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Estimation of parameters μ_2 and σ_2 of extended morgenstern type bivariate bilal distribution by concomitants of order statistics and ranked set sampling

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Abstract

In this article we have derived the best linear unbiased estimator (BLUE) of the parameters μ_2 and σ_2 involved in the six parameter Extended Morgenstern type bivariate Bilal distribution (EMTBBD) by ranked set sampling (RSS). The efficiency comparison of the proposed estimators with that of the estimators using concomitants of order statistics is also made.

Keywords and Phrases: Morgenstern family of distributions, extended morgenstern type bivariate bilal distribution, best linear unbiased estimation, concomitants of order statistics, concomitants of ranked set sampling

1. Introduction

The concept of ranked set sampling was first introduced by McIntyre(1952) as a process of improving the precision of the sample mean as an estimator of the population mean. Ranked set sampling as described in McIntyre (1952) is applicable whenever ranking of a set of sampling units can be easily by judgment method, for details, see, Chen *et al.*, (2004) ^[4]. The process of ranked set sampling involves selecting n sets of units, each of size n , and ordering the units of each of the set by judgment method or by using values obtained on the units by a relatively inexpensive method, without making actual measurement on the units. Then the unit ranked as one from the first set is actually measured, the unit ranked as two from the second set is measured. The process continues in this way until the unit ranked as n from the set is measured. Then the observations resulting out of actual measurements made on the units chosen as described in the above Process is known as ranked set sample.

Ranking by a judgment method is not recommendable in case there is ambiguity in discriminating the rank of one unit with another (for example, ranking based on intensity of yellowish color of newborn babies, as mentioned in Chen *et al.*, (2004, p.7)) ^[4], become difficult, as we known that probability of ties in this situation is not zero as the characteristic under consideration is not continuous). In certain situation, one may prefer exact measurement of some easily measurable variable associated with the study variable rather than ranking the units by a crude judgment method. Suppose the variable of interest say, Y , is difficult or much expensive to measure, but an auxiliary variable X correlated with Y is readily measurable and can be ordered exactly. In this case as an alternative to McIntyre's (1952) method of ranked set sampling, Stokes (1977) ^[17] used an auxiliary variable for the ranking of the sampling units. Stokes (1977) ^[17] using auxiliary variate is as follows: Choose independent units, arrange them randomly into n sets each with n units and observe the value of the auxiliary variable X on each of these units. In the first set, that units for which the measurement on the auxiliary variable is the smallest is chosen. In the second set, that units for which the measurement on the auxiliary variable is the second smallest is chosen. The

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procedure is repeated until in the last set, that unit for which the measurement on the auxiliary variable is the largest is chosen. The measurements on the Y variate of the resulting new set of n units chosen by the above method gives a ranked set sample as suggested by Stokes(1977) ^[17]. If $X_{(r)r}$ is the observation measured on auxiliary variable X from the unit chosen from the r^{th} set then we write $Y_{[r]r}$ to denote the corresponding measurement made on the study variable Y on this unit so that $Y_{[r]r}$, $r = 1, 2, \dots, n$ form the ranked set sampling. Clearly $Y_{[r]r}$ is the concomitants of r^{th} order statistic arising from the r^{th} sample. By using the expression given by Yang(1977) for the mean and variance of concomitants of order statistics, we have the following . For $1 \leq r \leq n$,

$$E[Y_{[r]r}] = E[E(Y / X_{(r)r})] \quad (1.1)$$

and

$$Var[Y_{[r]r}] = Var[E(Y / X_{(r)r})] + E[Var(Y / X_{(r)r})] \quad (1.2)$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ for $r \neq s$ are drawn from two independent samples we have,

$$Cov[Y_{[r]r}, Y_{[s]s}] = 0, \quad r \neq s. \quad (1.3)$$

Stokes (1977) ^[17] has suggested the ranked set sample mean as an estimator for the mean of the study variable Y , when an auxiliary variable X is used for ranking the sample units, under the assumption that (X, Y) follows a bivariate normal distribution. Barnett and Moore (1997) ^[2] have improved the estimator of Stokes (1977) ^[17] by deriving the Best linear unbiased estimator (BLUE) of the mean of the study variate Y , based on ranked set sample obtained on the study variate Y .

