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Properties of quasi-similarity on θ -operators

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Abstract

Class of operators in Hilbert spaces have recently been studied in relation to quasi-similarity. Quasi-similarity has been shown to be an equivalence relation. Further, it has been established that normality, hyponormality, M -hyponormality, w -hyponormality, log-hyponormality and (p, k) -quasihyponormality are all preserved by quasi-similarity. It has also been demonstrated how quasi-similarity relates to other equivalence relations such as similarity and almost similarity. Specifically, similarity implies quasi-similarity which in turn implies almost similarity under certain conditions. That is, if $A, B \in B(H)$ are normal operators where H is an infinite dimensional Hilbert space such that A and B are quasi-similar, then A and B are almost similar. However, there exist other operators whose quasi-similarity properties have not been established. In this study, we focus on θ -operators and determine their quasi-similarity properties.

Keywords: θ -operators, quasi-similarity, quasi-affinity, Hilbert spaces, bounded linear operators

1. Introduction

Let H denote a complex Hilbert space and $B(H)$ denote a Banach algebra of bounded linear operators on a Hilbert space H . An operator $T \in B(H)$ is a structure preserving map or a symbol that represents a transformation that can be performed on a function. If $P \in B(H)$, then P^* denotes the adjoint of an operator P and P^{-1} denotes the inverse of P . If $A, B \in B(H)$ then $(AB)^{-1} = B^{-1}A^{-1}$. Kiprop *et al.* (2021)^[2] demonstrated that normal operators are θ -operators. Kamau *et al.* (2024)^[6] noted that quasi-similarity preserves normality of an operator. Kiprop *et al.* (2021)^[2] determined the properties of θ -operators on unitary equivalence and almost similarity of an operator. Kibe *et al.* (2024)^[3] extended the work of Kiprop *et al.* (2021)^[2] to the class of unitary quasi-equivalence and demonstrated that unitary quasi-equivalence relation preserves θ -operators.

Definition 1.1: A Hilbert space H (Halmos, 2017)^[5]. A Hilbert space H is a complete inner product space.

Definition 1.2: Quasi-affinity operator (Nzimbi *et al.*, 2016)^[8] An operator $X \in B(H, K)$ is said to be quasi-affinity or quasi-invertible if it is injective with dense range.

Definition 1.3: Quasi-similar operators (Nagy, 2010)^[7]. Two operators S and T of bounded linear operators $B(H)$ and $B(K)$ on Hilbert spaces H and K respectively are said to be quasi-similar if there exist quasi-affinity operators $X \in B(H, K)$ and $Y \in B(K, H)$ satisfying the equations $XS = TX$ and $SY = YT$.

2. Methodology

In order to achieve the desired results, the following outcomes were useful:

Theorem 2.1: (Kiprop *et al.*, 2021)^[2]. If T is self adjoint, then

$$T^*T(T^* + T) = (T^* + T)T^*T.$$

Example 2.2: Consider the operator $T = \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix}$ on the two-dimensional Hilbert space H^2 , we establish that the operator T is a θ -operator.

Solution: Given $T = \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix}$, then $T^* = \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix}$. We need to determine whether $T^*T(T^* + T) = (T^* + T)T^*T$.

$$T^*T = \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} 50 & 16 \\ 16 & 82 \end{pmatrix}$$

$$T^* + T = \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} 14 & 2 \\ 2 & 18 \end{pmatrix}$$

$$T^*T(T^* + T) = \begin{pmatrix} 50 & 16 \\ 16 & 82 \end{pmatrix} \begin{pmatrix} 14 & 2 \\ 2 & 18 \end{pmatrix} = \begin{pmatrix} 732 & 388 \\ 388 & 1508 \end{pmatrix}$$

$$(T^* + T)T^*T = \begin{pmatrix} 14 & 2 \\ 2 & 18 \end{pmatrix} \begin{pmatrix} 50 & 16 \\ 16 & 82 \end{pmatrix} = \begin{pmatrix} 732 & 388 \\ 388 & 1508 \end{pmatrix}$$

Hence

$$T^*T(T^* + T) = (T^* + T)T^*T.$$

Theorem 2.3: (Helmberg, 2008). Let A and B be bounded linear operators on a Hilbert space H . Then

1. $A^{**} = A$.
2. $(\alpha A)^* = \alpha A^*$.
3. $(A + B)^* = A^* + B^*$.
4. $(AB)^* = B^*A^*$.
5. If A is invertible, then so is A^* and $(A^*)^{-1} = (A^{-1})^*$.

