

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

NAAS Rating (2026): 4.49

Maths 2026; 11(8): 174-180

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<https://www.mathsjournal.com>

Received: 18-06-2026

Accepted: 21-07-2026

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Geometric construction of biangular grassmannian frames

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DOI: <https://www.doi.org/10.22271/math.2026.v11.i8b.2679>

Abstract

Grassmannian frames represent optimal line packings in complex Hilbert spaces, serving as vital structures in quantum state tomography, signal coding, and compressed sensing. While Equiangular Tight Frames achieve optimal coherence bounds, their existence is severely constrained, prompting the study of frames with small angle set cardinalities, such as biangular and triangular configurations. In this paper, we establish an algebraic-geometric foundation for classifying unit-norm tight Grassmannian frames by embedding frame vectors into projective space via the second Veronese map. By reformulating tightness, unit-norm, and biangularity requirements as polynomial ideals, computational commutative algebra enables direct verification of existence through Krull dimension analysis and Gröbner bases. We have proved that unit-norm tight biangular $(M, 2)$ frames do not exist in $\mathbb{C}^2$ for any $M \geq 2$. Consequently, any tight Grassmannian frame consisting of $M \geq 4$ vectors in $\mathbb{C}^2$ must be at least triangular.

Keywords: Geometric construction of biangular grassmannian frames, optimal line packings in hilbert spaces, embedding complex vectors into veronese varieties, reformulating frame requirements as polynomial ideals, existence verification using computational gröbner bases, non-existence of tight biangular $\mathbb{C}^2$ frames

1. Introduction

Finite frame theory provides redundant expansions of vectors in finite-dimensional Hilbert spaces, offering resilience against noise, erasures, and channel errors in signal processing and quantum communications [14, 7]. In quantum information theory, quantum state tomography, coding theory, and spherical packings, significant attention is devoted to frames that minimise maximal pairwise cross-correlations, known as *Grassmannian frames*.

While Equiangular Tight Frames (ETFs) represent optimal line packings achieving the Welch bound, their existence is severely restricted by dimensional and structural constraints [15]. When ETFs do not exist, one naturally turns to K -angular frames with small angle set cardinalities ($K = 2, 3$). In this paper, we develop a rigorous computational algebraic geometry framework to analyse unit-norm tight biangular ($K = 2$) Grassmannian frames over the complex field $\mathbb{C}$. By embedding complex projective lines into spaces of rank-one Hermitian operators via the second Veronese map, we reformulate frame constraints as polynomial ideals and solve the existence question using computational commutative algebra.

The remainder of this paper is structured as follows. In Section 2, we collect foundational definitions from finite frame theory, spherical embeddings, and algebraic geometry over $\mathbb{C}$. In Section 3, we formulate frame vectors as projective points of the Veronese variety and define system polynomial ideals. In Section 4, we provide concrete worked algebraic examples ($M = 2, 3, 5$). In Section 5, we prove the main non-existence theorem for tight biangular frames in $\mathbb{C}^2$.

2. Preliminaries and Notations

Throughout this paper, we work exclusively over the complex field $\mathbb{C}$. Let $\mathcal{H} = \mathbb{C}^m$ denote the m -dimensional complex Hilbert space equipped with the standard inner product $\langle x, y \rangle =$

$y^*x = \sum_{i=1}^m x_i \overline{y_i}$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$. Let $\{e_j\}_{j=1}^m$ denote the canonical orthonormal basis for $\mathbb{C}^m$, and I_m denote the $m \times m$ identity matrix.

2.1 Finite Frames in Complex Vector Spaces

Definition 2.1 (Finite Frames and Matrix Operators). A set of vectors $\Phi = \{v_j\}_{j=1}^M \subset \mathbb{C}^m$ is a finite frame for $\mathbb{C}^m$ if $\text{span}\{v_1, \dots, v_M\} = \mathbb{C}^m$. Equivalent characterisations are given by

- a. **Frame bounds:** There exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \|x\|^2 \leq \sum_{j=1}^M |\langle x, v_j \rangle|^2 \leq \beta \|x\|^2 \quad \text{for all } x \in \mathbb{C}^m.$$

- b. **Synthesis operator:** The $m \times M$ matrix whose columns are the frame vectors

$$T^* = [v_1 \quad v_2 \quad \dots \quad v_M] \in \mathbb{C}^{m \times M}.$$

- c. **Analysis operator:** The adjoint matrix $T = (T^*)^* \in \mathbb{C}^{M \times m}$ given by

$$T = \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_M^* \end{bmatrix} \in \mathbb{C}^{M \times m}.$$

- d. **Frame operator:** The self-adjoint positive-definite operator $S = T^*T \in \mathbb{C}^{m \times m}$

$$S = T^*T = \sum_{j=1}^M v_j v_j^* \in \mathbb{C}^{m \times m}.$$

Definition 2.2 (Unit-Norm and Tight Frames). Let $\Phi = \{v_j\}_{j=1}^M \subset \mathbb{C}^m$ be a frame.

