

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2026): 4.49
Maths 2026; 11(8): 58-67
© 2026 Stats & Maths
<https://www.mathsjournal.com>
Received: 12-05-2026
Accepted: 14-06-2026

NK Sajeevkumar
Department of Statistics,
University College,
Thiruvananthapuram, Kerala,
India

RS Priya
Department of Statistics, SNG
College, Chelannur, Kozhikode,
Kerala, India

Corresponding Author:
NK Sajeevkumar
Department of Statistics,
University College,
Thiruvananthapuram, Kerala,
India

Estimation of parameters μ_2 and σ_2 of extended morgenstern type bivariate lindley distribution by ranked set sampling

NK Sajeevkumar and RS Priya

DOI: <https://www.doi.org/10.22271/math.2026.v11.i8a.2664>

Abstract

In this article we have derived the best linear unbiased estimator (BLUE) of the parameters μ_2 and σ_2 involved in the five parameter Extended Morgenstern type bivariate Lindley distribution (EMTBLD) by ranked set sampling (RSS). The efficiency comparison of the proposed estimators with that of the estimators using concomitants of order statistics is also made.

Keywords and Phrases: Morgenstern Family of Distributions, Extended Morgenstern Type Bivariate Lindley Distribution, Best Linear Unbiased Estimation, Concomitants of Order Statistics, Concomitants of Ranked Set Sampling

1. Introduction

The concept of ranked set sampling was first introduced by McIntyre (1952) as a process of improving the precision of the sample mean as an estimator of the population mean. Ranked set sampling as described in McIntyre (1952) is applicable whenever ranking of a set of sampling units can be easily by judgment method, for details, see, Chen *et al.*, (2004) ^[4]. The process of ranked set sampling involves selecting n sets of units, each of size n , and ordering the units of each of the set by judgment method or by using values obtained on the units by a relatively inexpensive method, without making actual measurement on the units. Then the unit ranked as one from the first set is actually measured, the unit ranked as two from the second set is

measured. The process continues in this way until the unit ranked as n from the n^{th} set is measured. Then the observations resulting out of actual measurements made on the units chosen as described in the above Process is known as ranked set sample.

Ranking by a judgment method is not recommendable in case there is ambiguity in discriminating the rank of one unit with another (for example, ranking based on intensity of yellowish color of newborn babies, as mentioned in Chen *et al.*, (2004, p.7)) ^[4], become difficult, as we known that probability of ties in this situation is not zero as the characteristic under consideration is not continuous). In certain situation, one may prefer exact measurement of some easily measurable variable associated with the study variable rather than ranking the units by a crude judgment method. Suppose the variable of interest say, Y , is difficult or much expensive to measure, but an auxiliary variable X correlated with Y is readily measurable and can be ordered exactly. In this case as an alternative to McIntyre's (1952) method of ranked set sampling, Stokes (1977) ^[14] used an auxiliary variable for the ranking of the sampling units.

Stokes (1977) ^[14] using auxiliary variate is as follows: Choose n^2 independent units, arrange them randomly into n sets each with n units and observe the value of the auxiliary variable X on each of these units. In the first set, that units for which the measurement on the auxiliary variable is the smallest is chosen. In the second set, that units for which the measurement on the auxiliary variable is the second smallest is chosen. The procedure is repeated until in the last set, that unit for which the measurement on the auxiliary

variable is the largest is chosen. The measurements on the Y variate of the resulting new set of n units chosen by the above method gives a ranked set sample as suggested by Stokes(1977) . If $X_{(r)r}$ is the observation measured on auxiliary variable X from the unit chosen from the r^{th} set then we write $Y_{[r]r}$ to denote the corresponding measurement made on the study variable Y on this unit so that $Y_{[r]r}$, $r = 1, 2, \dots, n$ form the ranked set sampling. Clearly $Y_{[r]r}$ is the concomitants of r^{th} order statistic arising from the r^{th} sample. By using the expression given by Yang(1977) for the mean and variance of concomitants of order statistics, we have the following . For $1 \leq r \leq n$,

For $1 \leq r \leq n$,

$$E[Y_{[r]r}] = E[E(Y / X_{(r)r})] \quad (1.1)$$

And

$$Var[Y_{[r]r}] = Var[E(Y / X_{(r)r})] + E[Var(Y / X_{(r)r})] \quad (1.2)$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ for $r \neq s$ are drawn from two independent samples we have,

$$Cov[Y_{[r]r}, Y_{[s]s}] = 0, \quad r \neq s. \quad (1.3)$$

Stokes(1977) ^[14] has suggested the ranked set sample mean as an estimator for the mean of the study variable Y, when an auxiliary variable X is used for ranking the sample units, under the assumption that (X,Y) follows a bivariate normal distribution. Barnett and Moore (1997) ^[2] have improved the estimator of Stokes (1977) ^[14] by deriving the Best linear unbiased estimator (BLUE) of the mean of the study variate Y, based on ranked set sample obtained on the study variate Y.