1.1 Morgenstern family of distributions.

The most important aspect to be considered in a modeling problem is regarding the flexibility of the chosen family, in the sense that it should contain a wide variety of models capable of representing any data situation. Another consideration is the sort of prior information available in the choice of the model. In problems involving several random variables, the analyst may make reasonable assumption about the marginal distributions. Then the question is to construct a joint distribution function with a set of given marginal. The Morgenstern family of distributions assumes importance in such contexts as a highly flexible system. Let X and Y be two random variables with cdf's given by $F_X(x)$ and $F_Y(y)$ respectively with pdf's $f_X(x)$ and $f_Y(y)$ and jointly distributed with cdf $F(x, y)$ given by (see, Kotz *et al.*, (2000), p.52).

$$F(x, y) = F_X(x)F_Y(y)\{1 + \alpha(1 - F_X(x))(1 - F_Y(y))\}, \quad -1 \leq \alpha \leq 1. \quad (1.4)$$

Then the family of distributions having the above form of cdf is called the Morgenstern family of distributions (MFD). The system provides a very general expression of a bivariate distribution from which member can be derived by substituting expression of any desired set of marginal distributions. The joint pdf corresponding to the cdf defined in (1.4) is given by,

$$f(x, y) = f_X(x)f_Y(y)\{1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))\}, \quad -1 \leq \alpha \leq 1. \quad (1.5)$$

The parameter α is considered as the association parameter, the two random variables are independent when the value of α is

zero and the maximum correlation attained between X and Y is $\frac{\alpha}{3}$. For suitable choice of $F_X(x)$ and $F_Y(y)$, the MFD is

primarily useful as an alternative to the standard bivariate normal distribution (See, Conway (1983) ^[5], p.28-31). The distribution theory of concomitance of order statistics from MFD are discussed in Scaria and Nair(1999) ^[16].

Chako and Thomas (2008) derived the BLUE of a parameter of a three parameter Morgenstern type bivariate exponential distribution using concomitance of order statistics. Sajeevkumar and Sumi(2023) ^[15] derived the BLUE's of the parameters μ_2

and σ_2 of a five parameter Morgenstern type bivariate exponential distribution (MTBED) using concomitants of ranked set sampling. Irshad *et al.* (2019) ^[7] estimate a parameter of a three parameter Morgenstern type bivariate Lindly distribution

(MTBLD) using ranked set sampling. ¹and Priya(2026a) ^[11] derived the BLUE's of the parameters μ_2 and σ_2 of a five parameter

Morgenstern type bivariate Lindly distribution (MTBLD) using ranked set sampling. Arun *et al.* (2023) ^[1] derived the BLUE of a parameter of a three parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using concomitance of

order statistics. Sajeevkumar and Priya (2026b) ^[11, 13, 13, 14] estimate the BLUE's of the parameters μ_2 and σ_2 of a five parameter

Morgenstern type bivariate moment exponential distribution (MTBMED) using Ranked set sampling. Irshad *et al.* (2021) ^[6]

estimated a parameter of a four parameter extended morgenstern type bivariate Lindley distribution (EMTBLD) using

concomitants of order statistics. Sajeevkumar and Priya (2026c) ^[11, 12, 13, 14] estimated the parameters μ_2 and σ_2 of a six

parameter Extended Morgenstern type bivariate Lindly distribution (EMTBLD) using concomitants of ranked set sampling.

Irshad *et al.* (2023) estimated a parameter of three parameter Morgenstern type bivariate Bilal distribution (MTBBD) using ranked

set sampling. Sajeevkumar and Priya(2026d) ^[11, 12, 13, 14] estimated the parameters μ_2 and σ_2 involved in a five parameter

Morgenstern type bivariate Bilal distribution (MTBBD) using ranked set sampling. Irshad *et al.* (2022) ^[8] estimated a parameter of

a four parameter extended Morgenstern type bilal distribution by concomitants of order statistics and ranked set sampling. In this work our aim is to estimate the parameters μ_2 and σ_2 involved in a six parameter extended Morgenstern type bilal distribution by concomitants of order statistics and ranked set sampling. This work is an extension of the work done by Irshad et.al (2022) [8].

Estimation of the parameters μ_2 and σ_2 involved in embdd using RSS.

An important member of Morgenstern family of distribution is (MFD) is Extended Morgenstern type bivariate Bilal distribution (EMTBDD). The bivariate random variable (X,Y) is said to have six parameter EMTBDD, if its pdf is given by

$$f(x,y) = \frac{36}{\sigma_1 \sigma_2} \left(1 - e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} \right) e^{-2\left(\frac{x-\mu_1}{\sigma_1}\right)} \left(1 - e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} \right) e^{-2\left(\frac{y-\mu_2}{\sigma_2}\right)} \{ 1 +$$

$$\alpha \left[2 \left(3 - 2e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} \right) e^{-2\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1 \right] \left[2 \left(3 - 2e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} \right) e^{-2\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1 \right] +$$