- a. Φ is unit-norm if $\|v_j\| = 1$ for all $j \in \{1, \dots, M\}$.
b. Φ is α -tight if its frame operator satisfies $S = \alpha I_m$ for some scalar $\alpha > 0$. Taking the trace of S for a unit-norm frame yields

$$\text{Tr}(S) = \text{Tr}\left(\sum_{j=1}^M v_j v_j^*\right) = \sum_{j=1}^M \|v_j\|^2 = M = \alpha \text{Tr}(I_m) = \alpha m,$$

Which forces the tightness constant to be $\alpha = \frac{M}{m}$.

- c. For a unit-norm $\frac{M}{m}$ -tight frame, every frame vector $v_j \in \Phi$ satisfies the Parseval-type energy identity ^[15]

$$\sum_{l=1}^M |\langle v_j, v_l \rangle|^2 = \frac{M}{m} \quad \text{for every } j \in \{1, \dots, M\}.$$

- d. **The gramian matrix** of Φ is the $M \times M$ matrix $G = TT^* \in \mathbb{C}^{M \times M}$, with entries $G_{jk} = v_k^* v_j = \langle v_j, v_k \rangle$. A unit-norm frame Φ is $\frac{M}{m}$ -tight if and only if G satisfies the spectral polynomial matrix equation

$$G^2 - \frac{M}{m}G = 0.$$

2.2 Grassmannian Frames, Coherence Bounds, and Angularity. Let $\Omega_{M,m}(\mathbb{C})$ denote the set of all M -element unit-norm frames in $\mathbb{C}^m$.

Definition 2.3 (Frame Coherence and Grassmannian Constant). Let $\Phi = \{v_j\}_{j=1}^M \in \Omega_{M,m}(\mathbb{C})$.

- a. The **frame correlation** (or **coherence**) of Φ is

$$\mu(\Phi) := \max_{1 \leq j \neq l \leq M} |\langle v_j, v_l \rangle|.$$

- b. The **Grassmannian constant** for the pair (M, m) over $\mathbb{C}$ is ^[7]

$$\mu_{M,m}(\mathbb{C}) := \inf_{\Phi \in \Omega_{M,m}(\mathbb{C})} \mu(\Phi) = \inf_{\Phi \in \Omega_{M,m}(\mathbb{C})} \left(\max_{1 \leq j \neq l \leq M} |\langle v_j, v_l \rangle| \right).$$

- c. A frame $\Phi \in \Omega_{M,m}(\mathbb{C})$ is a **Grassmannian frame** if $\mu(\Phi) = \mu_{M,m}(\mathbb{C})$.

Theorem 2.4 (Lower Bounds for Grassmannian Coherence ^[15, 7]). For any unit-norm frame $\Phi \in \Omega_{M,m}(\mathbb{C})$

- a. **Welch bound**

$$\mu(\Phi) \geq \sqrt{\frac{M-m}{m(M-1)}}.$$

A frame achieves the Welch bound if and only if it is an Equiangular Tight Frame (ETF). This requires $M \leq m^2$.

- b. **Orthoplectic bound:** When $M > m^2$, the Welch bound cannot be saturated, and the coherence satisfies

$$\mu(\Phi) \geq \frac{1}{\sqrt{m}}.$$

Remark 2.5. In two-dimensional complex Hilbert space ($\mathcal{H} = \mathbb{C}^2$), the Welch bound limit $M \leq m^2 = 4$ governs equiangular tight frames ($K = 1$), as exemplified by the $M = 3$ Mercedes-Benz frame. When $M > 4$, the orthoplectic bound applies ($\mu(\Phi) \geq 1/\sqrt{2}$), saturating at $M = 2(m^2 - 1) = 6$ vectors embedded as a regular octahedron in $\mathbb{R}^3$. As established in Section 5, any unit-norm tight frame with $M \geq 4$ in $\mathbb{C}^2$ bypasses $K = 2$ (biangularity) and strictly requires $K \geq 3$ (triangularity).

Definition 2.6 (Frame Angles, K -Angularity, and Equidistribution). Let $\Phi = \{v_j\}_{j=1}^M \in \Omega_{M,m}(\mathbb{C})$.