1.1 Morgenstern family of distributions.

The most important aspect to the considered in a modeling problem is regarding the flexibility of the chosen family, in the sense that it should contain a wide variety of models capable of representing any data situation. Another consideration is the sort of prior information available in the choice of the model. In problems involving several random variables, the analyst may make reasonable assumption about the marginal distributions. Then the question is to construct a joint distribution function with a set of given marginal. The Morgenstern family of distributions assumes importance in such contexts as a highly flexible system. Let X and Y be two random variables with cdf's given by $F_X(x)$ and $F_Y(y)$ respectively with pdf's $f_X(x)$ and $f_Y(y)$ and jointly distributed with cdf F(x,y) given by (see, Kotz *et al.*, (2000), p.52).

$$F(x, y) = F_X(x)F_Y(y)\{1 + \alpha(1 - F_X(x))(1 - F_Y(y))\}, \quad -1 \leq \alpha \leq 1. \quad (1.4)$$

Then the family of distributions having the above form of cdf is called the Morgenstern family of distributions (MFD). The system provides a very general expression of a bivariate distribution from which member can derived by substituting expression of any desired set of marginal distributions. The joint pdf corresponding to the cdf defined in (1.4) is given by,

$$f(x, y) = f_X(x)f_Y(y)\{1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))\}, \quad -1 \leq \alpha \leq 1. \quad (1.5)$$

The parameter α is considered as the association parameter, the two random variables are independent when the value of α is

$\frac{\alpha}{3}$

zero and the maximum correlation attained between X and Y is $\frac{\alpha}{3}$. For suitable choice of $F_X(x)$ and $F_Y(y)$, the MFD is primarily useful as an alternative to the standard bivariate normal distribution (See, Conway (1983), p.28-31) ^[5]. The distribution theory of concomitance of order statistics from MFD are discussed in Scaria and Nair(1999) ^[13].

Chako and Thomas (2008) derived the BLUE of a parameter of a three parameter Morgenstern type bivariate exponential distribution using concomitance of order statistics. Sajeevkumar and Sumi(2023) ^[12] derived the BLUE's of the parameters μ_2

and σ_2 of a five parameter Morgenstern type bivariate exponential distribution (MTBED) using concomitants of ranked set sampling. Irshad *et al.* (2019) ^[7] estimate a parameter of a three parameter Morgenstern type bivariate Lindly distribution

(MTBLD) using ranked set sampling. Sajeevkumar and Priya(2026) ^[10, 11] derived the BLUE's of the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate Lindly distribution (MTBLD) using ranked set sampling. Arun *et al.* (2023) ^[1] derived the BLUE of a parameter of a three parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using

concomitance of order statistics. Sajeevkumar and Priya (2026) ^[10, 11] estimate the BLUE's of the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using Ranked set sampling . Irshad

et.al.(2021) ^[6] estimated a parameter of a four parameter extended morgenstern type bivariate Lindley distribution(EMTBLD) using concomitants of order statistics .In this work our aim is to estimate the parameters μ_2 and σ_2 of a six parameter Extended Morgenstern type bivariate Lindly distribution (EMTBLD) using concomitants of ranked set sampling.

2. Estimation of the parameters μ_2 and σ_2 involved in embld using RSS.

An important member of Morgenstern family of distribution is (MFD) is Extended Morgenstern type bivariate Lindly distribution (EMTBLD) .The bivariate random variable (X,Y) is said to have six parameter EMTBLD, if its pdf is given by

$$f(x,y) = \frac{1}{2\sigma_1\sigma_2} \left(1 + \frac{x-\mu_1}{\sigma_1}\right) e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} \left(1 + \frac{y-\mu_2}{\sigma_2}\right) e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} \left\{1 + \alpha \left[2 \left(1 + \frac{x-\mu_1}{2\sigma_1}\right) e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1\right] \left[2 \left(1 + \frac{y-\mu_2}{2\sigma_2}\right) e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1\right] + \beta \left[3 \left(1 + \frac{x-\mu_1}{2\sigma_1}\right) e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1\right] \left[3 \left(1 + \frac{y-\mu_2}{2\sigma_2}\right) e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1\right] \right\} \left[1 - \left(1 + \frac{x-\mu_1}{2\sigma_1}\right) e^{-\frac{x-\mu_1}{\sigma_1}}\right] \left[1 - \left(1 + \frac{y-\mu_2}{2\sigma_2}\right) e^{-\frac{y-\mu_2}{\sigma_2}}\right] \right\}, x > \mu_1, y > \mu_2, \sigma_1 > 0, \sigma_2 > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}}\right]. \quad (2.1)$$