$$\beta \left[3 \left(3 - 2e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} \right) e^{-2\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1 \right] \left[3 \left(3 - 2e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} \right) e^{-2\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1 \right] \}$$

$$\left[\left(3 - 2e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} \right) e^{-2\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1 \right] \left[\left(3 - 2e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} \right) e^{-2\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1 \right] \}, \quad x > \mu_1, y > \mu_2, \sigma_1 > 0, \sigma_2 > 0,$$

$$|\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right]. \quad (2.1)$$

The pdf defined in (2.1) is an extension of the four parameter EMTBLD defined in Irshad et.al (2021) [6]. The joint pdf of standard EMTBLD is obtained by making the substitution

$$U = \frac{X - \mu_1}{\sigma_1} \quad \text{and} \quad V = \frac{Y - \mu_2}{\sigma_2} \quad \text{in} \quad (2.1) \quad \text{and} \quad \text{it} \quad \text{is} \quad \text{given} \quad \text{by,} \quad g(u,v) =$$

$$36e^{-2u} (1 - e^{-u}) e^{-2v} (1 - e^{-v}) \{ 1 + \alpha [2e^{-2u} (3 - 2e^{-u}) - 1] [2e^{-2v} (3 - 2e^{-v}) - 1] +$$

$$\beta [3e^{-2u} (3 - 2e^{-u}) - 1] [3e^{-2v} (3 - 2e^{-v}) - 1] [e^{-2u} (3 - 2e^{-u}) - 1] [e^{-2v} (3 - 2e^{-v}) - 1] \},$$

$$u > 0, v > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \quad (2.2)$$

Suppose X and Y are the two characteristics of interest on the units of a population and (X,Y) has a distribution with pdf given in (2.1). Suppose there are n independent random samples each of size n drawn from (2.1) and in each sample, units are ranked according to the values of the observations measured on the easily measurable variable X on the units. Though the variable Y is of primary interest, we assume that it is difficult to measure. Then to deal with the inference problem on the distribution of the variable Y we obtain a ranked set sample by measuring the value of Y on the r^{th} ordered unit of the r^{th} sample, $r=1,2,\dots,n$.

We write $Y_{[r]r}$ to denote this observation measured on Y of the r^{th} ordered unit of the r^{th} set, $r=1,2,\dots,n$. Then clearly $Y_{[r]r}$ is distributed as the concomitants of r^{th} order statistic in each sample, units are ranked according to the value of the observations measured on the easily measurable variable U on the unit. We obtain a ranked set sample by measuring the value of V on the r^{th} ordered unit of the r^{th} sample, We write $V_{[r]r}$ to denote this observation measured on V of the r^{th} ordered unit of the r^{th} set, $r=1,2,\dots,n$. Then clearly $V_{[r]r}$ is distributed as the concomitants of r^{th} order statistic of a random sample of size n arising from (2.2). Then the pdf $g_{[r]r}(v)$ of $V_{[r]r}$, $r=1,2,\dots,n$ is given by

$$g_{[r]r}(v) = 6(1 - e^{-v})e^{-2v} + \alpha \frac{(n-2r+1)}{2(n+1)} [12e^{-2v} \{1 - e^{-v}\} \{2e^{-2v}(3 - 2e^{-v}) - 1\}] +$$

$$\beta \frac{(2n-3r+1)r}{3(n+1)(n+2)} \{ [18(1 - e^{-v})e^{-2v}] [1 - e^{-2v}(3 - 2e^{-v})] [3e^{-2v}(3 - 2e^{-v}) - 1] \},$$

$$v > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \quad (2.3)$$

Using (2.3), for $1 \leq r \leq n$,

$$E[V_{[r]r}] = \frac{5}{6} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{19}{60} \right) - \beta \left(\frac{62}{315} \right) \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] = \xi_r \quad (2.4)$$

And

$$\begin{aligned} \text{Var}[V_{[r]r}] &= \frac{13}{36} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{253}{1800} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{140071}{793800} \right) - \frac{361}{3600} \alpha^2 \frac{(n-2r+1)^2}{(n+1)^2} - \\ &\frac{3844}{99225} \beta^2 \frac{(2n-3r+1)^2 r^2}{(n+1)^2 (n+2)^2} - \frac{1178}{9450} \alpha (n-2r+1) \beta \frac{(2n-3r+1)r}{(n+1)^2 (n+2)} = e_{r,r} \quad (2.5) \end{aligned}$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[V_{[r]r}, V_{[s]s}] = 0 = e_{r,s}, \quad r \neq s, \quad r, s = 1, 2, \dots, n \quad (2.6)$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ are drawn for $r \neq s$ are drawn from two independent samples.