Lemma 2.4: (Kiprop et al., 2021) ^[2]. Suppose $E, F \in B(H)$ such that E is unitary equivalent to F and E is a θ -operator, then F is also a θ -operator.

Lemma 2.5: (Kiprop et al., 2021) ^[2]. Suppose $K, L \in B(H)$ such that K is almost similar to L and K is a θ -operator, then L is also a θ -operator.

Lemma 2.6: (Kibe et al., 2024) ^[3]. Suppose $S, T \in B(H)$ such that S is unitary quasi-equivalent to T and S is a θ -operator, then T is also a θ -operator.

3. Main Results

Theorem 3.1

Suppose $A, B \in B(H)$ such that A is quasi-similar to B and A is a θ -operator, then B is also a θ -operator.

Proof. Since A and B are quasi-similar, then there exists two operators X and Y that are quasi-affinity/ quasi-invertible such that

$$XA = BX \tag{1}$$

And

$$AY = YB \tag{2}$$

Considering equation (1), since X is quasi-invertible then pre-multiplying both sides by X^{-1} we have

which follows that,

$$X^{-1}XA = X^{-1}BX \quad A = X^{-1}BX$$

$$A^* = (X^{-1}BX)^* = X^*B^*(X^{-1})^*$$

Since A is a θ -operator, then by definition

$$A^*A(A^* + A) = (A^* + A)A^*A$$

$$X^*B^*(X^{-1})^*X^{-1}BX(X^*B^*(X^{-1})^* + X^{-1}BX) = (X^*B^*(X^{-1})^* + X^{-1}BX)X^*B^*(X^{-1})^*X^{-1}BX \tag{3}$$

But from Theorem 2.3 since X is invertible, then $(X^{-1})^* = (X^*)^{-1}$ and equation 3 becomes

$$X^*B^*(X^*)^{-1}X^{-1}BX(X^*B^*(X^*)^{-1} + X^{-1}BX) = (X^*B^*(X^*)^{-1} + X^{-1}BX)X^*B^*(X^*)^{-1}X^{-1}BX \quad (4)$$

Given two operators $A, B \in B(H)$, we have,

$$(AB)^{-1} = B^{-1}A^{-1} \quad (5)$$

and equation (4) becomes

$$\begin{aligned} X^*B^*(XX^*)^{-1}BX(X^*B^*(X^*)^{-1} + X^{-1}BX) = \\ (X^*B^*(X^*)^{-1} + X^{-1}BX)X^*B^*(XX^*)^{-1}BX \end{aligned} \quad (6)$$

But X is quasi-invertible, hence $XX^* = I$ and equation (6) becomes

$$X^*B^*BX(X^*B^*(X^*)^{-1} + X^{-1}BX) = (X^*B^*(X^*)^{-1} + X^{-1}BX)X^*B^*BX \quad (7)$$

which follows that

$$\begin{aligned} X^*B^*BXX^*B^*(X^*)^{-1} + X^*B^*BXX^{-1}BX = \\ X^*B^*(X^*)^{-1}X^*B^*BX + X^{-1}BXX^*B^*BX \end{aligned} \quad (8)$$

Since X is invertible, then $XX^{-1} = X^{-1}X = I$, $XX^* = I$ and from Theorem

2.3 $(AB)^* = B^*A^*$, then equation (8) becomes

$$X^*B^*BIB^*(X^*)^{-1} + X^*B^*BIBX = X^*B^*(X^{-1})^*X^*B^*BX + X^{-1}BIB^*BX \quad (9)$$

which follows that,

$$X^*B^*BB^*(X^*)^{-1} + X^*B^*BBX = X^*B^*(XX^{-1})^*B^*BX + X^{-1}BB^*BX \quad (10)$$

$$X^*B^*BB^*(X^*)^{-1} + X^*B^*BBX = X^*B^*B^*BX + X^{-1}BB^*BX \quad (11)$$

Pre-multiplying and post-multiplying both sides of equation (11) by X and

X^{-1} respectively, we get

$$XX^*B^*BB^*(X^*)^{-1}X^{-1} + XX^*B^*BBXX^{-1} = XX^*B^*B^*BXX^{-1} + XX^{-1}BB^*BXX^{-1} \quad (12)$$

Applying equation (5) in equation (12), we get

$$XX^*B^*BB^*(XX^*)^{-1} + XX^*B^*BBXX^{-1} = XX^*B^*B^*BXX^{-1} + XX^{-1}BB^*BXX^{-1} \quad (13)$$

But $XX^{-1} = I$ and $XX^* = I$, hence equation (13) becomes

$$IB^*BB^*I + IB^*BBI = IB^*B^*BI + IB^*BBI \quad (14)$$

$$B^*BB^* + B^*BB = B^*B^*B + BB^*B \quad (15)$$

Since B^*B is common on both sides of equation (15), we have

$$B^*B(B^* + B) = (B^* + B)B^*B$$

which implies that B is also a θ -operator.