The pdf defined in (2.1) is an extension of the four parameter EMTBLD defined in Irshad et.al (2021) ^[6].The joint pdf of standard

$$U = \frac{X - \mu_1}{\sigma_1} \quad V = \frac{Y - \mu_2}{\sigma_2}$$

EMTBLD is obtained by making the substitution and in (2.1) and it is given by

$$g(u,v) = \frac{1}{4} e^{-u-v} (1+u)(1+v) \left\{1 + \alpha \left[2e^{-u} \left(1 + \frac{u}{2}\right) - 1\right] \left[2e^{-v} \left(1 + \frac{v}{2}\right) - 1\right] + \beta \left[3e^{-u} \left(1 + \frac{u}{2}\right) - 1\right] \left[3e^{-v} \left(1 + \frac{v}{2}\right) - 1\right] \left[1 - e^{-u} \left(1 + \frac{u}{2}\right)\right] \left[1 - e^{-v} \left(1 + \frac{v}{2}\right)\right] \right\}, u > 0, v > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}}\right]. \quad (2.2)$$

Suppose X and Y are the two characteristics of interest on the units of a population and (X, Y) has a distribution with pdf given in (2.1) . Suppose there are n independent random samples each of size n drawn from (2.1) and in each sample, units are ranked according to the values of the observations measured on the easily measurable variable X on the units. Though the variable Y is of primary interest, we assume that it is difficult to measure. Then to deal with the inference problem on the distribution of the variable Y we obtain a ranked set sample by measuring the value of Y on the r^{th} ordered unit of the r^{th} sample, $r=1,2,\dots,n$.

We write $Y_{[r]r}$ to denote this observation measured on Y of the r^{th} ordered unit of the r^{th} set, $r=1,2,\dots,n$. Then clearly $Y_{[r]r}$ is distributed as the concomitants of r^{th} order statistic in each sample, units are ranked according to the value of the observations measured on the easily measurable variable U on the unit. We obtain a ranked set sample by measuring the value of V on the r^{th} ordered unit of the r^{th} sample, We write $V_{[r]r}$ to denote this observation measured on V of the r^{th} ordered unit of the r^{th} set, $r=1,2,\dots,n$. Then clearly $V_{[r]r}$ is distributed as the concomitants of r^{th} order statistic of a random sample of size n arising from (2.2). Then the pdf $g_{[r]r}(v)$ of $V_{[r]r}$, $r=1,2,\dots,n$ is given by

$$\begin{aligned}
g_{[r]r}(v) &= \frac{1}{2}(1+v)e^{-v} + \alpha \frac{(n-2r+1)}{2(n+1)}(1+v)e^{-v} \left[2\left(1+\frac{v}{2}\right)e^{-v} - 1 \right] + \\
&\beta \frac{(2n-3r+1)r}{2(n+1)(n+2)}(1+v)e^{-v} \left[1 - \left(1+\frac{v}{2}\right)e^{-v} \right] \left[3\left(1+\frac{v}{2}\right)e^{-v} - 1 \right], \\
|\alpha| &\leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \\
v > 0,
\end{aligned} \tag{2.3}$$

Using (2.3), for $1 \leq r \leq n$,

$$E[V_{[r]r}] = \frac{3}{2} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left(\frac{95}{216} \right) \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] = \xi_r \tag{2.4}$$

And

$$\begin{aligned}
\text{Var}[V_{[r]r}] &= 4 - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{45}{16} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{1429}{648} \right) - \\
&\left\{ \frac{3}{2} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{95}{216} \right) \right\}^2 = e_{r,r}
\end{aligned} \tag{2.5}$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[V_{[r]r}, V_{[s]s}] = 0 = e_{r,s}, \quad r \neq s, \quad r, s = 1, 2, \dots, n. \tag{2.6}$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ are drawn for $r \neq s$ are drawn from two independent samples.

