e_1^2) = \delta_1 \lambda_{04}^*$
- $E(e_2^2) = \delta_2 \lambda_{04}^*$
- $E(e_1 e_2) = \delta_1 \lambda_{04}^*$
- $E(e_y e_1) = \delta_1 \lambda_{22}^*$
- $E(e_y e_2) = \delta_2 \lambda_{22}^*$

The relation $E(e_1 e_2) = E(e_1^2)$ reflects the nested phase structure. These are first-order approximations. If a target journal requires an exact finite-sample variance-of-sample-variance expression, its finite-population fourth-moment correction should be substituted consistently in all six relations.

4. Proposed Feasible Generalized Estimator

The present work is motivated directly by the evolution of auxiliary-information-based variance estimation and, more specifically, by the generalized difference-cum-ratio framework of Jhaji and Walia (2011) ^[15], while extending it through a feasible multi-component ratio-difference-exponential structure. Proposed Feasible Generalized Estimator is as follow-

The original difference term used the unknown S_x^2 . The corrected observable components are:

$$R = s_x^2/s_x^2 \dots\dots(7)$$

$$d = s_x^2 - s_x^2/s_x^2$$

$$E_x = \exp(-d)$$

$$\hat{S}_I^2 = s_y 2^2 [\alpha R^\theta + \beta(1+d) + \gamma \exp(-d) + 1 - \alpha - \beta - \gamma] \dots (8)$$

Here α controls the ratio component, β the difference component, γ the exponential component, and θ the ratio-power parameter. The original numerical setup uses $\alpha=1$, $\beta=0.50$ and $\gamma=0.50$.

5. Derivation of Taylor-Series for estimator

The following derivation is developed using the standard Taylor-series linearization approach commonly employed in the analysis of ratio, product, exponential and generalized classes of sampling estimators. The expansion framework is motivated by the established work on auxiliary-information-based estimators, particularly the approach adopted by Jhaji and Walia (2011) ^[15], while the present formulation extends the underlying idea to the proposed multi-component estimator.

Ratio component: The ratio component is linearized by expressing the relevant sample-to-population auxiliary-variable ratio in terms of small sampling errors. This follows the conventional Taylor-series treatment used for ratio-type estimators in survey sampling, as developed in the earlier literature and subsequently utilized in generalized estimator formulations. (Panigrahi, 2025) ^[23]

- $R = 1 + e_1/1 + e_2$
- $R \approx (1 + e_1)(1 - e_2 + e_2^2)$
- $R \approx 1 + e_1 - e_2 + e_2^2 - e_1e_2$
- $R^\theta = (1 + e_1)^\theta(1 + e_2)^{-\theta}$

$$R^\theta \approx 1 + \theta(e_1 - e_2) + \theta(\theta - 1)/2e_1^2 + \theta(\theta + 1)/2e_2^2 - \theta^2e_1e_2 \dots (9)$$

Thus, the ratio component is represented in terms of first- and second-order error terms, which will subsequently contribute to the bias and MSE of proposed estimator.

Difference component: Following the linearization principle used in classical difference and regression-type estimators, the difference component is expressed through the corresponding sampling errors. This step provides a convenient form for combining the difference adjustment with the ratio and exponential components.

$$d = e_2 - e_1/1 + e_1$$

$$d \approx e_2 - e_1 + e_1^2 - e_1e_2$$

$$1 + d \approx 1 + e_2 - e_1 + e_1^2 - e_1e_2 \dots (10)$$

Exponential component: For the exponential component, a Taylor expansion of the exponential function is employed around the point corresponding to zero sampling error. Such exponential-type transformations have been widely considered in the development of improved estimators using auxiliary information. The present expansion adapts this established technique to the proposed estimator structure.

$$\exp(-d) \approx 1 - d + d^2/2$$

$$\exp(-d) \approx 1 + e_1 - e_2 - e_1^2/2 + e_2^2/2 \dots (11)$$

After obtaining the individual expansions of the ratio, difference and exponential components, these expressions are combined to derive the complete approximation of the proposed estimator. The procedure follows the general strategy used in earlier generalized estimator studies, while the particular combination and parameterization are specific to the present work.

