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Evaluation of double infinite integral for the generalized polynomial set $R_n(x, y)$ of two variables

Rupesh Kumar SinghDOI: <https://www.doi.org/10.22271/math.2026.v11.i8b.2674>**Abstract**

Present paper deals with the integral involving of $R_n(x, y)$. In this paper, a number of double infinite integral in which the integrand is a product of the polynomial set $R_n(x, y)$ and some special functions like Meijer's G-function, the exponential functions and the Legendre function etc., have been evaluated. Some integrals involving of these functions have most important role in Statistical problem. Most of the integrals have been evaluated in two legs firstly, the double integrals have to be transformed to a single one with the help of one of the known results ^[3] and then these integrals can finally be evaluated on making use of known results from tables of integral transform Vol.2 ^[4], after making proper substitution. AMS Subject Classification: Special Function-33c

Keywords: Generalized polynomial, infinite integral, generating relation, Legendre function**1. Introduction**Singh R.K ^[6] defined the polynomial set $R_n(x, y)$ by means of the generating relation.

$$\left(\xi + \lambda_1 y^{-m_1} t^{m_1} \right)^\sigma F_1 \left[\begin{matrix} (a_p); (b_u); b^1; \\ \lambda_1 x^e t^e, \lambda_2 y^{-m_2} t^{m_2} \\ (c_q); \end{matrix} \right] = \sum_{n=0}^{\infty} R_{n,e;m_1;m_2}^{\lambda;\lambda_1;\lambda_2;\xi;\sigma;(a_p);(b_u);b^1}(x,y) t^n \quad \dots$$

(1.1)

Where $\lambda; \lambda_1; \lambda_2; \sigma; \xi;$ are real and m_1, m_2 , and e are positive integers.The left hand side of (1.1) contains generalized Appell function [1] of two variables by notation of Burchnall and Chaundy ^[2].

After little simplification (1.1) gives

$$R_n(x, y) = \sum_{\substack{h,k \geq 0 \\ m_1 h + m_2 k \leq n}} \frac{\Delta(h, k)}{(n - m_1 h - m_2 k)!}$$

... (1.2)

Where

$$\Delta(h, k) = \frac{[(a_p)]_{n-m_1 h - (m_2-1)k} [(b_u)]_{n-m_1 h - m_2 k} (b^1)_k}{[(c_q)]_{n-m_1 h - (m_2-1)k} h! \xi^h}$$

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$$\frac{(-\sigma)_h (-\lambda_1)^h (\lambda x^e)^{n-m_1h-m_2k} \lambda_2^{m_2k}}{k! y^{m_1h+m_2k} (n-m_1h-m_2k)!}$$

2. Lemmas and Notations

I. Lemmas

The following lemmas which are the generalized form of the lemma due to Rainville ^[5] have been used throughout the paper

$$(i) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} A(k, n - mk)$$

$$(ii) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + mk)$$

$$(iii) (A) \quad \sum_{\substack{n, k \geq 0 \\ mh+m_2k \leq n}} \bar{\Delta}(h, k) \neq \sum_{h=0}^{\left[\frac{n}{m_1}\right]} \sum_{k=0}^{\left[\frac{n}{m_2}\right]} \bar{\Delta}(h, k)$$

$$(B) \quad \sum_{\substack{h, k \geq 0 \\ mh+m_2k \leq n}} \frac{\bar{\Delta}(h, k)}{(n-m_1h-m_2k)!} = \sum_{n=0}^{\left[\frac{n}{m_1}\right]} \sum_{k=0}^{\left[\frac{n}{m_2}\right]} \frac{\bar{\Delta}(h, k)}{(n-m_1h-m_2s)!}$$

II. Notations:

- A** (i) $n = 1, 2, 3, \dots, n-1, n$.
(ii) $(ap) = a_1, a_2, a_3, \dots, ap$.
(iii) $[(ap)] = a_1, a_2, \dots, ap$.

$$(iv) \quad \left[(a_p) \right]_n = \prod_{i=1}^p (a_i)_n = (a_1)_n (a_2)_n (a_3)_n \dots (a_p)_n.$$

$$\mathbf{B} \quad (i) \quad \Delta(a; b) = \frac{b}{a}, \frac{b+1}{a}, \frac{b+2}{a}, \dots, \frac{b+a-1}{a}.$$

$$(ii) \quad \Delta_k[a(i); b] = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-2}{a}$$

$$(iii) \quad \Delta_k[m; (a_p)] = \left(\frac{a_i + r - 1}{m} \right)_k \quad \begin{matrix} r = 1, 2, \dots, m \\ i = 1, 2, \dots, p \end{matrix}$$