Let $(X_i, Y_i), i=1, 2, \dots$ be a ranked set sample of size n drawn from a population with pdf defined by (2.1). Clearly we have

$X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i=1, 2, \dots, n$. Then using (2.4) to (2.6), we have for $1 \leq r \leq n$,

$$E(Y_{[r]r}) = \mu_2 + \sigma_2 \xi_r \quad (2.7)$$

$$\text{Var}[Y_{[r]r}] = \sigma_2^2 e_{r,r} \quad (2.8)$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[Y_{[r]r}, Y_{[s]s}] = 0, \quad r \neq s \quad (2.9)$$

Where ξ_r , $e_{r,r}$ and $e_{r,s}$ are defined by (2.4), (2.5) and (2.6) respectively. Clearly from (2.4) to (2.6) it follows that ξ_r , $e_{r,r}$ and $e_{r,s}$ are known constants provided α is known. If we write

$\underline{Y}_{[n]} = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$, $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$, then we can write

$$E[\underline{Y}_{[n]}] = \underline{1} \mu_2 + \underline{\xi} \sigma_2, \quad (2.10)$$

Where $\underline{1}$ is a column vector of n ones and $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$. And

$$\text{Var}[\underline{Y}_{[n]}] = E \sigma_2^2 \quad (2.11)$$

Now we derive the BLUE's of μ_2 and σ_2 involved in (2.1) and it is given in the following theorem.

Theorem 2.1:

Let $\underline{Y}_n = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be the vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of concomitants of ranked set sample arising (2.2). Then the BLUE's of the parameters μ_2^* of μ_2 and σ_2^* of σ_2 is given by

$$\mu_2^* = -\frac{\underline{\xi}' E^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}') E^{-1}}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \underline{Y}_n \quad (2.12)$$

$$\sigma_2^* = \frac{\underline{1}' E^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}') E^{-1}}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \underline{Y}_n \quad (2.13)$$

And corresponding variances are given by

$$\text{Var}(\mu_2^*) = \frac{(\underline{\xi}' E^{-1} \underline{\xi}) \sigma_2^2}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \quad (2.14)$$

and

$$\text{Var}(\sigma_2^*) = \frac{(\underline{1}' E^{-1} \underline{1}) \sigma_2^2}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \quad (2.15)$$

Proof: Let $\underline{Y}_{[n]} = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be a vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of vector of concomitants of ranked set sample arising from (2.2), then by using (2.10) and (2.11) together defines a generalized Gauss-Markov set up. Then by Generalized Gauss-Markov theorem we get the BLUE's of μ_2 and σ_2 and their corresponding variances.

Estimation of the parameters μ_2 and σ_2 involved in emtbdd using concomitants of order statistics.

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be the observations of random sample of size n drawn from (2.1). The Y component associated with r^{th} order statistic $X_{r:n}$ is defined as the concomitants of r^{th} order statistic and it is denoted by $Y_{[r:n]}, r = 1, 2, \dots, n$. Also let $(U_i, V_i), i = 1, 2, \dots, n$ be the sequence of independent observations drawn from (2.2). Let $V_{[r:n]}$ be the concomitants of r^{th} order statistic $U_{r:n}$ arising from (2.2), then the pdf of $V_{[r:n]}$ and the joint pdf of $V_{[r:n]}$ and $V_{[s:n]}$ for $1 \leq r < s \leq n$ are respectively given in (3.1) and (3.2).

$$g_{[r:n]}^*(v) = 6(1 - e^{-v})e^{-2v} + \alpha \frac{(n - 2r + 1)}{2(n + 1)} [12e^{-2v} \{1 - e^{-v}\} \{2e^{-2v}(3 - 2e^{-v}) - 1\}] +$$

$$\beta \frac{(2n - 3r + 1)r}{3(n + 1)(n + 2)} \{ [18(1 - e^{-v})e^{-2v}] [1 - e^{-2v}(3 - 2e^{-v})] [3e^{-2v}(3 - 2e^{-v}) - 1] \}$$