- a. The **frame angle set** A_Φ is the set of distinct pairwise inner product magnitudes

$$A_\Phi := \{|\langle v_j, v_l \rangle| : 1 \leq j \neq l \leq M\}.$$

- b. Φ is **K -angular** if $|A_\Phi| = K \in \mathbb{N}$. In particular

- $K = 1$: **Equiangular** ($A_\Phi = \{c_1\}$);
 - $K = 2$: **Biangular** ($A_\Phi = \{c_1, c_2\}$ with $0 \leq c_1 < c_2 < 1$);
 - $K = 3$: **Triangular** ($A_\Phi = \{c_1, c_2, c_3\}$).
- c. A K -angular frame Φ with angle set $\{c_1, \dots, c_K\}$ is **equidistributed** if there exist integers $m_1, \dots, m_K \in \mathbb{N}$ (the *multiplicities*) such that for every vector $v_j \in \Phi$,

$$|\{l \in \{1, \dots, M\} \setminus \{j\} : |\langle v_j, v_l \rangle| = c_k\}| = m_k \quad \text{for all } k \in \{1, \dots, K\}.$$

Note that $\sum_{k=1}^K m_k = M - 1$.

Following ^[15], we recall some relevant results for completeness.

Proposition 2.7 (Equidistribution and Parity of Biangular Tight Frames). Let $\Phi = \{v_j\}_{j=1}^M \in \Omega_{M,m}(\mathbb{C})$ be a biangular tight frame with angles c_1, c_2 .

- Φ is automatically equidistributed with multiplicities m_1, m_2 satisfying $m_1 + m_2 = M - 1$ and $m_1 c_1^2 + m_2 c_2^2 = \frac{M-m}{m}$.
- If M is odd, the multiplicities m_1 and m_2 must both be even positive integers.

2.3 Conway-Hardin-Sloane Spherical Embedding. The spherical embedding technique of Conway, Hardin, and Sloane is used to map complex frame vectors to higher-dimensional real spheres.

Theorem 2.8 (Spherical Embedding of Tight Frames). Let $\Phi = \{v_j\}_{j=1}^M \in \Omega_{M,m}(\mathbb{C})$ be a unit-norm frame in $\mathbb{C}^m$, and set $D = m^2 - 1$.

- The frame vectors map isometrically to the sphere of radius $\sqrt{\frac{m-1}{m}}$ in $\mathbb{R}^D$ via $v_j \mapsto y_j = v_j v_j^* - \frac{1}{m} I_m \in \text{Herm}_0(m, \mathbb{C}) \cong \mathbb{R}^D$, satisfying $|\langle v_j, v_l \rangle|^2 = \frac{1}{m} + \frac{m-1}{m} \langle y_j, y_l \rangle$.
- If Φ is a tight frame, the embedded vectors sum to zero in $\mathbb{R}^D$: $\sum_{j=1}^M y_j = \sum_{j=1}^M \left(v_j v_j^* - \frac{1}{m} I_m \right) = \frac{M}{m} I_m - \frac{M}{m} I_m = 0$.

2.4 Gröbner Bases and Hilbert's Nullstellensatz

Definition 2.9 (Complex Projective Space). Let $\mathcal{H} = \mathbb{C}^m$. Complex projective space $\mathbb{CP}^{m-1}$ is the quotient space

$$\mathbb{CP}^{m-1} := (\mathbb{C}^m \setminus \{0\}) / \mathbb{C}^\times, \quad v \mapsto [v] = \{\lambda v : \lambda \in \mathbb{C}^\times\}.$$

For $v = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$, $[v] = [z_0 : \dots : z_{m-1}]$ denotes its homogeneous coordinates.

Definition 2.10 (Polynomial Ideals and Varieties). Let $R = \mathbb{C}[z_{10}, \dots, z_{M1}]$ be the polynomial ring in $2M$ variables representing vector coordinates in $\mathbb{C}^2$.

- An **affine algebraic variety** associated with an ideal $I = (f_1, \dots, f_n) \subset R$ is $\mathbb{V}(I) := \{v \in \mathbb{C}^{2M} : f_i(v) = 0 \text{ for all } i = 1, \dots, n\}$.
- A **projective algebraic variety** $\mathbb{V}(I^h) \subset \mathbb{CP}^{2M-1}$ is defined by a homogeneous ideal I^h .
- Hilbert's Weak Nullstellensatz** ^[19]: The algebraic variety $\mathbb{V}(I)$ is empty ($\mathbb{V}(I) = \emptyset$) if and only if $1 \in I$, which occurs if and only if the reduced Gröbner basis of I under any monomial ordering is $\{1\}$.
- In terms of Krull dimension, $\mathbb{V}(I) = \emptyset \Leftrightarrow \dim(R/I) = -1$.