$$(iv) \quad \square(a; b \pm c \pm d) = \square(a; b + c + d) \square(a; b + c - d), \\ \square(a; b - c + d), \square(a; b - c - d).$$

$$\mathbf{C} \quad (i) \quad \Delta_k[a, b] = \prod_{r=1}^a \left(\frac{b+r-1}{a} \right)_k = \left(\frac{b}{a} \right)_k, \left(\frac{b+1}{a} \right)_k, \dots, \left(\frac{b+a-2}{a} \right)_k$$

$$(ii) \quad \Delta_k[m; (a_p)] = \prod_{i=1}^p \prod_{r=1}^m \left(\frac{a_i + r - 1}{m} \right)_k.$$

$$\mathbf{D} \quad (i) \quad \Gamma[(a_p)] = \prod_{i=1}^p \Gamma(a_i).$$

$$(ii) \quad \Gamma\left[a + \frac{(m)}{m}\right] = \prod_{r=1}^m \Gamma\left(a + \frac{r}{m}\right).$$

$$(iii) \quad \Gamma[(a; b)] = \prod_{r=1}^a \Gamma\left(\frac{b+r-1}{a}\right).$$

$$(vi) \quad \Gamma[\Delta(m; (a_p))] = \prod_{i=1}^p \prod_{r=1}^m \left(\frac{a_i + r - 1}{m}\right).$$

$$E \quad (i) \quad \Gamma_*(a \pm b) = \Gamma(a+b) \Gamma(a-b).$$

$$(ii) \quad \Gamma_{**}(a+1) = \Gamma(a+b) \Gamma(a-b).$$

3. Evaluation of Double Infinite Integral

We proceed to evaluate the double integral

$$I = \int_0^\infty \int_0^\infty x^D y^E P(x+y) R_n(\alpha x^{g_1}, \beta y^{g_2}) dx dy$$

which on using (1.2), becomes

$$\begin{aligned} I &= \xi^\sigma \sum_{h=0}^\infty \sum_{k=0}^\infty \frac{[(a_p)]_{n-m_1h-(m_2-1)k} [(b_u)]_{n-m_1h-m_2k} (b')_k}{[(c_q)]_{n-m_1h-(m_2-1)k} \xi^h h! k!} \\ &\times \frac{(-\sigma)_h (-\lambda_1)^h (\lambda \alpha x^{eg_1})^{n-m_1h-m_2k} \lambda_2 k}{(\beta y^{g_2})^{m_1h+m_2k} (n-m_1h-m_2k)!} \int_0^\infty \int_0^\infty x^D y^E f(x+y) dx dy \\ &= \xi^\sigma \sum_{h=0}^\infty \sum_{k=0}^\infty \frac{[(a_p)]_{n-m_1h-(m_2-1)k} [(b_u)]_{n-m_1h-m_2k} (b')_k}{[(c_q)]_{n-m_1h-(m_2-1)k} \xi^h h! k!} \\ &\times \frac{(-\sigma)_h (-\lambda_1)^h (\lambda \alpha)^{n-m_1h-m_2k} \lambda_2 k}{(\beta)^{m_1h+m_2k} (n-m_1h-m_2k)!} \\ &\times \int_0^\infty \int_0^\infty x^{D+eg_1(n-m_1h-m_2k)} y^{E-m_1g_2h-m_2g_2k} F(x+y) dx dy \end{aligned} \quad \dots (3.1)$$

From [3]

$$\times \int_0^\infty \int_0^\infty x^a y^b F(x+y) dx dy = \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+1)} \int_0^\infty z^{a+b+1} F(z) dz \quad \dots (3.2)$$

Then, we immediately get

$$I = \xi^\sigma \sum_{h=0}^\infty \sum_{k=0}^\infty \frac{[(a_p)]_{n-m_1h-(m_2-1)k} [(b_u)]_{n-m_1h-m_2k} (b')_k}{[(c_q)]_{n-m_1h-(m_2-1)k} \xi^h h! k!}$$

$$\begin{aligned} & \times \frac{(-\sigma)_h (-\lambda_1)^h (\lambda\alpha)^n \lambda_2^k \Gamma(D+1+neg_1-eg_1m_1h-eg_1m_2k)}{(\lambda\alpha\beta)^{m_1h+m_2k} (n-m_1h-m_2k)!} \\ & \times \frac{\Gamma(E+1-m_1g_2h-m_2g_2k)}{\Gamma(D+E+2+neg_1(eg_1+g_2)m_1h-(eg_1+g_2)m_2k)} \\ & \times \int_0^\infty F(z) z^{D+E+1+neg_1-(eg_1+g_2)m_1h-(eg_1+g_2)m_2k} dz \end{aligned} \quad \dots (3.3)$$