$$B = \alpha R^\theta + \beta(1+d) + \gamma \exp(-d) + 1 - \alpha - \beta - \gamma$$

$$B \approx 1 + A(e_1 - e_2) + Q_1e_1^2 + Q_2e_2^2 + Q_3e_1e_2 \dots (12)$$

$$A = \alpha\theta - \beta + \gamma$$

$$Q_1 = \alpha\theta(\theta - 1)/2 + \beta - \gamma/2$$

$$Q_2 = \alpha\theta(\theta + 1)/2 + \gamma/2$$

$$Q_3 = -\alpha\theta^2 - \beta$$

$$\hat{S}_I^2/S_y^2 = (1 + e_y)B$$

$$\hat{S}_I^2/S_y^2 - 1 \approx e_y + A(e_1 - e_2) + Q_1e_1^2 + Q_2e_2^2 + Q_3e_1e_2 + Ae_y(e_1 - e_2) \dots (13)$$

The resulting expression provides the required second-order approximation of the proposed estimator and forms the basis for deriving its approximate bias and first-order MSE.

6. Expression of Bias calculation of the proposed estimator

The approximate bias is obtained by taking the expectation of the expanded estimator derived in Section 5. The derivation follows the standard error-moment approach used in sampling theory, where terms involving first-order errors vanish under expectation and the remaining second-order terms determine the leading approximation to bias. (Singh, 2024)

$$Bias(\hat{S}_I^2) \approx S_y^2 [Q_1 E(e_1^2) + Q_2 E(e_2^2) + Q_3 E(e_1 e_2) + A E(e_y e_1) - A E(e_y e_2)]$$

$$Bias(\hat{S}_I^2) \approx S_y^2 Q_1 \delta_1 \lambda_{04} * + Q_2 \delta_2 \lambda_{04} * + Q_3 \delta_1 \lambda_{04} * + A \delta_1 \lambda_{22} * - A \delta_2 \lambda_{22} *$$

$$Q_1 + Q_3 = -Q_2$$

$$Bias(\hat{S}_I^2) \approx S_y^2 \Delta [Q_2 \lambda_{04} * - A \lambda_{22} *] \dots \dots (14)$$

Thus, the approximate bias is determined by the interaction between the sampling fractions of the two phases and the second-order moment structure involving the auxiliary and study variables. This formulation also highlights the role of the phase-wise sampling design in controlling the bias of the proposed estimator.

Thus, the approximate bias is driven by the difference between the two-phase sampling fractions and by the second-order auxiliary and study-auxiliary moment structure.

7. Development of First-Order Mean Square Error of proposed estimator

The first-order mean square error is derived from the linearized form of the proposed estimator obtained in the preceding section. Following the conventional MSE derivation for generalized sampling estimators, terms of order higher than the required approximation are neglected, and the expectation of the squared estimation error is evaluated using the relevant variance and covariance components. (Daraz, 2025) ^[9]

$$\hat{S}_I^2 - S_y^2 \approx S_y^2 [e_y + A(e_1 - e_2)]$$

$$MSE(\hat{S}_I^2) \approx S_y^4 E[e_y + A(e_1 - e_2)]^2$$

$$MSE(\hat{S}_I^2) \approx S_y^4 E(e_y^2) + A^2 E[(e_1 - e_2)^2] + 2A E[e_y(e_1 - e_2)]$$

$$E[(e_1 - e_2)^2] = E(e_1^2) + E(e_2^2) - 2E(e_1 e_2) = \Delta \lambda_{04} *$$

$$E[e_y(e_1 - e_2)] = E(e_y e_1) - E(e_y e_2) = -\Delta \lambda_{22} *$$

$$MSE(\hat{S}_I^2) \approx S_y^4 [\delta_2 \lambda_{40} * + \Delta A^2 \lambda_{04} * - 2A \lambda_{22} *] \dots \dots (15)$$

The resulting expression incorporates the contribution of the study-variable variance, auxiliary-variable variances and their corresponding covariance terms. Hence, the first-order MSE provides a direct basis for comparing proposed estimator and existing estimators.