3. Frame Vectors As Projective Points of a Veronese Variety

Let $\mathcal{H} = \mathbb{C}^{m+1}$ and consider a frame $\Phi = \{v_j\}_{j=1}^M \subset \mathcal{H}$. Each frame vector $v_j = (z_{j0}, \dots, z_{jm}) \in \mathbb{C}^{m+1}$ defines a rank-one Hermitian projection operator $P_{v_j} = v_j v_j^*$

$$P_{v_j} := v_j v_j^* = \begin{bmatrix} z_{j0} \overline{z_{j0}} & z_{j0} \overline{z_{j1}} & \dots & z_{j0} \overline{z_{jm}} \\ z_{j1} \overline{z_{j0}} & z_{j1} \overline{z_{j1}} & \dots & z_{j1} \overline{z_{jm}} \\ \vdots & \vdots & \ddots & \vdots \\ z_{jm} \overline{z_{j0}} & z_{jm} \overline{z_{j1}} & \dots & z_{jm} \overline{z_{jm}} \end{bmatrix}_{(m+1) \times (m+1)}.$$

Since scaling v_j by a complex scalar $\lambda \in U(1)$ leaves $P_{\lambda v_j} = |\lambda|^2 P_{v_j} = P_{v_j}$ invariant, each operator P_{v_j} (denoted simply by P_j) represents a unique projective point in $\mathbb{P}(\text{Herm}(m+1, \mathbb{C})) \cong \mathbb{CP}^{(m+1)^2-1}$.

Definition 3.1 (Veronese Embedding). The second Veronese embedding $v_2: \mathbb{CP}^m \rightarrow \mathbb{CP}^{(m+1)^2-1}$ is defined by ^[20]

$$v_2: [z_{j0} : \dots : z_{jm}] \mapsto [z_{ji} \overline{z_{jk}}]_{0 \leq i, k \leq m}.$$

The image $v_2(\mathbb{CP}^m)$ corresponds to rank-one Hermitian operators and is realised as the algebraic variety defined by the vanishing of all 2×2 minors of P_j

$$\begin{aligned} v_2(\mathbb{CP}^m) &= \mathbb{V} \left(\bigwedge^2 P_j \right) \\ &= \{P_j \in \mathbb{CP}^{(m+1)^2-1} : P_j^2 \sim P_j, \text{rank}(P_j) = 1\}. \end{aligned}$$

Remark 3.2. The Veronese embedding translates key geometric and frame-theoretic properties into algebraic geometry

- Hermitian rank-one operators corresponding to frame vectors $\{v_1, \dots, v_M\}$ are projective points lying on the Veronese variety $v_2(\mathbb{CP}^m) \subset \mathbb{CP}^{(m+1)^2-1}$.
- The inner product magnitude between two frame vectors satisfies

$$|\langle v_j, v_l \rangle|^2 = \text{Tr}(P_j P_l).$$

Consequently, a frame $\Phi = \{v_j\}_{j=1}^M$ is biangular with angle set $A_\Phi = \{c_1, c_2\}$ if and only if the set of operators $\{P_1, \dots, P_M\} \subset v_2(\mathbb{CP}^m)$ forms a two-distance set under the trace metric

$$\text{Tr}(P_j P_l) \in \{c_1^2, c_2^2\} \quad \text{for all } j \neq l.$$

- The $\frac{M}{m+1}$ -tightness condition $\sum_{j=1}^M v_j v_j^* = \frac{M}{m+1} I_{m+1}$ translates into a linear operator identity on the Veronese variety

$$\sum_{j=1}^M P_j = \frac{M}{m+1} I_{m+1}.$$

3.1 Polynomial Ideals Defining Biangular Tight Frame Varieties: To convert the existence of biangular tight frames into a decision problem in computational commutative algebra, we express all conditions as a system of polynomial equations in the polynomial ring

$$R = \mathbb{C}[z_{10}, z_{11}, \dots, z_{M0}, z_{M1}].$$

For a collection of M vectors in $\mathbb{C}^2$ ($m = 1$), we define the following sub-ideals

- Unit-norm ideal** (I_{norm}): Imposes $\|v_j\|^2 = 1$ for all $j \in \{1, \dots, M\}$

$$I_{\text{norm}} = (z_{j0}^2 + z_{j1}^2 - 1: j = 1, \dots, M).$$

2. **Biangular angle ideal (I_{angle}):** For distinct pairs $1 \leq j < l \leq M$, define the inner product polynomial $\text{inner}_{jl} = (z_{j0}z_{l0} + z_{j1}z_{l1})^2$. The biangular condition $|\langle v_j, v_l \rangle|^2 \in \{c_1^2, c_2^2\}$ yields

$$I_{\text{angle}} = ((\text{inner}_{jl} - c_1^2)(\text{inner}_{jl} - c_2^2): 1 \leq j < l \leq M).$$

3. **Gauge-fixing ideal (I_{fix}):** By unitarily rotating the frame, we may fix the first frame vector to $v_1 = (1, 0)^T$

$$I_{\text{fix}} = (z_{10} - 1, z_{11}).$$

The full ideal defining the configuration variety of $(M, 2)$ biangular frames is given by the sum of these ideals

$$I = I_{\text{norm}} + I_{\text{angle}} + I_{\text{fix}} \subset R.$$

4. Concrete Worked Algebraic Examples

To build geometric intuition and demonstrate the algebraic framework, we present explicit worked examples for small vector configurations in $\mathbb{C}^2$.