$$, v > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \quad (3.1)$$

and for $1 \leq r < s \leq n$

$$\begin{aligned}
f_{[r,s;n]}(v_1, v_2) &= 36e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2}) + \alpha \frac{(n-2r+1)}{2(n+1)} * \\
&\{72e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[2e^{-2v_1}(3-2e^{-v_1})-1]\} + \\
&\alpha \frac{(n-2s+1)}{2(n+1)} \{72e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[2e^{-2v_2}(3-2e^{-v_2})-1]\} + \\
&\beta \frac{(2n-3r+1)r}{3(n+1)(n+2)} \{108e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[1-e^{-v_2}][1-e^{-2v_1}(3-2e^{-v_1})][3e^{-2v_1}(3-2e^{-v_1})-1]\} + \\
&\beta \frac{(2n-3s+1)s}{3(n+1)(n+2)} \{108e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[1-e^{-2v_2}(3-2e^{-v_2})][3e^{-2v_2}(3-2e^{-v_2})-1]\} + \\
&\frac{\alpha^2}{4} \left[\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} \right] \{144e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2}) * \\
&[2e^{-2v_1}(3-2e^{-v_1})-1][2e^{-2v_2}(3-2e^{-v_2})-1]\} + \\
&\frac{\alpha\beta}{6} \left\{ -\frac{2r}{n+1} + \frac{[3r(r+1)+4r(n-s+1)]}{(n+1)(n+2)} - \frac{6r(r+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\} * \\
&\{216e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[1-e^{-2v_1}(3-2e^{-v_1})][3e^{-2v_1}(3-2e^{-v_1})-1][2e^{-2v_1}(3-2e^{-v_1})-1]\} \\
&+ \frac{\alpha\beta}{6} \left\{ -\frac{s}{n+1} + \frac{[2r(s+1)+3s(n-s+1)]}{(n+1)(n+2)} - \frac{6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\} * \\
&\{216e^{-2v_1}(1-e^{-v_1})e^{-2v_2}(1-e^{-v_2})[1-e^{-2v_2}(3-2e^{-v_2})][3e^{-2v_2}(3-2e^{-v_2})-1][2e^{-2v_1}(3-2e^{-v_1})-1]\} + \\
&\frac{\beta^2}{9} \left\{ -\frac{2r(s+1)}{(n+1)(n+2)} + \frac{[3r(r+1)(s+2)+6r(s+1)(n-s+1)]}{(n+1)(n+2)(n+3)} - \frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)} \right\} * \\
&[e^{-2v_1}(3-2e^{-v_1})-1][e^{-2v_2}(3-2e^{-v_2})-1][3e^{-2v_1}(3-2e^{-v_1})-1][3e^{-2v_2}(3-2e^{-v_2})-1]
\end{aligned}$$

$$v_1 > 0, v_2 > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \quad (3.2)$$

Using (3.1) and (3.2), we have ,

$$E[V_{[r]r}] = \frac{5}{6} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{19}{60} \right) - \beta \left(\frac{62}{315} \right) \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] = \xi_r \quad (3.3)$$

and for $1 \leq r < s \leq n$

$$Var[V_{[r]r}] = \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{253}{1800} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{140071}{793800} \right) - \frac{361}{3600} \alpha^2 \frac{(n-2r+1)^2}{(n+1)^2} -$$

$$\frac{3844}{99225} \beta^2 \frac{(2n-3r+1)^2 r^2}{(n+1)^2 (n+2)^2} - \frac{1178}{9450} \frac{\alpha(n-2r+1)\beta(2n-3r+1)r}{(n+1)^2 (n+2)} = b_{r,r} \quad (3.4)$$

Also for $1 \leq r < s \leq n$

$$\begin{aligned} \text{Cov}[V_{[r:n]}, V_{[s:n]}] &= \frac{361}{3600} \alpha^2 r \left[\frac{n-2s+1}{(n+1)^2} - \frac{(n-2s)}{(n+1)(n+2)} \right] + \frac{3844}{99225} \beta^2 \left[-\frac{2r(s+1)}{(n+1)(n+2)} + \right. \\ &\quad \left. \frac{3r(r+1)(s+2)+6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)} - \frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)} - \frac{rs(2n-3r+1)(2n-3s+1)}{(n+1)^2 (n+2)^2} \right] \\ &+ \frac{589}{9450} \alpha \beta \left[-\frac{(2r+s)}{(n+1)} + \frac{r(3r+2s+5)+(n-s+1)(4r+3s)}{(n+1)(n+2)} - \frac{6r(n-s+1)(r+s+2)}{(n+1)(n+2)(n+3)} \right] \quad (3.5) \end{aligned}$$

Let $(X_i, Y_i), i=1, 2, \dots, n$ be sample of size n from a population with pdf defined in (2.1). Clearly $X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i=1, 2, \dots, n$.

Then for $1 \leq r \leq n$,

$$E[Y_{[r:n]}] = \mu_2 + \sigma_2 \xi_r \quad (3.6)$$

$$\text{Var}[Y_{[r:n]}] = \sigma_2^2 b_{r,r} \quad (3.7)$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[Y_{[r:n]}, Y_{[s:n]}] = \sigma_2^2 b_{r,s} \quad (3.8)$$

Where $\xi_r, b_{r,r}$ and $b_{r,s}$ are respectively defined in (3.3), (3.4) and (3.5).