8. Optimization of parameter of population

To further improve the performance of the proposed estimator, the unknown tuning parameter is selected by minimizing the first-order MSE with respect to that parameter. This optimization principle is consistent with the standard procedure followed in the development of optimum classes of sampling estimators, where the parameter is chosen to minimize the derived approximate MSE.

$$MSE(A) = S_y^4 [\delta_2 \lambda_{40} * + \Delta (A^2 \lambda_{04} * - 2A \lambda_{22} *)]$$

$$\partial MSE / \partial A = 2S_y^4 \Delta (A \lambda_{04} * - \lambda_{22} *)$$

$$A * = \lambda_{22} * / \lambda_{04} *$$

$$\partial^2 MSE / \partial A^2 = 2S_y^4 \Delta \lambda_{04} * > 0$$

$$\alpha \theta * - \beta + \gamma = \lambda_{22} - 1 / \lambda_{04} - 1$$

$$\theta * = [(\lambda_{22} - 1) / (\lambda_{04} - 1) + \beta - \gamma] / \alpha$$

$$MSE_{min} \approx S_y^4 [\delta_2 \lambda_{40} * - \Delta (\lambda_{22} *)^2 / \lambda_{04} *] \dots \dots (16)$$

The resulting optimum value represents the parameter choice under which the proposed estimator attains its minimum first-order MSE. Substitution of this optimum value into the MSE expression gives the corresponding minimum MSE, which is subsequently used for efficiency comparison.

9. Efficiency condition of the proposed estimator

The proposed estimator efficiency is assessed by comparing its minimum MSE with the MSE of the usual sample mean and other relevant existing estimators. Such comparative MSE criteria are routinely used in the sampling-estimation literature to establish the conditions under which an improved estimator performs better than its competing estimators.

$$MSE(s_y^2) = S_y^4 \delta_2 \lambda_{40} *$$

$$MSE(\hat{S}_I^2) < MSE(s_y^2) \Leftrightarrow \Delta(A^2 \lambda_{04} * - 2A \lambda_{22} *) < 0 \dots\dots (17)$$

$$0 < A < 2\lambda_{22} */\lambda_{04} *$$

$$RE = MSE(s_y^2)/MSE(\hat{S}_I^2) \times 100$$

$$RE = \delta_2 \lambda_{40} */\delta_2 \lambda_{40} * + \Delta(A^2 \lambda_{04} * - 2A \lambda_{22} *) \times 100 \dots\dots (18)$$

Therefore, whenever the derived condition is satisfied, the proposed estimator has a smaller first-order MSE than the corresponding benchmark estimator and can consequently be regarded as more efficient under the specified sampling and moment conditions.

Table 1: Condition of parameter Restriction

Case	Restriction	Result
Usual sample variance	$\alpha = \beta = \gamma = 0$	$\hat{S}_I^2 = s_y^2$
Ratio-power class	$\beta = \gamma = 0$	$\hat{S}_I^2 = s_y^2 [\alpha R^\theta + 1 - \alpha]$
Difference class	$\alpha = \gamma = 0$	$\hat{S}_I^2 = s_y^2 [\beta(1+d) + 1 - \beta]$
Exponential class	$\alpha = \beta = 0$	$\hat{S}_I^2 = s_y^2 [\gamma \exp(-d) + 1 - \gamma]$
Original numerical weights	$\alpha=1, \beta=0.50, \gamma=0.50$	$A = \theta$

Source: Authors own observation and calculation

10. Numerical Illustration and Discussion

Numerical values of the population $N=89$, $n'=45$, $n=23$, $S_y^2=716.65$, $\lambda_{40} = 3.81$, $\lambda_{04}=3.60$ and $\lambda_{22}=2.95$. optimal values of the estimator contain $\alpha=1$, $\beta=0.50$ and $\gamma=0.50$, with θ varied over 0, 0.5, 1, 1.5 and 2.