Which, for brevity, can be thrown into the form,

$$I = \sum_{h=0}^{\left[\frac{n}{m_1}\right]} \sum_{k=0}^{\left[\frac{n}{m_2}\right]} A_n(D, E) \int_0^\infty F(z) z^{D+E+1+ng_1+(eg_1+g_2)m_1h-(eg_1+g_2)m_2h} dz \quad \dots (3.4)$$

$$A_n(D, E) = \frac{\left[(a_p)\right]_{n-m_1h-(m_2-1)k} \left[(b_u)\right]_{n-m_1h-m_2k} (b')_k \xi^\sigma}{\left[(c_q)\right]_{n-m_1h-(m_2-1)k} \xi^h h! k!}$$

Where

$$\begin{aligned} & \times \frac{(-\sigma)_h (-\lambda_1)^h (\lambda\alpha)^n \lambda_2^k \Gamma(D+1+neg_1-eg_1m_1h-eg_1m_2k)}{(\lambda\alpha\beta)^{m_1h+m_2k} (n-m_1h-m_2k)!} \\ & \times \frac{\Gamma(E+1-m_1g_2h-m_2g_2k)}{\Gamma(D+E+2+neg_1(eg_1+g_2)m_1h-(eg_1+g_2)m_2k)} \quad \dots (3.5) \end{aligned}$$

Numerous interesting results, from (3.4), may be achieve on setting different functions for $F(z)$. Some of them are given below

Relation 1: On making the substitution

$$F(z) = e^{-B(x+y)} G_{E,F}^{e,f} \left(D(x+y) \left| \begin{matrix} (P_E) \\ (R_F) \end{matrix} \right. \right)$$

In (3.4) and then integrating the right hand side with the help of [4], we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty x^D y^E e^{-B(x+y)} G_{E,F}^{e,f} \left(D(x+y) \left| \begin{matrix} (P_E) \\ (R_F) \end{matrix} \right. \right) R_n(\alpha x^{g_1}, \beta y^{g_2}) dx dy \\ & = \sum_{h=0}^{\left[\frac{n}{m_1}\right]} \sum_{k=0}^{\left[\frac{n}{m_2}\right]} A_n(D, E) B^{1+D+E+neg_1-(eg_1+g_2)m_1h-(eg_1+g_2)m_2k} \\ & \times G_{E+1,F}^{e,f+1} \left(\frac{D}{E} \left| \begin{matrix} m_2(eg_1+g_2)k + (eg_1+g_2)h - neg_1 - E - D - 1, \\ (R_F) \end{matrix} \right. \right) \begin{matrix} (P_E) \end{matrix} \right) \quad \dots (3.6) \end{aligned}$$

Provided that

$$E+F < 2(e+f), |\arg D| < \left(e+f - \frac{D}{2} - \frac{F}{2}\right) \pi, |\arg B| < \frac{\pi}{2}$$

and $\operatorname{Re}(R_j + D + E - ng_2 + 2) > 0, j = 1, 2, \dots, e$

$$F(z) = G_{E,F}^{e,f} \left(z \left| \begin{matrix} (P_E) \\ (R_F) \end{matrix} \right. \right)$$