Also, since Y is quasi-invertible then multiplying equation (2) from the right by Y^{-1} we have

which follows that,

$$AYY^{-1} = YBY^{-1}$$

$$A = YBY^{-1}$$

$$A^* = (YBY^{-1})^* = (Y^{-1})^*B^*Y^*$$

Since A is a θ -operator, then by definition

$$A^*A(A^* + A) = (A^* + A)A^*A$$

$$\begin{aligned}
& (Y^{-1})^* B^* Y^* Y B Y^{-1} ((Y^{-1})^* B^* Y^* + Y B Y^{-1}) = \\
& ((Y^{-1})^* B^* Y^* + Y B Y^{-1}) (Y^{-1})^* B^* Y^* Y B Y^{-1}
\end{aligned} \tag{16}$$

But from Theorem 2.3 since Y is invertible, then $(Y^{-1})^* = (Y^*)^{-1}$ and equation (16) becomes

$$\begin{aligned}
& (Y^*)^{-1} B^* Y^* Y B Y^{-1} ((Y^*)^{-1} B^* Y^* + Y B Y^{-1}) = \\
& ((Y^*)^{-1} B^* Y^* + Y B Y^{-1}) (Y^*)^{-1} B^* Y^* Y B Y^{-1}
\end{aligned} \tag{17}$$

But Y is quasi-invertible, hence $Y^* Y = I$ and equation (17) becomes $(Y^*)^{-1} B^* B Y^{-1} ((Y^*)^{-1} B^* Y^* + Y B Y^{-1}) =$

$$\begin{aligned}
& ((Y^*)^{-1} B^* Y^* + Y B Y^{-1}) (Y^*)^{-1} B^* B Y^{-1} \\
& (Y^*)^{-1} B^* B Y^{-1} (Y^*)^{-1} B^* Y^* + (Y^*)^{-1} B^* B Y^{-1} Y B Y^{-1} = \\
& (Y^*)^{-1} B^* Y^* (Y^*)^{-1} B^* B Y^{-1} + Y B Y^{-1} (Y^*)^{-1} B^* B Y^{-1}
\end{aligned} \tag{18}$$

Since Y is invertible, then $Y Y^{-1} = Y^{-1} Y = I$ and from Theorem 2.3 and equation (5), then equation (18) becomes

$$\begin{aligned}
& (Y^*)^{-1} B^* B (Y^* Y)^{-1} B^* Y^* + (Y^*)^{-1} B^* B B Y^{-1} = (Y^*)^{-1} B^* Y^* (Y^{-1})^* B^* B Y^{-1} + Y B (Y^* Y)^{-1} B^* B Y^{-1} \\
& (Y^*)^{-1} B^* B B^* Y^* + (Y^*)^{-1} B^* B B Y^{-1} = (Y^*)^{-1} B^* B^* B Y^{-1} + Y B B^* B Y^{-1}
\end{aligned} \tag{19}$$

Pre-multiplying and post-multiplying both sides of equation (19) by Y^{-1} and Y respectively, we get

$$\begin{aligned}
& Y^{-1} (Y^*)^{-1} B^* B B^* Y^* Y + Y^{-1} (Y^*)^{-1} B^* B B Y^{-1} Y = \\
& Y^{-1} (Y^*)^{-1} B^* B^* B Y^{-1} Y + Y^{-1} Y B B^* B Y^{-1} Y \\
& (Y^* Y)^{-1} B^* B B^* Y^* Y + (Y^* Y)^{-1} B^* B B Y^{-1} Y = (Y^* Y)^{-1} B^* B^* B Y^{-1} Y + Y^{-1} Y B B^* B Y^{-1} Y
\end{aligned}$$

But $Y^{-1} Y = I$ and $Y^* Y = I$, hence

$$\begin{aligned}
& I B^* B B^* I + I B^* B B I = I B^* B^* B I + I B B^* B I \\
& B^* B B^* + B^* B B = B^* B^* B + B B^* B
\end{aligned} \tag{20}$$

Since $B^* B$ is common on both sides of equation (20), we have

$$B^* B (B^* + B) = (B^* + B) B^* B$$

which implies that B is also a θ -operator.

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