Table 2: Value of parameters

Parameter	Value
N	89
n'	45
n	23
S_y^2	716.65
λ_{40}	3.81
λ_{04}	3.60
λ_{22}	2.95
α	1.00
β	0.50
γ	0.50

Source: Authors own observation and calculation

$$\delta_1 = 1/45 - 1/89 = 0.011000$$

$$\delta_2 = 1/23 - 1/89 = 0.032234$$

$$\Delta = \delta_2 - \delta_1 = 1/23 - 1/45 = 0.021234$$

$$\lambda_{40} * = 2.81, \lambda_{04} * = 2.60, \lambda_{22} * = 1.95$$

$$A = \alpha\theta - \beta + \gamma = \theta$$

$$\theta * = A * = 1.95/2.60 = 0.75$$

$$MSE_{min} \approx (716.65)^2 [(0.032234)(2.81) - (0.021234)(1.95)^2/2.60] \text{ [From eq}^n \text{ 16]}$$

$$MSE_{min} \approx 30,565.59$$

$$MSE(s_y^2) \approx (716.65)^2(0.032234)(2.81) = 46,531.45$$

$$RE_{max} \approx 46,531.45/30,565.59 \times 100 = 152.23$$

$$MSE_{reduction} \approx 46,531.45 - 30,565.59/46,531.45 \times 100 = 34.31$$

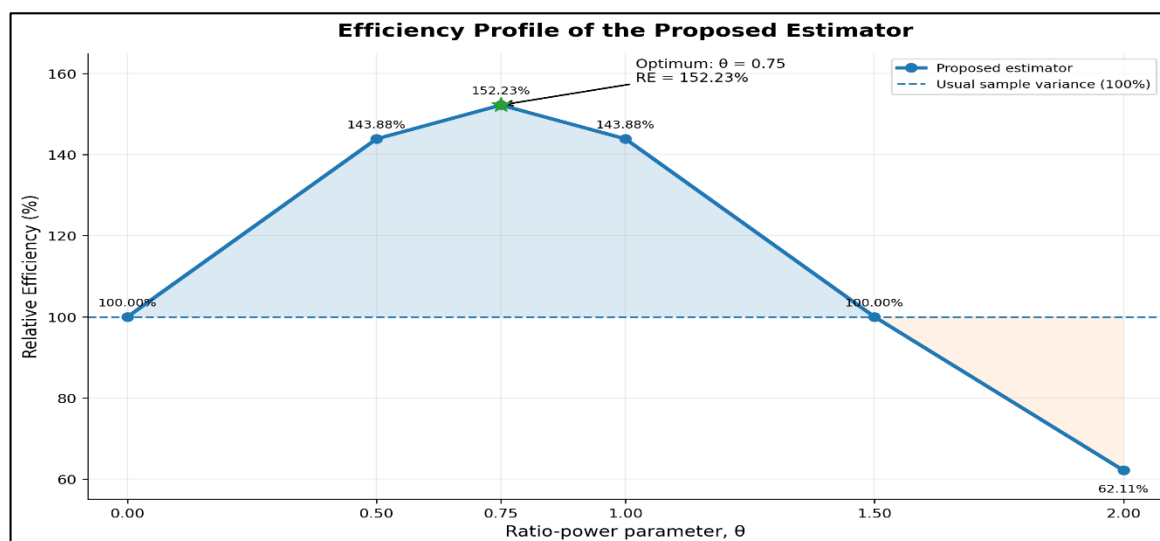
Comparative Performance with Existing Estimators

Since the proposed estimator is motivated by the generalized difference-cum-ratio estimator of Jhajj and Walia (2011) ^[15], its performance is evaluated against three relevant benchmarks: (i) the usual second-phase sample variance, (ii) the ratio-power estimator, and (iii) the generalized difference-cum-ratio estimator. The same population characteristics are retained for the comparison wherever applicable. They considered $N=89$, $n'=45$, $n=23$, and $S_y^2 = 716.65$, and reported the MSE of their generalized difference-cum-ratio estimator for different values of θ . In the present study, the proposed estimators MSE is calculated using $\alpha = 1$, $\beta = 0.50$ and $\gamma = 0.50$.