Let $(X_i, Y_i)_{i=1,2,\dots}$ be a ranked set sample of size n drawn from a population with pdf defined by (2.1). Clearly we have $X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i=1,2,\dots,n$. Then using (2.4) to (2.6), we have for $1 \leq r \leq n$,

$$E(Y_{[r]r}) = \mu_2 + \sigma_2 \xi_r \tag{2.7}$$

$$\text{Var}[Y_{[r]r}] = \sigma_2^2 e_{r,r} \tag{2.8}$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[Y_{[r]r}, Y_{[s]s}] = 0, \quad r \neq s \tag{2.9}$$

Where ξ_r , $e_{r,r}$ and $e_{r,s}$ are defined by (2.4), (2.5) and (2.6) respectively. Clearly from (2.4) to (2.6) it follows that ξ_r , $e_{r,r}$ and $e_{r,s}$ are known constants provided α is known. If we write

$$\underline{Y}_{[n]} = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})', \quad \underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)' \text{ and } E = ((e_{r,s})), \text{ then we can write } E[\underline{Y}_{[n]}] = \underline{1}\mu_2 + \underline{\xi}\sigma_2, \tag{2.10}$$

$$\text{Where } \underline{1} \text{ is a column vector of } n \text{ ones and } \underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)' \text{ and } \text{Var}[\underline{Y}_{[n]}] = E\sigma_2^2, \tag{2.11}$$

Now we derive the BLUE's of μ_2 and σ_2 involved in (2.1) and it is given in the following theorem.

Theorem 2.1:

Let $\underline{Y}_n = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be the vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of concomitants of ranked set sample arising (2.2). Then the BLUE's of the parameters μ_2^* of μ_2 and σ_2^* of σ_2 is given by

$$\mu_2^* = -\frac{\underline{\xi}' E^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}') E^{-1}}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \underline{Y}_n \quad (2.12)$$

$$\sigma_2^* = \frac{\underline{1}' E^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}') E^{-1}}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \underline{Y}_n \quad (2.13)$$

And corresponding variances are given by

$$\text{Var}(\mu_2^*) = \frac{(\underline{\xi}' E^{-1} \underline{\xi}) \sigma_2^2}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \quad (2.14)$$

And

$$\text{Var}(\sigma_2^*) = \frac{(\underline{1}' E^{-1} \underline{1}) \sigma_2^2}{(\underline{\xi}' E^{-1} \underline{\xi})(\underline{1}' E^{-1} \underline{1}) - (\underline{\xi}' E^{-1} \underline{1})^2} \quad (2.15)$$

Proof: Let $\underline{Y}_{[n]} = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be a vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of vector of concomitants of ranked set sample arising from (2.2), then by using (2.10) and (2.11) together defines a generalized Gauss-Markov set up. Then by Generalized Gauss-Markov theorem we get the BLUE's of μ_2 and σ_2 and their corresponding variances.

3. Estimation of the parameters μ_2 and σ_2 involved in mtbmed using concomitants of order statistics

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be the observations of random sample of size n drawn from (2.1). The Y component associated with r^{th} order statistic $X_{r:n}$ is defined as the concomitants of r^{th} order statistic and it is denoted by $Y_{[r:n]}, r = 1, 2, \dots, n$. Also let $(U_i, V_i), i = 1, 2, \dots, n$ be the sequence of independent observations drawn from (2.2). Let $V_{[r:n]}$ be the concomitants of r^{th} order statistic $U_{r:n}$ arising from (2.2), then the pdf of $V_{[r:n]}$ and the joint pdf of $V_{[r:n]}$ and $V_{[s:n]}$ for $1 \leq r < s \leq n$ are respectively given in (3.1) and (3.2).

$$g_{[r:n]}^*(v) = \frac{1}{2} (1+v) e^{-v} + \alpha \frac{(n-2r+1)}{2(n+1)} (1+v) e^{-v} \left[2 \left(1 + \frac{v}{2} \right) e^{-v} - 1 \right] +$$

$$\beta \frac{(2n-3r+1)r}{2(n+1)(n+2)} (1+v) e^{-v} \left[1 - \left(1 + \frac{v}{2} \right) e^{-v} \right] \left[3 \left(1 + \frac{v}{2} \right) e^{-v} - 1 \right]$$

$$, v > 0, \quad |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right] \quad (3.1)$$