Relation 2: On setting

In (3.4) and then using ^[4], we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty x^D y^E G_{E,F}^{e,f} \left((x+y) \left| \begin{matrix} (P_E) \\ (R_F) \end{matrix} \right. \right) R_n(\alpha x^{g_1}, \beta y^{g_2}) dx dy \\ &= \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} A_n(D, E) \frac{\prod_{j=1}^e \Gamma\{(R_j) + D + E + 2 + neg_1 - (eg_1 + g_2)m_1h - (eg_1 + g_2)m_2k\}}{\prod_{j=e+1}^f \Gamma\{-(R_j) - D - E - 1 - neg_1 + (eg_1 + g_2)m_1h + (eg_1 + g_2)m_2k\}} \\ & \times \frac{\sum_{j=1}^f \Gamma\{-(P_j) - D - E - 1 - neg_1 + (eg_1 + g_2)m_1h + (eg_1 + g_2)m_2k\}}{\sum_{j=f+1}^E \Gamma\{-(P_j) + D + E + 2 + neg_1 - (eg_1 + g_2)m_1h - (eg_1 + g_2)m_2k\}} \\ &= \xi^\sigma \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} \frac{\left[(a_p) \right]_{n-m_1h-(m_2-1)k} \left[(b_u) \right]_{n-m_1h-m_2k} (b')_k (\lambda\alpha)^n}{\left[(c_q) \right]_{n-m_1h-(m_2-1)k} \xi^h h! k! (\lambda\alpha\beta)^{m_1h+m_2k}} \\ & \times \frac{(-\sigma)_h (-\lambda_1)^h \lambda_2^k \Gamma(D+1+neg_1-eg_1m_1h-eg_1m_2k)}{(n-m_1h-m_2k)! \Gamma(D+E+2+neg_1-(eg_1+g_2)m_1h-(eg_1+g_2)m_2k)} \\ & \times \frac{\Gamma(E+1-m_1g_1h-m_2g_2k) \Gamma\{(Re)+D+E+2+neg_1-(eg_1+g_2)m_1h+(eg_1+g_2)m_2k\}}{\Gamma\{-(R_F)-(e)-D-E-1-neg_1+(eg_1+g_2)m_1h+(eg_1+g_2)m_2k\}} \\ & \times \frac{\Gamma\{-(P_F)-D-E-1-neg_1+(eg_1+g_2)m_1h+(eg_1+g_2)m_2k\}}{\Gamma\{-(P_E):(f)+D+E+2+neg_1-(eg_1+g_2)m_1h-(eg_1+g_2)m_2k\}} \dots (3.7) \\ &= \frac{\left[(a_p) \right]_n \left[(b_u) \right]_n \xi^\sigma (\lambda\alpha)^n \Gamma(D+1+neg_1) \Gamma(E+1) \Gamma\{(Re)+D+E+2+neg_1\}}{\left[(c_q) \right]_n n! \Gamma(D+E+2+neg_1) \Gamma(-P_E):(f)+D+E+2+neg_1} \\ & \times \frac{\Gamma\{(P)-D-E-1-neg_1\}}{\Gamma\{(R_F):(f)-B-E-1-neg_1\}} \\ & \times \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \frac{\Delta_h[m_1; -n] \Delta_h[m_1; 1-(c_q)-n]}{\Delta_h[m_1; 1-(a_p)-n] \Delta_h[m_1; 1-(b_u)-n]} \\ & \times \frac{\Delta_h[m_1eg_1; -D-E-1-neg_1] \Delta_h[m_1eg_1; -D-E-1-neg_1-(P)-n]}{\Delta_h[m_1eg_1; -D-neg_1] \Delta_h[m_1eg_1; -(R_F)-D-E-1-neg_1]} \\ & \times \frac{\Delta_h[m_1eg_1; -D-E-1-neg_1(P_E):(f)] (-1)^{em_1g_1(E-F-e)} (m_1eg_1)^{meg_1(E+F+D)}}{\Delta_h[m_1eg_1; -D-E-neg_1(R_F):(c)] \Delta_h[em_1g_1; -E] (\lambda\alpha\beta)^{m_1h}} \end{aligned}$$

$$\begin{aligned}
& \times \frac{(-m)^{m_1(2-p+1+q)h}}{h!} \\
& \times F \left[\begin{array}{l} \Delta(m_2; -n+m_1h), \Delta(m_2-1; 1-(c_q)-n+m_1h), b', \\ \Delta(m_2 \overline{eg_1+g_2}; -D-E-1-neg_1+(eg_1+g_2)^{m_1}(R_f), \\ \Delta(m_2 \overline{eg_1+g_2}; -D-E-1-neg_1+(eg_1+g_2), (P_E); \\ \Delta(\overline{eg_1+g_2} m_2; -D-E-1-neg_1+(eg_1+g_2)m_1); \\ \lambda_2(-1)^{m_2(eg_1+g_2)(E-f)} (-m_2)^{m_2(1+u)} \{-(m_2-1)\}^{(m_2-1)(q-p)} \\ \hline g_2^{m_2g_2} (\lambda x^e y)^{m_2} (\lambda \alpha \beta)^{m_2} (g_1 e + g_2)^{m_2} (g_1 e + g_2)^{(F-E-1)} (eg_1)^{eg_1 m_2} \\ \Delta(m_2-1; 1-(a_p)-n+m_1h), \Delta(m_2; 1-(b_u)-n+m_1h), \\ \Delta(eg_1 m_2; -E), \Delta(eg_1 m_2; -D-neg_1+eg_1 h), \\ \Delta(\overline{eg_1+g_2} m_2; -D-E-2-neg_1+(eg_1+g_2)h-(R_e)); \\ \Delta(\overline{eg_1+g_2} m_2; -D-E-1-neg_1+(eg_1+g_2)h-(P_f)); \end{array} \right] \dots (3.8)
\end{aligned}$$