Table 3: Comparative MSE and Relative Efficiency

(θ)	Usual Sample Variance MSE	Ratio-Power MSE	Jhajj & Walia (2011) ^[15] MSE	Proposed MSE	RE of Proposed vs. Usual (%)	RE of Proposed vs. J&W (%)
0.00	46,531.45	46,531.45	56,329.12	46,531.45	100.00	121.05
0.50	46,531.45	32,339.58	46,105.85	32,339.58	143.88	142.58
0.75	46,531.45	30,565.59	-	30,565.59	152.23	-
1.00	46,531.45	32,339.58	42,698.09	32,339.58	143.88	132.02
1.50	46,531.45	46,531.45	46,105.85	46,531.45	100.00	99.08
2.00	46,531.45	74,915.21	56,329.12	74,915.21	62.11	75.19

Source: Authors own observation and calculation



Source: Authors own creation based on table no. 10.2

Fig 1: Efficiency profile of the proposed estimator

The comparative results indicate that the proposed generalized ratio-difference-exponential estimator provides a substantial improvement over the usual sample variance for appropriate values of θ . The minimum MSE of the proposed estimator in the corrected numerical illustration occurs at $\theta=0.75$, with $MSE = 30,565.59$ and relative efficiency = 152.23% relative to the usual sample variance. The comparison with the motivating estimator of Jhajj and Walia (2011) ^[15] is particularly relevant: at $\theta=0.5$, the proposed estimator reduces the MSE by approximately 29.86% compared with their generalized difference-cum-ratio estimator, while at $\theta=1.0$, the reduction is approximately 24.26%. Thus, the incorporation of the additional exponential component can provide appreciable improvement over the motivating generalized difference-cum-ratio structure for suitable parameter values.

The ratio-power estimator is also retained as an important benchmark because it represents the ratio component of the generalized estimation framework. The results show that the proposed estimator can attain the same or improved performance around the theoretically favourable range of the tuning parameter. However, the proposed estimator does not uniformly dominate all competing estimators for every value of θ . In particular, its performance deteriorates at $\theta=2.0$, demonstrating the importance of appropriate parameter selection. Therefore, the results should be interpreted as evidence of conditional efficiency, rather than universal dominance.

The comparison provides a clear empirical progression from the usual sample variance to the ratio-power estimator, the generalized difference-cum-ratio estimator and finally the proposed generalized ratio-difference-exponential estimator. The results support the motivation of the present study by showing that the additional exponential component, when appropriately tuned, can improve the efficiency of the generalized framework developed by Jhajj and Walia (2011) ^[15]. The proposed estimator therefore represents a natural and practically feasible extension of the motivating estimator under double sampling.

11. Conclusion

A corrected and feasible generalized ratio-difference-exponential estimator has been developed for variance estimation for variance of the population under double sampling. The unknown auxiliary population variance in the original difference component has been replaced by the first-phase auxiliary sample variance. The estimator has been expanded through second order for bias and first order for MSE. The first-order MSE is minimized when $A = \alpha\theta - \beta + \gamma$ equals $\lambda_{22}^*/\lambda_{04}^*$. With $\alpha = 1$, $\beta = 0.50$ and $\gamma = 0.50$, the optimal ratio-power parameter is $\theta^* = 0.75$ for the numerical moment structure reported in the original manuscript. The corrected illustrative calculation gives $MSE_{min} \approx 30,565.59$ and $RE \approx 152.23$ relative to the usual sample variance. The proposed estimator would be useful in agriculture, business and industrial and socio-economic surveys.