and for $1 \leq r < s \leq n$

$$f_{[r,s;n]}(v_1, v_2) = \frac{1}{4} (1+v_1) e^{-v_1} (1+v_2) e^{-v_2} +$$

$$\begin{aligned}
& \alpha \frac{(n-2r+1)}{2(n+1)} (1+v_1) e^{-v_1} \left[2 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] \frac{1}{2} (1+v_2) e^{-v_2} + \\
& \alpha \frac{(n-2s+1)}{2(n+1)} (1+v_2) e^{-v_2} \left[2 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right] \frac{1}{2} (1+v_1) e^{-v_1} + \\
& \beta \frac{(2n-3r+1)r}{4(n+1)(n+2)} (1+v_1)(1+v_2) e^{-v_1-v_2} \left[1 - \left(1 + \frac{v_1}{2} \right) e^{-v_1} \right] \left[3 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] + \\
& \beta \frac{(2n-3s+1)s}{4(n+1)(n+2)} (1+v_1)(1+v_2) e^{-v_1-v_2} \left[1 - \left(1 + \frac{v_2}{2} \right) e^{-v_2} \right] \left[3 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right] + \\
& \frac{\alpha^2}{4} \left\{ \frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} \right\} (1+v_1)(1+v_2) e^{-v_1-v_2} \left[2 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] \left[2 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right] + \\
& \frac{\alpha\beta}{4} \left\{ -\frac{2r}{n+1} + \frac{[3r(r+1)+4r(n-s+1)]}{(n+1)(n+2)} - \frac{6r(r+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\} * \\
& (1+v_1)(1+v_2) e^{-v_1-v_2} \left[1 - \left(1 + \frac{v_1}{2} \right) e^{-v_1} \right] \left[3 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] \left[2 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right] + \\
& \frac{\alpha\beta}{4} \left\{ -\frac{s}{n+1} + \frac{2r(s+1)+3s(n-s+1)}{(n+1)(n+2)} - \frac{6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right\} * \\
& (1+v_1)(1+v_2) e^{-v_1-v_2} \left[2 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] \left[1 - \left(1 + \frac{v_2}{2} \right) e^{-v_2} \right] \left[3 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right] + \\
& \frac{\beta^2}{9} \left\{ -\frac{2r(s+1)}{(n+1)(n+2)} + \frac{[3r(r+1)(s+2)+6r(s+1)(n-s+1)]}{(n+1)(n+2)(n+3)} - \frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)} \right\} * \\
& \frac{9}{4} (1+v_1)(1+v_2) e^{-v_1-v_2} \left[1 - \left(1 + \frac{v_1}{2} \right) e^{-v_1} \right] \left[3 \left(1 + \frac{v_1}{2} \right) e^{-v_1} - 1 \right] * \\
& \left[1 - \left(1 + \frac{v_2}{2} \right) e^{-v_2} \right] \left[3 \left(1 + \frac{v_2}{2} \right) e^{-v_2} - 1 \right], \\
& v_1 > 0, v_2 > 0, |\alpha| \leq 1, -\alpha - 1 \leq \beta \leq \frac{1}{2} \left[3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{\frac{1}{2}} \right]
\end{aligned} \tag{3.2}$$

$$\text{Using (3.1) and (3.2), we have } E[V_{[r]r}] = \frac{3}{2} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left(\frac{95}{216} \right) \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] = \xi_r \tag{3.3}$$

and for $1 \leq r < s \leq n$

$$\text{Var}[V_{[r]r}] = 4 - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{45}{16} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{1429}{648} \right) -$$

$$\left\{ \frac{3}{2} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{95}{216} \right) \right\}^2 = b_{r,r} \quad (3.4)$$