Let $Y_{[n]} = (Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]})'$ be the vector of concomitants of order statistics arising from (2.1). Let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $B = (b_{r,s})$ respectively the mean vector and dispersion matrix of $Y_{[n]}$. Then we can write $E[Y_{[n]}] = \underline{1}\mu_2 + \underline{\xi}\sigma_2$, where $\underline{1}$ is a column vector of n ones and $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $V[Y_{[n]}] = B\sigma_2^2$.

Now using generalized Gauss markov theorem BLUE's of μ_2 and σ_2 involved in (2.1) are given by

$$\hat{\mu}_2 = -\frac{\underline{\xi}' B^{-1} (\underline{1}\underline{\xi}' - \underline{\xi}\underline{1}) B^{-1}}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} Y_{[n]} \quad (3.9)$$

$$\hat{\sigma}_2 = \frac{\underline{1}' B^{-1} (\underline{1}\underline{\xi}' - \underline{\xi}\underline{1}) B^{-1}}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} Y_{[n]} \quad (3.10)$$

and corresponding variances are given by

$$\text{Var}(\hat{\mu}_2) = \frac{(\underline{\xi}' B^{-1} \underline{\xi}) \sigma_2^2}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} \quad (3.11)$$

$$Var(\hat{\sigma}_2) = \frac{(\underline{1}' B^{-1} \underline{1}) \sigma_2^2}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} \quad (3.12)$$

Computation section

We have evaluated the coefficients of the BLUE μ_2^* of μ_2 given in (2.12) and evaluated the coefficients of the BLUE σ_2^* of σ_2 given in (2.13) using ranked set sampling, for different values of n, α and $\beta = 0.50$, are given in Table 1. Also evaluated the coefficient of the BLUE $\hat{\mu}_2$ of μ_2 given in (3.9) and evaluated the coefficient of the BLUE $\hat{\sigma}_2$ given in (3.10) using concomitants of order statistics for different values of n, α and $\beta = 0.50$, are given in Table 2. Also we have evaluated $Var(\mu_2^*)$ given in (2.14), $Var(\sigma_2^*)$ given in (2.15), $Var(\hat{\mu}_2)$ given in (3.11), $Var(\hat{\sigma}_2)$ given in (3.12) and evaluated the relative efficiency RE_1 of μ_2^* relative to $\hat{\mu}_2$, the relative efficiency RE_2 of σ_2^* relative to $\hat{\sigma}_2$ for different values of n, α and $\beta = 0.50$, are given Table 3.

Table 1: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.25$ and $\beta = 0.50$

Estimator	n	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
μ_2^*	2	10.23744	-9.23744								
	3	6.4946	0.87054	-6.36513							
	4	4.65323	2.28205	-1.06866	-4.86662						
	5	3.58045	2.39003	0.57270	-1.59971	-3.94347					
	6	2.88808	2.22399	1.14276	-0.21442	-1.72441	-3.31600				
	7	2.4085	2.01095	1.32593	0.42857	-0.60293	-1.70913	-2.86190			
	8	2.05925	1.80874	1.35383	0.73554	0.00352	-0.79944	-1.64387	-2.51755		
	9	1.79441	1.63095	1.31733	0.87769	0.34276	-0.25687	-0.89691	-1.56205	-2.24731	
	10	1.58763	1.47775	1.25537	0.93440	0.53444	0.07696	-0.41915	-0.94009	-1.47757	-2.02973

Table 2: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.50$ and $\beta = 0.50$.

Estimator	n	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
μ_2^*	2	6.52291	-5.52291								
	3	4.37553	0.28619	-3.66172							
	4	3.27176	1.25217	-0.77590	-2.74803						
	5	2.60113	1.42388	0.19156	-1.01326	-2.20331					
	6	2.15229	1.39525	0.57589	-0.24181	-1.04095	-1.84068				
	7	1.83188	1.31151	0.73407	0.14479	-0.43528	-1.00543	-1.58153			
	8	1.59229	1.21669	0.79216	0.35024	-0.09028	-0.52293	-0.95125	-1.38693		
	9	1.40693	1.12521	0.80268	0.46139	0.11656	-0.22417	-0.55959	-0.89377	-1.23524	
	10	1.25932	1.04158	0.79009	0.52032	0.24424	-0.03095	-0.30220	-0.57019	-0.83855	-1.11365

Table 3: Coefficients of the BLUE's σ_2^* using RSS for n=2(1)10, $\alpha = 0.25$ and $\beta = 0.50$.