Example 4.1 (Orthonormal Basis: $M = 2$ in $\mathbb{C}^2$). Consider $M = 2$ vectors in $\mathbb{C}^2$ given by the canonical basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The rank-one Hermitian projection operators on $v_2(\mathbb{CP}^1) \subset \mathbb{CP}^3$ are

$$P_1 = v_1 v_1^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = v_2 v_2^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Checking frame properties

- a. **Tightness:** $P_1 + P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = \frac{2}{2} I_2$, so Φ is 1-tight (Parseval frame).
- b. **Inner product & trace:** $\text{Tr}(P_1 P_2) = \text{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 = |\langle v_1, v_2 \rangle|^2$.
- c. **Angularities:** The frame angle set is $A_\Phi = \{0\}$, which has cardinality $|A_\Phi| = 1$. Thus, Φ is strictly **equiangular** ($K = 1$), not biangular ($K = 2$).

Example 4.2 (Mercedes-Benz Frame: $M = 3$ in $\mathbb{C}^2$). Consider the 3-vector frame in $\mathbb{C}^2$ (the Mercedes-Benz ETF)

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}.$$

The corresponding Hermitian rank-one projection matrices are

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}.$$

Checking algebraic properties

- a. **Tightness**

$$P_1 + P_2 + P_3 = \begin{bmatrix} 1 + 1/4 + 1/4 & 0 \\ 0 & 0 + 3/4 + 3/4 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} = \frac{3}{2} I_2.$$

- b. **Trace inner products:** For all $j \neq k$, $\text{Tr}(P_j P_k) = |\langle v_j, v_k \rangle|^2 = 1/4$.
- c. **Angularities:** The angle set is $A_\Phi = \{1/2\}$, which has cardinality $|A_\Phi| = 1$. Thus Φ is equiangular ($K = 1$).
- d. **Parity constraint:** By Proposition 7, for odd $M = 3$, any biangular tight frame requires multiplicities m_1, m_2 to be even integers summing to $M - 1 = 2$. The only solutions are $(m_1, m_2) = (2, 0)$ or $(0, 2)$, which confirms algebraically that $M = 3$ tight frames cannot be biangular ($K = 2$).

Example 4.3 (Non-Tight Biangular Frame: $M = 5$ in $\mathbb{C}^2$). To demonstrate that biangularity $K = 2$ is possible without tightness, consider the 5-vector frame $\Phi_{\text{non}} = \{v_1, \dots, v_5\} \subset \mathbb{C}^2$ given by ^[15]

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad v_5 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

The five vectors are distinct unit vectors. Computing the pairwise inner-product magnitudes for distinct vectors gives

$$\begin{aligned} |\langle v_1, v_2 \rangle| &= |\langle v_1, v_3 \rangle| = |\langle v_1, v_4 \rangle| = |\langle v_1, v_5 \rangle| = \frac{1}{\sqrt{2}}, \\ |\langle v_2, v_3 \rangle| &= |\langle v_4, v_5 \rangle| = 0, \\ |\langle v_2, v_4 \rangle| &= |\langle v_2, v_5 \rangle| = |\langle v_3, v_4 \rangle| = |\langle v_3, v_5 \rangle| = \frac{1}{\sqrt{2}}. \end{aligned}$$

Hence, the set of distinct off-diagonal inner-product magnitudes is

$$A_{\Phi_{\text{non}}} = \left\{0, \frac{1}{\sqrt{2}}\right\}, \quad |A_{\Phi_{\text{non}}}| = 2.$$

Thus, Φ_{non} is biangular.

On the other hand, its frame operator is

$$\begin{aligned} S &= \sum_{j=1}^5 v_j v_j^* \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \neq \frac{5}{2} I_2. \end{aligned}$$

Therefore, Φ_{non} is a biangular ($K = 2$), unit-norm frame in $\mathbb{C}^2$ that is not tight. This example demonstrates that biangularity alone does not imply tightness, and hence the tightness condition is the essential additional constraint in the nonexistence of tight biangular frames consisting of five vectors in $\mathbb{C}^2$.

Example 4.4 (Tight Triangular Frame: $M = 5$ in $\mathbb{C}^2$). Consider the orthoplectic 5-vector frame $\Phi_{\text{tri}} = \{v_1, \dots, v_5\} \subset \mathbb{C}^2$ from ^[15]

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_4 = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_5 = \begin{bmatrix} 1/\sqrt{2} \\ (-1-i)/2 \end{bmatrix}.$$

This frame satisfies $S = \sum_{j=1}^5 v_j v_j^* = \frac{5}{2} I_2$ (tight) and has frame angle set $A_{\Phi_{\text{tri}}} = \{\frac{1}{\sqrt{2}}, \frac{1}{2}, 0\}$. Its angle set cardinality is $|A_{\Phi_{\text{tri}}}| = 3$ (triangular).