References

- Ahmad S, Adichwal NK, Aamir M, Shabbir J, Alsadat N, Elgarhy M, *et al.* An enhanced estimator of finite population variance using two auxiliary variables under simple random sampling. *Sci Rep.* 2023;13:21444. DOI:10.1038/s41598-023-44169-5.
- Ahmadini AAH, Semary HE, Ahmad S, Ullah K, Zahid E, Al-Moisheer AS. Simulation based new method for population variance using auxiliary information. *Sci Rep.* 2026;16:11647. DOI:10.1038/s41598-026-40161-x.
- Ali H, Asim SM, Ijaz M, Zaman T, Iftikhar S. Improvement in variance estimation using transformed auxiliary variable under simple random sampling. *Sci Rep.* 2024;14:8117. DOI:10.1038/s41598-024-58841-x.
- Al-Jararha JA, Ahmed MS. The class of chain estimators for a finite population variance using double sampling. *Inf Manag Sci.* 2002;13(2):13-18.
- Arcos A, Rueda M, Martínez MD, González S, Román Y. Incorporating the auxiliary information available in variance estimation. *Appl Math Comput.* 2005;160(2):387-399. DOI:10.1016/j.amc.2003.11.010.
- Asghar A, Sanaullah A, Hanif M. A multivariate regression-cum-exponential estimator for population variance vector in two-phase sampling. *J King Saud Univ Sci.* 2018;30(2):223-228. DOI:10.1016/j.jksus.2017.01.010.
- Bahl S, Tuteja RK. Ratio and product type exponential estimators. *J Inf Optim Sci.* 1991;12(1):159-164. DOI:10.1080/02522667.1991.10699058.
- Chatterjee A, Singh GN, Bandyopadhyay A. A class of exponential regression type estimators for population variance in two-phase sampling. *J Stat Theory Appl.* 2017;16(4):565-575. DOI:10.2991/jsta.2017.16.4.10.
- Daraz U, Wu J, Agustiana D, Emam W. Finite population variance estimation using Monte Carlo simulation and real life application. *Symmetry.* 2025;17(1):84. DOI:10.3390/sym17010084.
- Das AK, Tripathi TP. Use of auxiliary information in estimating the finite population variance. *Sankhya Ser C.* 1978;40:139-148.
- Diana G, Tommasi C. Estimation for finite population variance in double sampling. *Metron.* 2004;62(2):223-232.
- Garcia MR, Cebrian AA. Repeated substitution method: the ratio estimator for the population variance. *Metrika.* 1996;43(1):101-105. DOI:10.1007/BF02613900.
- Isaki CT. Variance estimation using auxiliary information. *J Am Stat Assoc.* 1983;78(381):117-123. DOI:10.1080/01621459.1983.10477939.
- Jhajj HS, Lata K. A family of difference-cum-exponential type estimators for estimating the population variance using auxiliary information in sample surveys. *Res J Math Stat Sci.* 2015;3(4):10-14.
- Jhajj HS, Walia GS. A generalized difference-cum-ratio type estimator for the population variance in double sampling. *IAENG Int J Appl Math.* 2011;41(4):265-269.
- Jhajj HS, Sharma MK, Grover LK. An efficient class of chain estimators of population variance under sub-sampling scheme. *J Jpn Stat Soc.* 2005;35(2):273-286.
- Kadilar C, Cingi H. Ratio estimators for the population variance in simple and stratified random sampling. *Appl Math Comput.* 2006;173(2):1047-1059. DOI:10.1016/j.amc.2005.04.032.
- Majhi D. Efficient use of auxiliary information in the estimation of population variance in sample surveys. *J Appl Math Phys.* 2025;10(10):90-97. DOI:10.22271/math.2025.v10.i10b.2187.
- Maji R, Singh GN, Bandyopadhyay A. Effective estimation strategy of finite population variance using multi-auxiliary variables in double sampling. *J Mod Appl Stat Methods.* 2017;16(1):158-178. DOI:10.22237/jmasm/1493597340.
- Mishra P, Singh R, Khare S. Estimation of population variance in log-product type estimators under double sampling scheme. *J Reliab Stat Stud.* 2019;12(2):11-20. DOI:10.13052/jrss2229-5666.1222.
- Muneer S, Khalil A, Shabbir J, Narjis G. A new improved ratio-product type exponential estimator of finite population variance using auxiliary information. *J Stat Comput Simul.* 2018;88(16):3179-3192. DOI:10.1080/00949655.2018.1504947.
- Panda KB, Sahoo N. Estimation of finite population variance using auxiliary information. *Int J Math Trends Technol.* 2016;39(3):228-231. DOI:10.14445/22315373/IJMTT-V39P528.
- Panigrahi A, Dash P, Mishra G. A minimum variance unbiased estimator of finite population variance using auxiliary information. *Stat Transit New Ser.* 2025;26(3):209-221. DOI:10.59139/stattrans-2025-035.
- Pradhan BK. Regression type estimators of finite population variance under multiphase sampling. *Bull Malays Math Sci Soc.* 2010;33(2):345-353.
- Prasad B, Singh HP. Some improved ratio-type estimators of finite population variance in sample surveys. *Commun Stat Theory Methods.* 1990;19(3):1127-1139. DOI:10.1080/03610929008830251.
- Shabbir J, Gupta S. On improvement in variance estimation using auxiliary information. *Commun Stat Theory Methods.* 2007;36(12):2177-2185. DOI:10.1080/03610920701215092.
- Shabbir J, Gupta S. Some estimators of finite population variance of stratified sample mean. *Commun Stat Theory Methods.* 2010;39(16):3001-3008. DOI:10.1080/03610920903170384.