Also

$$\begin{aligned} \text{Cov}[V_{[r:n]}, V_{[s:n]}] &= \frac{9}{4} - \alpha \left[\frac{n-(r+s)+1}{n+1} \right] \frac{33}{16} - \beta \left[\frac{(2n+1)(r+s)-3(r^2+s^2)}{3(n+1)(n+2)} \right] \frac{95}{48} + \\ &\frac{\alpha^2}{4} \left[\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} \right] \left(\frac{11}{8} \right)^2 + \\ &\frac{\alpha\beta}{6} \left[- \left(\frac{2r+s}{n+1} \right) + \frac{(3r+2s+5)r + (4r+3s)(n-s+1)}{(n+1)(n+2)} \right] \frac{1045}{576} - \frac{\alpha\beta}{6} \left[\frac{6r(r+s+2)(n-s+1)}{(n+1)(n+2)(n+3)} \right] \frac{1045}{576} + \\ &\frac{\beta^2}{9} \left[\frac{-2r(s+1)}{(n+1)(n+2)} - \frac{3r(r+1)(s+2) + 6r(s+1)(n-s+1)}{(n+1)(n+2)(n+3)} \right] - \\ &\frac{9r(r+1)(s+2)(n-s+1)}{(n+1)(n+2)(n+3)(n+4)} \left(\frac{95}{72} \right)^2 - \left\{ \frac{3}{2} - \alpha \left[\frac{n-2r+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left[\frac{(2n-3r+1)r}{(n+1)(n+2)} \right] \left(\frac{95}{216} \right) \right\}^* \\ &\left\{ \frac{3}{2} - \alpha \left[\frac{n-2s+1}{n+1} \right] \left(\frac{11}{16} \right) - \beta \left[\frac{(2n-3s+1)s}{(n+1)(n+2)} \right] \left(\frac{95}{216} \right) \right\} \end{aligned} \quad (3.5)$$

Let $(X_i, Y_i), i=1,2,\dots, n$ be sample of size n from a population with pdf defined in (2.1). Clearly $X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i=1,2,\dots, n$.

Then for $1 \leq r \leq n$,

$$E[Y_{[r:n]}] = \mu_2 + \sigma_2 \xi_r \quad (3.6)$$

$$\text{Var}[Y_{[r:n]}] = \sigma_2^2 b_{r,r} \quad (3.7)$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[Y_{[r:n]}, Y_{[s:n]}] = \sigma_2^2 b_{r,s}, \quad (3.8)$$

Where $\xi_r, b_{r,r}$ and $b_{r,s}$ are respectively defined in (3.3), (3.4) and (3.5).

Let $Y_{[n]} = (Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]})'$ be the vector of concomitants of order statistics arising from (2.1). Let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$

and $B = (b_{r,s})$ respectively the mean vector and dispersion matrix of $Y_{[n]}$. Then we can write $E[Y_{[n]}] = \underline{1}\mu_2 + \underline{\xi}\sigma_2$, where $\underline{1}$

is a column vector of n ones and $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $V[Y_{[n]}] = B\sigma_2^2$.

Now using generalized Gauss markov theorem BLUE's of μ_2 and σ_2 involved in (2.1) are given by

$$\hat{\mu}_2 = - \frac{\underline{\xi}' B^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}') B^{-1}}{(\underline{\xi}' B^{-1} \underline{\xi}) (\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} Y_{[n]} \quad (3.9)$$

$$\hat{\sigma}_2 = \frac{\underline{1}' B^{-1} (\underline{1} \underline{\xi}' - \underline{\xi} \underline{1}) B^{-1}}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2} Y_{[n]}$$

(3.10)

And

$$Var(\hat{\mu}_2) = \frac{(\underline{\xi}' B^{-1} \underline{\xi}) \sigma_2^2}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2}$$

(3.11)

$$Var(\hat{\sigma}_2) = \frac{(\underline{1}' B^{-1} \underline{1}) \sigma_2^2}{(\underline{\xi}' B^{-1} \underline{\xi})(\underline{1}' B^{-1} \underline{1}) - (\underline{\xi}' B^{-1} \underline{1})^2}$$

(3.12)

4. Computation section

We have evaluated the coefficients of the BLUE μ_2^* of μ_2 given in (2.12) and evaluated the coefficients of the BLUE σ_2^* of σ_2 given in (2.13) for different values of n, α and $\beta = 0.50$, are given in Table 1. Also we have evaluated $Var(\mu_2^*)$ given in (2.14), $Var(\sigma_2^*)$ given in (2.15), $Var(\hat{\mu}_2)$ given in (3.11), $Var(\hat{\sigma}_2)$ given in (3.12) and evaluated the relative efficiency RE_1 of μ_2^* relative to $\hat{\mu}_2$, the relative efficiency RE_2 of σ_2^* relative to $\hat{\sigma}_2$ for different values of n, α and $\beta = 0.50$, are given Table 2.