Example 4.5 (Non-Existence Proof for $M = 5$ via Veronese Embedding). We prove directly via the Veronese embedding $v_2: \mathbb{CP}^1 \rightarrow \mathbb{CP}^3$ that no unit-norm tight biangular (5,2) Grassmannian frame exists in $\mathbb{C}^2$.

Under the second Veronese map, each frame vector $v_j \in \mathbb{C}^2$ corresponds to a rank-one Hermitian projection operator on the Veronese variety

$$P_j = v_j v_j^* \in v_2(\mathbb{CP}^1) \subset \text{Herm}(2, \mathbb{C}).$$

The trace metric evaluates vector inner product magnitudes as $\text{Tr}(P_j P_l) = |\langle v_j, v_l \rangle|^2$, and $\frac{5}{2}$ -tightness imposes the linear operator identity on the Veronese variety

$$\sum_{j=1}^5 P_j = \frac{5}{2} I_2.$$

a. **Trace energy identity:** Taking the Hilbert-Schmidt trace norm squared of both sides of (1)

$$\begin{aligned} \text{Tr} \left(\left(\sum_{j=1}^5 P_j \right)^2 \right) &= \sum_{j=1}^5 \text{Tr}(P_j^2) + 2 \sum_{1 \leq j < l \leq 5} \text{Tr}(P_j P_l) \\ &= 5(1) + 2 \sum_{1 \leq j < l \leq 5} |\langle v_j, v_l \rangle|^2. \end{aligned}$$

Evaluating the right-hand side yields $\text{Tr} \left(\left(\frac{5}{2} I_2 \right)^2 \right) = \frac{25}{4} \text{Tr}(I_2) = \frac{25}{2}$. Equating both sides gives the exact sum of off-diagonal trace inner products

$$5 + 2 \sum_{1 \leq j < l \leq 5} |\langle v_j, v_l \rangle|^2 = \frac{25}{2} \Rightarrow \sum_{1 \leq j < l \leq 5} |\langle v_j, v_l \rangle|^2 = \frac{15}{4}.$$

b. **Multiplicity & spectral contradiction:** By Proposition 2.7, any biangular tight frame with odd $M = 5$ requires equal multiplicities $m_1 = m_2 = 2$. Tightness requires $2c_1^2 + 2c_2^2 = \frac{5-2}{2} = \frac{3}{2} \Rightarrow c_1^2 + c_2^2 = \frac{3}{4}$. Decomposing $P_j = y_j + \frac{1}{2} I_2$ into traceless components $y_j \in \text{Herm}_0(2, \mathbb{C}) \cong \mathbb{R}^3$, (1) forces $\sum_{j=1}^5 y_j = 0$. Evaluating $0 = \|\sum_{j=1}^5 y_j\|^2$ yields

$$\begin{aligned} 0 &= \left\| \sum_{j=1}^5 y_j \right\|^2 = \sum_{j=1}^5 \|y_j\|^2 + 2 \sum_{1 \leq j < l \leq 5} \langle y_j, y_l \rangle \\ &= 5 \left(\frac{1}{2} \right) + 2 \sum_{1 \leq j < l \leq 5} (2|\langle v_j, v_l \rangle|^2 - 1) \\ &= \frac{5}{2} + 4 \left(\frac{15}{4} \right) - 20 = \frac{5}{2} + 15 - 20 = -\frac{5}{2}. \end{aligned}$$

The contradiction $0 = -5/2$ proves that no set of five rank-one operators $\{P_1, \dots, P_5\} \subset v_2(\mathbb{CP}^1)$ can simultaneously satisfy biangular trace metrics and the tightness linear operator identity. (Equivalently, in terms of polynomial ideals, $1 \in I_5 \subset \mathbb{C}[z_{10}, \dots, z_{51}]$, so $V(I_5) = \emptyset$ and $\dim I_5 = -1$).

5. Classification and Non-Existence Theorem in $\mathbb{C}^2$

We now present the general non-existence theorem for biangular tight Grassmannian frames in $\mathbb{C}^2$.

Theorem 18 (Non-Existence of Biangular Tight Frames in $\mathbb{C}^2$). Let $\Phi \in \Omega_{M,2}(\mathbb{C})$ be a unit-norm tight Grassmannian frame for $\mathbb{C}^2$. Then Φ cannot be biangular ($K = 2$) for any $M \geq 2$. Consequently, any unit-norm tight Grassmannian frame in $\mathbb{C}^2$ with $M \geq 4$ vectors must have an angle set cardinality $|A_\Phi| \geq 3$.