Where $0 \leq e \leq F$, $0 \leq f \leq E$, $(E+F) < 2(e+f)$, $D+E+ne+(Pf)+1$, $E+D+ng_1+(Rf;e)+1$, should not be positive integer and $\text{Re}(Pj-ng_2)+1 < \text{Re}(D+E) < \min. \text{Re } Rj+ng_1+E$; $1 \leq j \leq f$, $1 \leq j \leq e$; similarly, (3.6) can be also thrown into the form;

$$\begin{aligned}
& = \frac{\left[(a_p) \right]_n \left[(b_u) \right]_n \xi^\sigma (\lambda \alpha)^n \Gamma(D+1+neg_1) \Gamma(E+1)}{\left[(c_q) \right]_n n! \Gamma(D+E+2+neg_1)} \\
& \times \frac{\Gamma\{(Re)+D+E+2+neg_1\} \Gamma\{-(P_f)-D-E-1-neg_1\}}{\Gamma\{(-P_E):(e)^{+D+E+2+neg_1}\} \Gamma\{(R_f):(f)^{-D-E-1-neg_1}\}} \\
& \times \sum_{k=0}^{\left[\frac{n}{m_2} \right]} \frac{\Delta_k[m_2; -n+m_2h] \Delta_k[m_2; 1-(C_q)-n+m_1h]}{\Delta_k[m_2; 1-(a_p)-n+m_1h] \Delta_k[m_2; 1-(b_u)-n+m_1h]} \\
& \times \frac{\Delta_k\{\overline{eg_1+g_2} m_2; -D-E-1-neg_1+(eg_1+g_2)m_1-(R_j)\}}{\Delta_k[eg_1 m_2; -E] \Delta_k[eg_1 m_2; -D-neg_1+eg_1 h]} \\
& \times \frac{\Delta_k\{m_2 \overline{eg_1+g_2}; -D-E-1-neg_1+(eg_1+g_2)^h-(P_E)\}}{\Delta_k\{eg_1+g_2 m_2; -D-E-2-neg_1+(eg_1+g_2)^h-(R_e)\}} \\
& \times \frac{\Delta_k\{\overline{eg_1+g_2} m_2; -D-E-1-neg_1+(eg_1+g_2)^h\} \lambda_2^k}{\Delta_k\{eg_1+g_2 m_2; -D-E-1-neg_1+(eg_1+g_2)^h-(P_f)\} g_2^{m_2g_2k}} \\
& \times \frac{(-1)^{m_2(eg_1+g_2)(E-f)k} (-m_2)^{m_2(1-u)k} \{(-m_2-1)\}^{(m_2-1)(q-p)k}}{(\lambda x^e y)^{m_2k} (\lambda_2 \alpha \beta)^{m_2k} (g_1 e + g_2)^{e_2(eg_1+g_2)(E-F-1)} (eg_1)^{eg_2 m_2 k}}
\end{aligned}$$

$$\times F \left[\begin{array}{c} \Delta(m_1; -n) \Delta(m_1; 1 - (C_q) - n) \Delta(m_1 e g_1; -D - E - 1 - n e g_1), \\ \Delta(m_1 e g_1; -D - E - 1 - n e g_1 - (P_i)), \\ \Delta(m_1 g_1 e; -D - E - 1 - n e g_1 - (P_E)_{(f)}), \\ \frac{\lambda_1 (-1)^{e m_1 g_1 (E - f - e)} (m e g_1)^{m_1 e g_1 (E - F + 1)} (-m_1)^{m_1 (q - 1 + u)}}{(\lambda \alpha)^{m_1}} \\ \Delta(m_1; 1 - (a_p) - n), \Delta(m_1; 1 - (b_u) - n), \Delta(e m_1 g_1; -E), \\ \Delta(m_1 e g_1; -D - n e g_1), \Delta(m_1 e g_1; -(R_e) - D - E - 1 - n e g_1), \\ \Delta(e m_1 g_1; -D - E - 1 - n e g_1 - (R_F)_{(e)}); \end{array} \right] \dots (3.9)$$

Relation 3: On making the substitution

$$F(z) = (1+z)^{-N} P_{N-1} \left(\frac{1-z}{1+z} \right)$$

In (3.4) and then [4], we achieve for ($m_2 > 1$)