Estimator	n	Coefficients									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
σ_2^*	2	-11.68497	11.68497								
	3	-7.39361	-0.64448	8.03810							
	4	-5.28444	-2.43743	1.58187	6.14000						
	5	-4.05753	-2.62690	-0.44684	2.15889	4.97239					
	6	-3.26695	-2.46784	-1.17039	0.45716	2.26850	4.1795				
	7	-2.02465	-1.40103	-0.71003	-0.00464	0.69095	1.37676	2.07265			
	8	-2.32262	-2.02010	-1.47354	-0.73187	0.14583	1.10876	2.12204	3.17150		
	9	-2.02153	-1.82365	-1.44653	-0.91897	-0.27756	0.44137	1.20901	2.00729	2.83057	
	10	-1.78674	-1.65334	-1.38568	-1.00033	-0.52066	0.02779	0.62262	1.24749	1.89269	2.55617

Table 4: Coefficients of the BLUE's σ_2^* using RSS for, $n=2(1)10$, $\alpha = 0.50$ and $\beta = 0.50$.

Estimator	n	Coefficients									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
σ_2^*	2	-7.22752	7.22752								
	3	-4.84829	0.05243	4.79586							
	4	-3.62321	-1.20436	1.2272	3.60037						
	5	-2.87857	-1.46868	0.00707	1.45302	2.88715					
	6	-2.38026	-1.47339	-0.49273	0.48702	1.44719	2.41216				
	7	-2.02465	-1.40103	-0.71003	-0.00464	0.69095	1.37676	2.07265			
	8	-1.75886	-1.30851	-0.80046	-0.27173	0.25586	0.77514	1.29089	1.81765		
	9	-1.55334	-1.21536	-0.82930	-0.42106	-0.00840	0.39996	0.80292	1.20579	1.61889	
	10	-1.38975	-1.12838	-0.82726	-0.50458	-0.17434	0.15513	0.48048	0.80271	1.12644	1.45955

Table 5: Coefficients of the BLUE's $\hat{\mu}_2$ using Concomitants of order statistics for $n=2(1)10$, $\alpha = 0.25$ and $\beta = 0.50$

Estimator	n	Coefficients									
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
$\hat{\mu}_2$	2	10.23744	9.23744								
	3	6.49267	0.87367	6.36634							
	4	4.65116	2.28519	1.06960	4.86676						
	5	3.57526	2.39296	0.57687	1.60010	3.94500					
	6	2.87968	2.2210	1.1417	-2.2146	1.7214	3.3065				
	7	2.40782	2.0112	1.3266	0.4280	-0.6033	1.7097	2.8609			
	8	2.05880	1.8097	1.3547	0.7352	0.0021	0.8000	1.6446	2.5161		
	9	1.79386	1.6314	1.3185	0.8779	0.3427	0.2577	0.8979	1.5621	-2.2460	
	10	1.58743	1.4784	1.2558	0.9345	0.5342	0.077	-0.417	0.9402	1.47757	2.02883

Table 6: Coefficients of the BLUE's $\hat{\mu}_2$ using Concomitants of order statistics for $n=2(1)10$, $\alpha = 0.50$ and $\beta = 0.50$

Estimator	n	Coefficients									
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
$\hat{\mu}_2$	2	6.52291	-5.52291								
	3	4.37233	0.29179	-3.66412							
	4	3.26827	1.25666	-0.77555	-2.74938						
	5	2.59841	1.42615	0.19330	-1.01387	-2.20399					
	6	2.14097	1.39293	0.57504	-0.24008	-1.03644	-1.93242				
	7	1.83061	1.31185	0.73497	0.14531	-0.43529	-1.00564	-1.58186			
	8	1.59149	1.21681	0.79267	0.35057	-0.09017	-0.52294	-0.95126	-1.38716		
	9	1.40643	1.12534	0.80294	0.46152	0.11656	-0.22416	-0.55947	-0.89363	-1.23552	
	10	1.25903	1.04174	0.79025	0.52032	0.24411	-0.03102	-0.30214	-0.56998	-0.83834	-1.11397

Table 7: Coefficients of the BLUE's $\hat{\sigma}_2$ using Concomitants of order statistics for $n=2(1)10$, $\alpha = 0.25$ and $\beta = 0.50$