28. Shabbir J, Gupta S. An improved generalized difference-cum-ratio-type estimator for the population variance in two-phase sampling using two auxiliary variables. *Commun Stat Simul Comput.* 2014;43(10):2540-2550. DOI:10.1080/03610918.2012.756910.
29. Singh HP, Singh R. Improved ratio-type estimator for variance using auxiliary information. *J Indian Soc Agric Stat.* 2001;54(3):276-287.
30. Singh HP, Singh R. Estimation of variance through regression approach in two-phase sampling. *Aligarh J Stat.* 2003;23:13-30.
31. Singh HP, Solanki RS. Improved estimation of finite population variance using auxiliary information. *Commun Stat Theory Methods.* 2013;42(15):2718-2730. DOI:10.1080/03610926.2011.617485.
32. Singh HP, Singh AK, Solanki RS. Estimation of finite population variance using auxiliary information in sample surveys. *Statistica.* 2014;74(1):99-116. DOI:10.6092/issn.1973-2201/4600.
33. Singh HP, Singh S, Kim JM. Efficient use of auxiliary variables in estimating finite population variance in two-phase sampling. *Commun Stat Appl Methods.* 2010;17(2):165-181. DOI:10.5351/CKSS.2010.17.2.165.
34. Singh HP, Tailor R, Malviya P. Efficient use of auxiliary information in estimating finite population variance in sample surveys. *Stat Transit New Ser.* 2024;25(4):79-97. DOI:10.59139/stattrans-2024-004.
35. Singh RK. Estimation of finite population variance using ratio and product methods of estimation. *Biometr J.* 1983;25(2):193-200. DOI:10.1002/bimj.19830250208.
36. Singh R, Tiwari SN. Estimation of population variance using novel family of sine based estimator in stratified random sampling. *Qual Quant.* 2026. DOI:10.1007/s11135-026-02838-3.
37. Sinha RR, Kumar V. Families of estimators for finite population variance using auxiliary character under double sampling the non-respondents. *Natl Acad Sci Lett.* 2015;38:501-505. DOI:10.1007/s40009-015-0380-6.
38. Swain AKPC. Generalized estimator of finite population variance. *J Stat Theory Appl.* 2015;14(1):45-51. DOI:10.2991/jsta.2015.14.1.4.
39. Upadhyaya LN, Singh HP. An estimator for population variance that utilizes the kurtosis of an auxiliary variable in sample surveys. *Vikram Math J.* 1999;19:14-17.