Proof. We analyse the cases for M systematically

- Case $M = 2$:** A unit-norm frame $\Phi = \{v_1, v_2\} \subset \mathbb{C}^2$ is tight if and only if it forms an orthonormal basis (Example 13). In this case, $|\langle v_1, v_2 \rangle| = 0$, so the angle set is $A_\Phi = \{0\}$, which has cardinality $|A_\Phi| = 1$. Hence Φ is equiangular ($K = 1$), not biangular ($K = 2$).
- Case $M = 3$:** Suppose $\Phi = \{v_1, v_2, v_3\} \subset \mathbb{C}^2$ is a unit-norm tight frame (Example 4.2). Since $M = 3$ is odd, by Proposition 2.7 (Proposition 5.3 in [15]), if Φ were biangular with angles c_1, c_2 and multiplicities m_1, m_2 , the multiplicities m_1, m_2 must both be even non-negative integers satisfying $m_1 + m_2 = M - 1 = 2$. The only non-negative even integers summing to 2 are $(m_1, m_2) = (2, 0)$ or $(0, 2)$, which forces all off-diagonal inner products to equal a single value c_1 . Thus Φ is equiangular ($K = 1$, as exemplified by the Mercedes-Benz frame), not biangular ($K = 2$).
- Case $M = 4$:** Suppose $\Phi = \{v_1, v_2, v_3, v_4\} \subset \mathbb{C}^2$ is a unit-norm tight biangular frame with angle set $A_\Phi = \{c_1, c_2\}$ where $0 \leq c_1 < c_2 < 1$. By Parseval tightness, every vector $v_j \in \Phi$ satisfies

$$\sum_{l \neq j} |\langle v_j, v_l \rangle|^2 = \frac{M - m}{m} = \frac{4 - 2}{2} = 1 \quad \text{for every } j \in \{1, 2, 3, 4\}.$$

Summing over all six pairwise off-diagonal inner products yields

$$\sum_{1 \leq j < l \leq 4} |\langle v_j, v_l \rangle|^2 = \frac{M(M - m)}{2m} = 2.$$

Under the Conway-Hardin-Sloane spherical embedding (Theorem 8), each vector v_j maps to $y_j = v_j v_j^* - \frac{1}{2} I_2 \in \mathbb{R}^3$ with norm $\|y_j\|^2 = 1/2$ and inner products satisfying $\langle y_j, y_l \rangle = 2|\langle v_j, v_l \rangle|^2 - 1$. Tightness forces the spherical vectors to sum to zero ($\sum_{j=1}^4 y_j = 0$), so $y_4 = -(y_1 + y_2 + y_3)$. Expanding the norm squared $\|y_4\|^2 = \|y_1 + y_2 + y_3\|^2$

$$\begin{aligned} \frac{1}{2} &= \sum_{j=1}^3 \|y_j\|^2 + 2 \sum_{1 \leq j < l \leq 3} \langle y_j, y_l \rangle \\ &= \frac{3}{2} + 2 \sum_{1 \leq j < l \leq 3} (2|\langle v_j, v_l \rangle|^2 - 1). \end{aligned}$$

Simplifying yields the exact subset inner product sum for $\{v_1, v_2, v_3\}$

$$\sum_{1 \leq j < l \leq 3} |\langle v_j, v_l \rangle|^2 = \frac{5}{4}.$$

Subtracting this subset sum from the total sum (2) determines the sum of inner products involving v_4

$$\sum_{j=1}^3 |\langle v_4, v_j \rangle|^2 = 2 - \frac{5}{4} = \frac{3}{4}.$$

However, Parseval tightness requires $\sum_{j=1}^3 |\langle v_4, v_j \rangle|^2 = 1$. The direct geometric contradiction $3/4 \neq 1$ proves that no complex vector configuration in $\mathbb{C}^2$ can simultaneously satisfy the frame tightness and biangularity polynomial constraints. Consequently, the configuration variety is empty ($\mathbb{V}(I_4) = \emptyset$). By Hilbert's Weak Nullstellensatz, $\mathbb{V}(I_4) = \emptyset$ if and only if $1 \in I_4$, which forces the coordinate ring quotient to be the zero ring ($R_4/I_4 = \{0\}$), yielding Krull dimension $\dim I_4 = -1$.

4. **Induction for $M \geq 4$:** Assume by induction that $\mathbb{V}(I_k) = \emptyset$ for some $k \geq 4$. For $M = k + 1$, the ideal of constraints I_{k+1} contains all constraint generators of I_k plus additional polynomial constraints imposed by the $(k + 1)$ -th frame vector. That is,

$$I_k \cdot R_{k+1} \subset I_{k+1} \subset R_{k+1}.$$

Since the zero-locus operator $\mathbb{V}(\cdot)$ is inclusion-reversing, we obtain

$$\emptyset = \mathbb{V}(I_k) \supseteq \mathbb{V}(I_{k+1}),$$

Which forces $\mathbb{V}(I_{k+1}) = \emptyset$. Hence, no $(M, 2)$ biangular tight Grassmannian frames exist in $\mathbb{C}^2$ for any $M \geq 4$.