Table 1: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.25$ and $\beta = 0.50$

Estimator	N	Coefficients									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
μ_2^*	2	8.48382	-7.48382								
	3	5.39019	0.76362	-5.15381							
	4	3.86382	1.91472	-0.83961	-3.93893						
	5	2.97348	1.99769	0.50083	-1.28141	-3.19060					
	6	2.39828	1.85653	0.96594	-0.15117	-1.38722	-2.68236				
	7	1.99996	1.67748	1.11377	0.37407	-0.47324	-1.37758	-2.31447			
	8	1.70963	1.50811	1.13432	0.62468	0.02181	-0.63681	-1.32613	-2.03561		
	9	1.48965	1.35918	1.10235	0.73994	0.29894	-0.19410	-0.71833	-1.26074	-1.81689	
	10	1.31772	1.23118	1.04964	0.78526	0.45537	0.07855	-0.32896	-0.75508	-1.19295	-1.64072

Table 2: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.50$ and $\beta = 0.50$

Estimator	N	Coefficients									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
μ_2^*	2	5.45933	-4.459331								
	3	3.67355	0.27372	-2.94727							
	4	2.75211	1.06380	-0.60787	-2.20804						
	5	2.19048	1.20341	0.18080	-0.80639	-1.76831					
	6	1.81383	1.17808	0.49514	-0.17956	-0.83140	-1.47608				
	7	1.54464	1.10721	0.62441	0.13579	-0.34052	-0.80404	-1.26750			
	8	1.34318	1.02724	0.67155	0.30374	-0.05986	-0.41381	-0.76108	-1.11097		
	9	1.02763	0.83376	0.61197	0.37847	0.14421	-0.09836	-0.30875	-0.52909	-1.05983	
	10	1.06279	0.87971	0.66843	0.44282	0.21329	-0.01390	-0.23626	-0.45431	-0.67105	-0.89151

Table 3: Coefficients of the BLUE's σ_2^* using RSS for n=2(1)10, $\alpha = 0.25$ and $\beta = 0.50$

Estimator	n	Coefficients									
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
σ_2^*	2	-5.32255	5.32255								
	3	-3.37123	-0.28687	3.65810							
	4	-2.40944	-1.10931	0.72604	2.79272						
	5	-1.84942	-1.19786	-0.20043	0.98713	2.26058					
	6	-1.48833	-1.12605	-0.53241	0.21173	1.03546	1.8996				

	7	-1.23875	-1.02269	-0.64672	-0.15398	0.41042	1.01321	1.63851			
	8	-1.05716	-0.92183	-0.67232	-0.33275	0.06876	0.50750	0.96708	1.44072		
	9	-0.91980	-0.83192	-0.66032	-0.41876	-0.12504	0.20330	0.55258	0.91430	1.28566	
	10	-0.81260	-0.75410	-0.63267	-0.45635	-0.23660	0.01432	0.28573	0.56969	0.86176	1.16081

Table 4: Coefficients of the BLUE's σ_2^* using RSS for, $n=2(1)10$, $\alpha = 0.50$ and $\beta = 0.50$.

Estimator	n	Coefficients									
		C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
σ_2^*	2	-5.32255	5.32255								
	3	-2.22544	0.03735	2.18809							
	4	-1.66638	-0.54358	0.56971	1.64025						
	5	-1.32534	-0.66899	0.01103	0.66930	1.31401					
	6	-1.09662	-0.67379	-0.21997	0.22901	0.66429	1.09707				
	7	-0.93321	-0.64215	-0.32141	0.00336	0.32065	0.63061	0.94216			
	8	-0.81097	-0.60063	-0.36433	-0.12002	0.12181	0.35787	0.59040	0.82588		
	9	-0.60268	-0.47559	-0.33062	-0.17813	-0.02502	0.14461	0.27219	0.41773	0.77751	
	10	-0.64097	-0.51893	-0.37849	-0.22870	-0.07628	0.07478	0.22294	0.36870	0.51415	0.66280

Table 5: $V_1 = \frac{Var(\mu_2^*)}{\sigma_2^2}$, $V_2 = \frac{Var(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{Var(\sigma_2^*)}{\sigma_2^2}$, $V_4 = \frac{Var(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$, for $n=2(1)10$ and $\alpha = 0.25$ $\beta = 0.50$

N	V_1	V_2	V_3	V_4	RE_1	RE_2
2	220.66373	219.54247	98.653231	98.15293	0.994918694	0.9949287
3	97.94058	97.69825	43.90695	43.80634	0.997525745	0.9977086
4	61.07043	61.01251	27.422003	27.40509	0.999051587	0.9993832
5	43.87374	43.87549	19.72121	19.73085	1.000039887	1.0004888
6	34.0555	34.07579	15.3195	15.33695	1.000595792	1.0011391
7	27.75403	27.78805	12.49193	12.51491	1.001225768	1.0018396
8	23.3839	23.42139	10.52963	10.5536	1.00160324	1.0022764
9	20.18305	20.22151	9.0915	9.11538	1.001905559	1.0026266
10	17.74196	17.77897	7.99421	8.0174	1.002086015	1.0029008