$$\begin{aligned} & \int_0^\infty \int_0^\infty x^D y^E (1+x+y)^{-N} P_{N-1} \left(\frac{1-x-y}{1+x+y} \right) R_n(ax^{g_1}, by^{g_2}) dx dy \\ &= \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} A_n(D, E) \frac{\Gamma\{D+E+2+n e g_1 + (e g_1 + g_2)^{m_1 h} - m_2(e g_1 + g_2)k\}}{\Gamma\{-D-E-n e g_1 + (e g_1 + g_2)^{m_1 h + m_2} + m_2(e g_1 + g_2)k\}} \\ & \times \frac{\Gamma\{N-D-E-1-n e g_1 + (e g_1 + g_2)m_1 h + (e g_1 + g_2)m_2 k\}^2}{\Gamma(N)^2} \\ &= \xi^\sigma \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} \frac{\left[(a_p) \right]_{n-m_1 h - (m_2-1)k} \left[(b_u) \right]_{n-m_1 h - m_2 k} (b')_k (\lambda \alpha)^n}{\left[(c_q) \right]_{n-m_1 h - (m_2-1)k} \xi^h h! k! (\lambda \alpha \beta)^{m_1 h + m_2 k}} \\ & \times \frac{(-\sigma)_h (-\lambda_1)^h \lambda_2^k \Gamma(D+1+n e g_1 - e g_1 m_1 h - e g_1 m_2 k)}{(n-m_1 h - m_2 k)! \Gamma(D+E+2+n e g_1 - (e g_1 + g_2)m_1 h - (e g_1 + g_2)m_2 k)} \\ & \times \frac{\Gamma(E+1-m_1 g_2 h - m_2 g_2 k) \Gamma(D+E+2+n e g_1 - (e g_1 + g_2)m_1 h (e g_1 + g_2)m_2 k)}{\Gamma(-D-E-1-n e g_1 + (e g_1 + g_2)m_1 h + (e g_1 + g_2)m_2 k)} \\ & \times \frac{\Gamma\{N-D-E-2-n e g_1 + (e g_1 + g_2)m_1 h + (e g_1 + g_2)m_2 k\}}{\Gamma(N)^2} \\ &= \frac{\left[(a_p) \right]_n \left[(b_u) \right]_n \xi^\sigma (\lambda \alpha)^n \Gamma(D+1+n e g_1) \Gamma(E+1)}{\left[(c_q) \right]_n n! \Gamma(D+E+2+n e g_1) \Gamma(-D-E-1-n e g_1)} \\ & \times \frac{\Gamma\{D+E+2+n e g_1\} \Gamma_{**}(N-D-E-2-n e g_1)}{\Gamma_{**} N} \\ & \times \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \frac{\Delta_h[m_1; -n] \Delta_h[m_1; 1 - (C_q) - n] \Delta_h[em, h; N-D-E-2-n e g_1]}{\Delta_h[m_1; 1 - (a_p) - n] \Delta_h[m_1; 1 - (b_u) - n] (\lambda \alpha \beta)^{m_1 h} h!} \end{aligned}$$

$$\begin{aligned}
& \times \frac{(-h)^{eg_1 m_1 (N)} (-m_1)^{m_1 (q-p+1+m)h}}{\Delta h [eg_1 h; -D - neg_1]} \\
& \times F \left[\begin{array}{c} \Delta(m_2; -n + m_1 h), \Delta(m_2 - 1; 1 - (C_q) - n + m_1 h), b', \\ \Delta^2(m_2 \overline{g_1 e + g_2}; N - D - E - 2 - neg_1 + (eg_1 + g_2)m_1), \\ \lambda_2(-m_2)^{m_2(1-u)} \left\{ -(m_2 - 1) \right\}^{(m_2 - 1)(q-p)} (-g_1 e - g_2)^{m_2(eg_1 + g_2)} \\ \hline (\lambda \alpha \beta)^{m_2} (eg_1)^{eg_1 m_2} g_2^{m_2 g_2} \\ \Delta(m_2 - 1; 1 - (a_p) - n + m_1 h), \Delta(m_2; 1 - (b_u) - n + m_1 h), \\ \Delta(eg_1 m_2; -E) \Delta(eg_1 m_2; -D - neg_1 + (eg_1 + g_2)m_1), \\ \Delta(m_2 \overline{eg_1 + g_2}; -D - E - 1 - neg_1 + \overline{eg_1 + g_2} m_1); \end{array} \right] \dots (3.10)
\end{aligned}$$

Where $\text{Re } neg_1 - 2 < \text{Re}(D + E) < \text{Re}(N + neg_2) - 2$ and $(c + d + ne3g_1 - N + L)$ is a non-positive integer.

Relation 4: On using (3.4) and [4], we arrive at the result (for $m_2 > 1$)