6. Reconstruction of Frame Vectors From Veronese Varieties

When the coordinate ring quotient R/I has finite non-zero Krull dimension, frame vector configurations can be reconstructed algebraically from the zero-dimensional solution variety $\mathbb{V}(I)$ or geometrically via spectral factorisation of points on the Veronese variety.

6.1 Algebraic reconstruction via variety solving: Let $I \subset R = \mathbb{C}[z_{10}, \dots, z_{M1}]$ be an ideal defining a valid frame configuration variety $\mathbb{V}(I)$. If $\dim(R/I) = 0$, the variety consists of a finite set of discrete points corresponding to isolated frame configurations up to global unitary transformations. Solving the system of polynomial equations $f_i(z) = 0$ for generators $f_i \in I$ yields the exact algebraic coordinates $(z_{j0}, z_{j1}) \in \mathbb{C}^2$ for each frame vector $v_j \in \Phi$.

6.2 Geometric reconstruction via spectral factorization: Alternatively, given a collection of rank-one projection operators $\{P_1, \dots, P_M\} \subset \nu_2(\mathbb{CP}^{m-1})$ lying on the Veronese variety, the pairwise trace inner products form the positive semi-definite Gramian matrix $G \in \mathbb{C}^{M \times M}$ with entries $G_{jl} = \text{Tr}(P_j P_l)$. By spectral decomposition, G factorises as

$$G = U \Lambda U^*,$$

Where $U \in \mathbb{U}(M)$ is unitary and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m, 0, \dots, 0) \in \mathbb{C}^{M \times M}$ is the diagonal matrix containing the non-zero eigenvalues of G . The synthesis matrix $T^* \in \mathbb{C}^{m \times M}$ containing the reconstructed frame vectors as its columns is uniquely recovered via

$$T^* = \Lambda^{1/2} U^* \in \mathbb{C}^{m \times M},$$

Where $\Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}, 0, \dots, 0) \in \mathbb{C}^{m \times M}$. The columns of T^* yield the exact unit-norm frame vectors $\{v_1, \dots, v_M\} \subset \mathbb{C}^m$ satisfying the prescribed geometric angularity and spectral tightness conditions.

Example 6.1. To illustrate the spectral factorisation procedure of Section 6, consider a configuration of $M = 3$ rank-one projection operators $\{P_1, P_2, P_3\} \subset \nu_2(\mathbb{CP}^1)$ on the Veronese variety representing the Mercedes-Benz ETF in $\mathbb{C}^2$.

a. **Gramian matrix construction:** Since $\|v_j\| = 1$ and $|\langle v_j, v_l \rangle| = 1/2$ for all $j \neq l$, the pairwise trace metric yields $\text{Tr}(P_j P_l) = |\langle v_j, v_l \rangle|^2 = 1/4$ for $j \neq l$, and $\text{Tr}(P_j^2) = 1$. The entrywise square root yields the positive semi-definite Gramian matrix $G \in \mathbb{C}^{3 \times 3}$

$$G = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix} \in \mathbb{C}^{3 \times 3}.$$

b. **Spectral factorisation ($G = U \Lambda U^*$):** Diagonalising G yields eigenvalues $\lambda_1 = 3/2$, $\lambda_2 = 3/2$, and $\lambda_3 = 0$. The rank $\text{rank}(G) = 2 = m$ confirms that the configuration spans $\mathbb{C}^2$, while the non-zero eigenvalues $\lambda_1 = \lambda_2 = 3/2 = M/m$ confirm Parseval tightness. An orthonormal matrix of eigenvectors $U \in \mathbb{O}(3)$ is

$$U = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \in \mathbb{O}(3).$$

c. **Synthesis matrix recovery ($T^* = \Lambda^{1/2} U^*$):** Form the non-zero diagonal square root matrix $\Lambda^{1/2} = \begin{bmatrix} \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \end{bmatrix} \in \mathbb{C}^{2 \times 3}$. Computing $T^* = \Lambda^{1/2} U^T \in \mathbb{C}^{2 \times 3}$

$$T^* = \begin{bmatrix} \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \in \mathbb{C}^{2 \times 3}.$$

d. **Reconstructed vector coordinates:** The columns of $T^* = [v_1 \ v_2 \ v_3]$ recover the exact unit-norm frame vectors in $\mathbb{C}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}.$$

This confirms that points on the Veronese variety allow complete, explicit geometric reconstruction of the unit-norm frame vectors.

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