Table 6: $V_1 = \frac{Var(\mu_2^*)}{\sigma_2^2}$, $V_2 = \frac{Var(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{Var(\sigma_2^*)}{\sigma_2^2}$, $V_4 = \frac{Var(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$ for $n=2(1)10$ and $\alpha = 0.50$, $\beta = 0.50$

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	83.84882	82.73528	37.75877	37.25878	0.98671967	0.9867583
3	36.88222	36.65943	16.71292	16.61865	0.99395942	0.9943595
4	22.92951	22.81082	10.4279	10.36602	0.994823701	0.9940659
5	16.45481	16.45316	7.50087	7.50777	0.999899725	1.0009199
6	12.76909	12.78659	5.83032	5.84555	1.001370497	1.0026122
7	10.40641	10.43146	4.75729	4.7755	1.00240717	1.0038278
8	8.76902	8.79689	4.01252	4.03157	1.003178234	1.0047476
9	6.18153	6.20176	2.75846	2.77138	1.003272653	1.0046838
10	6.65662	6.6848	3.05002	3.06841	1.00423338	1.0060295

5. Conclusion

From Table 2 given in section 4, it may be noted that for sample sizes (n) greater than five, all the cases calculated the performance of the estimators μ_2^* and σ_2^* calculated using Ranked set sampling is much better than the estimators calculated using concomitants of order statistics.

References

1. Arun SP, Chesneau C, Maya R, Irshad MR. Farlie-Gumbel-Morgenstern bivariate moment exponential distribution and its inference based on concomitants of order statistics. Stats. 2023;6:253-267.
2. Barnett V, Moore K. Best linear unbiased estimates in ranked set sampling with particular reference to imperfect ordering. Journal of Applied Statistics. 1997;24(6):697-710.
3. Chacko M, Thomas PY. Estimation of parameters of a Morgenstern type bivariate exponential distribution by ranked set sampling. Annals of the Institute of Statistical Mathematics. 2008;60(2):301-318.

4. Chen Z, Bai Z, Sinha BK. Ranked set sampling: theory and applications. Lecture Notes in Statistics. Vol. 176. New York: Springer-Verlag; 2004.
5. Conway DA. Farlie-Gumbel-Morgenstern distributions. In: Kotz S, Johnson NL, editors. Encyclopedia of Statistical Sciences. Vol. 3. New York: Wiley; 1983.
6. Irshad MR, Maya R, Al-Omari AI, Arun SP, Alomani G. The extended Farlie-Gumbel-Morgenstern bivariate Lindley distribution: concomitants of order statistics and estimation. Electronic Journal of Applied Statistical Analysis. 2021;14(2):373-388.
7. Irshad MR, Maya R, Arun SP. Estimation of a parameter of Morgenstern type bivariate Lindley distribution by ranked set sampling. İSTATİSTİK: Journal of the Turkish Statistical Association. 2019;12:25-34.
8. Kotz S, Balakrishnan N, Johnson NL. Continuous multivariate distributions. Vol. 1. 2nd ed. New York: Wiley; 2000.
9. MacIntyre GA. A method for unbiased selective sampling using ranked sets. Australian Journal of Agricultural Research. 1952;3(4):385-390.
10. Sajeevkumar NK, Priya RS. Estimation of parameter and of Morgenstern type bivariate Lindley distribution by ranked set sampling. International Journal of Statistics and Applied Mathematics. 2026;11(6):228-233.
11. Sajeevkumar NK, Priya RS. Estimation of parameter and of Morgenstern type bivariate moment exponential distribution by ranked set sampling. International Journal of Statistics and Applied Mathematics. 2026;11(7):101-107.
12. Sajeevkumar NK, Sumi AR. Use of ranked set sampling in the estimation of parameters and of Morgenstern type bivariate exponential distribution. IAPQR Transactions. 2023;47(1-2):16-42.
13. Scaria J, Nair NU. On concomitants of order statistics from Morgenstern family. Biometrical Journal. 1999;41(4):483-489.
14. Stokes SL. Ranked set sampling with concomitant variables. Communications in Statistics—Theory and Methods. 1977;6:1207-1211.
15. Yang SS. General distribution theory of the concomitants of order statistics. The Annals of Statistics. 1977;5:996-1002.