$$\begin{aligned}
& \int_0^\infty \int_0^\infty x^D y^E \left\{ B^2 (x+y)^2 \right\}^{-\frac{1}{2}} \left[\left\{ B^2 + (x+y)^2 \right\}^{\frac{1}{2}} + (x+y) \right]^{D'} \\
& \times R_n(ax^{2g_1}, by^{2g_2}) dx dy \\
& = \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} A_n(D, E) \frac{B^{D'+D+E+1+2neg_1} 4^{(eg_1+g_2)m_1 h + (eg_1+g_2)m_2 k}}{2^{D+E+2+2neg_1} B^{2(eg_1+g_2)m_1 h + 2(eg_1+g_2)m_2 k}} \\
& \times \frac{\Gamma\{D+E+2+2neg_1 - 2(eg_1+g_2)m_1 g_1 - 2(eg_1+g_2)m_2 k\}}{\Gamma\left\{\frac{D+E+3-D+2neg_1}{2} - (eg_1+g_2)m_1 g_1 - (eg_1+g_2)m_2 k\right\}} \\
& \times \frac{\Gamma\left(\frac{-D-E-D-1-2neg_1}{2} - (eg_1+g_2)m_1 h - (eg_1+g_2)m_2 k\right)}{1} \\
& = \xi^\sigma \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} \frac{\left[(a_p) \right]_{n-m_1 h(m_2-1)k} \left[(b_u) \right]_{n-m_1 h-m_2 k} (b')_k}{\left[(C_q) \right]_{n-m_1 h(m_2-1)k} \xi^h h! s! (\lambda \alpha \beta)^{m_1 h + m_2 k}} \\
& \times \frac{(\lambda \alpha)^n (\sigma)_k \lambda_1^h \Gamma(D+1+neg_1 - eg_1 h m_1 - eg_2 m_2 k)}{(n-m_1 h - m_2 k)! \Gamma(D+E+2+neg_1 - (eg_1+g_2)m_1 h - (eg_2+g_2)m_2 k)} \\
& \times \frac{\Gamma(E+1-m_1 g_1 h - m_2 g_2 k) B^{D'+D+E+1+neg_1}}{2^{D+E+2+2neg_1} B^{2(eg_1+g_2)m_1 h + 2(eg_1+g_2)m_2 k}} \\
& \times \frac{4^{(eg_1+g_2)m_1 h + (eg_1+g_2)m_2 k} \Gamma\left\{\frac{-D-E-D'-2ng_1 + (eg_1+g_2)m_1 h + (eg_1+g_2)m_2 k}{2}\right\}}{\Gamma\left\{\frac{D+E+3-D'+2neg_1}{2} - (eg_1+g_2)m_1 h - (eg_1+g_2)m_1 k\right\}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\left[(a_p) \right]_n \left[(b_u) \right]_n \xi^\sigma (\lambda \alpha)^n \Gamma(D+1+neg_1) \Gamma(E+1)}{\left[(c_q) \right]_n n! \Gamma(D+E+2+neg_1) \Gamma\left\{ \frac{1}{2}(D+E+3-D'+2ne_3g_1) \right\}} \\
&\times \frac{\Gamma\left\{ \frac{1}{2}(D+E+D'+1+2neg_1) \right\} B^{D'+D+E+2neg_1} 4^{(eg_1+g_2)m_1h}}{2^{D+E+2+2neg_1} B^{2(eg_1+g_2)m_1h}} \\
&\times \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \frac{\Delta_h[m_1; -n] \Delta_h[m_1; 1-(c_q)-n] (-\sigma_1)_h \lambda_1^h}{\Delta_h[m_1; 1-(a_p)-n] \Delta_h[m_1; 1-(b_u)-n] \xi^h h!} \\
&\times \frac{\Delta_h[m_1 eg_1; -D-E-1-neg_1] (-m_1)^{m_1(q-p+1+u)^h}}{\Delta_h[m_1 eg_1; -D-neg_1] \Delta_h[em_1 g_1; -E] (\lambda \alpha \beta)^{m_1 h}} \\
&\times \frac{\Delta_h \left[\overline{eg_1 + g_2 m_1}; -\frac{D}{2} - \frac{E}{2} \pm \frac{D'}{2} - \frac{1}{2} - 2neg_1 \right]}{\Delta_h \left[\overline{eg_1 + g_2 m_1}; \frac{D}{2} + \frac{E}{2} + \frac{D'}{2} - \frac{1}{2} + neg_1 \right]} \\
&\times F \left[\begin{array}{c} \Delta(m_2; -n+m_1 h), \Delta(m_2-1; 1-(C_q)-n+m_1 h), b', \\ \Delta(eg_1+g_2)m_2; -\frac{D}{2} - \frac{E}{2} \pm \frac{D'}{2} - \frac{1}{2} - neg_1(eg_1+g_2)m_1), \\ \Delta(\overline{eg_1+g_2})m_2; -D-E \pm 1 - neg_1(eg_1+g_2)m_1), \\ \lambda_2(-m_2)^{m_1(1-u)} \{-(m_2-1)\}^{(m_2-1)(q-p)} \{-(eg_1+g_2)\}^{(eg_1+g_2)m_2} 4^{(eg_1+g_2)m_2} \\ (\lambda \alpha \beta)^{m_2} \beta^{2(eg_1+g_2)m_2} (g_1 e)^{eg_1 m_2} g_2^{m_2 g_2} \\ \Delta(m_2-1; 1-(a_p)-n+m_1 h), \Delta(m_2; 1-(b_u)-n+m_1 h), \\ \Delta(eg_1 m_1; -E), \Delta(eg_1 m_1; -D-neg_1+eg_1 h), \\ \Delta\left(\overline{eg_1+g_2 m_2}; \frac{-D-E-D'+2neg_1}{2} + (eg_1+g_2)h\right); \end{array} \right] \dots (3.11)
\end{aligned}$$