Estimator	N	Coefficients									
		d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
$\hat{\sigma}_2$	2	-11.68497	11.68497								
	3	-7.39117	-0.64845	8.03962							
	4	-5.28165	-2.44141	1.58270	6.14037						
	5	-4.04990	-2.63075	-0.45307	2.15851	4.97520					
	6	-3.25688	-2.46496	-1.16917	0.45623	2.26384	4.16795				
	7	-2.71884	-2.24171	-1.41993	-0.34284	0.89536	2.22258	3.60538			
	8	-2.32154	-2.02053	-1.47455	-0.73254	0.14616	1.10962	2.12261	3.17076		
	9	-2.02033	-1.82373	-1.44818	-0.91943	-0.27745	0.44199	1.20973	2.00742	2.82998	
	10	-1.78599	-1.65363	-1.38636	-1.00092	-0.52085	0.02813	0.62329	1.24809	1.89277	2.55548

Table 8: Coefficients of the BLUE's $\hat{\sigma}_2$ using concomitants of order statistics for, $n=2(1)10$, $\alpha = 0.50$ and $\beta = 0.50$

Estimator	N	Coefficients									
		d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
$\hat{\sigma}_2$	2	-7.22752	7.22752								
	3	-4.84391	0.04476	4.79915							
	4	-3.61799	-1.21058	1.22579	3.60278						
	5	-2.87398	-1.47183	0.00378	1.45301	2.88902					
	6	-2.36611	-1.47182	-0.49354	0.48299	1.44057	2.40273				
	7	-2.02159	-1.40117	-0.71183	-0.00644	0.68994	1.37681	2.07429			
	8	-1.75637	-1.30815	-0.80148	-0.27314	0.25458	0.77434	1.29092	1.81929		
	9	-1.55124	-1.21485	-0.82977	-0.42200	-0.00950	0.39891	0.80215	1.20571	1.62059	
	10	-1.38794	-1.12781	-0.82741	-0.50514	-0.17514	0.15414	0.47949	0.80200	1.12650	1.4630

Table 9: $V_1 = \frac{Var(\hat{\mu}_2^*)}{\sigma_2^2}$, $V_2 = \frac{Var(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{Var(\hat{\sigma}_2^*)}{\sigma_2^2}$, $V_4 = \frac{Var(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$, for $n=2(1)10$ and $\alpha = 0.25$ $\beta = 0.50$

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	67.79733	67.61954	98.11117	97.85448	0.997378	0.997384
3	30.08576	30.03969	43.63765	43.57232	0.998469	0.998503
4	18.75737	18.73598	27.24307	27.21361	0.99886	0.998919
5	13.47499	13.4556	19.58871	19.56584	0.998561	0.998832
6	10.45923	10.42227	15.21422	15.16187	0.996466	0.996559
7	8.52364	8.51576	12.40452	12.39427	0.999076	0.999174
8	7.18172	7.17505	10.4556	10.447	0.999071	0.999177
9	6.19854	6.19195	9.02686	9.01836	0.998937	0.999058
10	5.44874	5.4436	7.9369	7.93035	0.999057	0.999175

Table 10: $V_1 = \frac{Var(\hat{\mu}_2^*)}{\sigma_2^2}$, $V_2 = \frac{Var(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{Var(\hat{\sigma}_2^*)}{\sigma_2^2}$, $V_4 = \frac{Var(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$ for $n=2(1)10$ and $\alpha = 0.50$, $\beta = 0.50$

N	V_1	V_2	V_3	V_4	RE_1	RE_2
2	25.5705	25.36444	37.22676	36.927959	0.991941495	0.99197349
3	11.25429	11.20861	16.46972	16.40604	0.995941103	0.99613351
4	6.99771	6.9817	10.27124	10.25137	0.997712109	0.99806547
5	5.02213	5.0153	7.38588	7.37938	0.998640019	0.99911994
6	3.89742	3.87664	5.7396	5.71244	0.994668268	0.99526796
7	3.17638	3.17491	4.68245	4.68342	0.999537209	1.00020716
8	2.6767	2.676	3.94892	3.95086	0.999738484	1.00049127
9	2.31085	2.31065	3.41128	3.41371	0.999913452	1.00071234
10	2.0319	2.03198	3.000984	3.003664	1.000039372	1.00089304

5 . Conclusion

From Table 3 given in section 4, it may be noted that for sample sizes $n=2(1)10$, $\alpha = 0.25$ and $\beta = 0.50$ all the cases calculated the efficiency of the estimators $\hat{\mu}_2$ and $\hat{\sigma}_2$ calculated using concomitants of order statistics is much better than the estimators calculated using ranked set sampling. Also for sample sizes $n=2(1)10$, $\alpha = 0.50$ and $\beta = 0.50$ case considered it may be noted that for $n \leq 6$ all the cases considered the efficiency of the estimators $\hat{\mu}_2$ and $\hat{\sigma}_2$ calculated using concomitants of order statistics is much better than the estimators calculated using ranked set sampling.

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