Where $\text{Re } B > 0$, $-z - \text{Re } 2ne_3g_1 < \text{Re}(D+E) < -\text{Re}(D'-1-\text{Re } 2neg_1)$ and $2neg_1 + D + E + 1$ is non-Positive integer.

Relation 5: For $m_2 > 1$

On setting

$$F(z) = e^{-(x+y)} L_n^{(f)}(x+y)$$

In (3.4) and then using [4], we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^D y^E e^{-(x+y)} L_n^{(f)}(x+y) R_n(\alpha x^{g_1}, by^{g_2}) dx dy \\
&= \sum_{h=0}^{\left[\frac{n}{m_1} \right]} \sum_{k=0}^{\left[\frac{n}{m_2} \right]} A_n(D, E) \frac{\Gamma\{f-D-E-1-neg_1+N+(eg_1+g_2)m_1h+(eg_1+g_2)m_1k\}}{\Gamma\{f-D-E-1-neg_1+(eg_1+g_2)m_1h+(eg_1+g_2)m_1k\}} \\
&\times \frac{\Gamma D+E+2+neg_1+(eg_1+g_2)m_1h+(eg_1+g_2)m_2k}{N!}
\end{aligned}$$

$$= \frac{\left[(a_p) \right]_n \left[(b_u) \right]_n \xi^\sigma (\lambda \alpha)^n \Gamma(D+1+neg_1) \Gamma(E+1) \Gamma(f-D-E-1-neg_1+N)}{\left[(c_q) \right]_n n! \Gamma(D+E+2+neg_1) N! \Gamma(f-D-E-1-neg_1)}$$

$$\sum_{h=0}^{\left[\frac{n}{m_1} \right]} \frac{\Delta_h[m_1; -n] \Delta_h[m_1; 1-(c_q)-n] \Delta_h(eg_1 m_1; -D-E-1-neg_1)}{\Delta_h[m_1; 1-(a_p)-n] \Delta_h[m_1; 1-(b_u)-n] \Delta_h(m_1 eg_1; -D-neg_1)}$$

$$\times \frac{\Delta_h[eg_1 m_1; -1-D-E-1+N-neg_1] \lambda_1^h (-m_1)^{m_1(q-p+1+u)^h}}{\Delta_h[em_1 g_1; -E] \Delta_h[eg_1 m_1; f-D-E-1-neg_1] \xi^h h! (\lambda \alpha \beta)^{m_1 h}}$$

$$\times F \left[\begin{array}{l} \Delta(m_2; -n+m_1 h), \Delta(m_2-1; 1-(C_q)-n+m_1 h), b', \\ \Delta(\overline{eg_1+g_2} m_2; -D-E-1-neg_1(eg_1+g_2)m_1 h), \\ \Delta(\overline{eg_1+g_2}); f-D-E-1+N-neg_1+(eg_1+g_2)m_1 h, \\ \frac{\lambda_2 (-m_2)^{m_1(1-u)} \{-(m_2-1)\}^{(m_2-1)(q-p)}}{(\lambda \alpha \beta)^{m_2} (eg_1)^{eg_1} g_2^{g_2 m_2} (eg_1+g_2)^{m_2(eg_1+g_2)}}, \\ \Delta(m_2-1; 1-(a_p)-n+m_1 h), \Delta(m_2; 1-(b_u)-n+m_1 h), \\ \Delta(eg_2 m_2; -E), \Delta(eg_1 m_2; -D-neg_1-(eg_1+g_2)h), \\ \Delta((eg_1+g_2)m_2; f-D-E-1-neg_1+(eg_1+g_2)m_1 h); \end{array} \right] \dots (3.12)$$

Provided that $\text{Re}(D+E-neg_2+2) > 0$, $+D+E+neg_1+N-f$ and $(D+E+neg_1-N+1)$ are non-positive ineteger and $L_n^{(f)}(x+y)$ is the well known Leguerre polynomial of order n .

Conclusion

In this paper, we have obtained many interesting new results for the polynomial set $Rn(x, y)$ followed by important and interesting particular cases. These are of at most important for scientist, Engineers and physical Sciences, because these occurs in the solution of statcal and physical problems.

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