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A new generalization of quasi rama distribution exploring its statistical properties and applications

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Abstract

This article approaches with a new version of quasi Rama distribution known as weighted quasi Rama distribution has been established by applying the weighted technique to baseline quasi Rama distribution. The newly executed distribution explored with its several structural features and its parameters are estimated based upon the technique of maximum likelihood estimation. Furthermore, the superiority and capability of a newly executed distribution has been demonstrated and inspected by examining a real data set to investigate its practicability and flexibility.

Keywords: Quasi rama distribution, weighted distribution, order statistics, reliability measures, entropy, parameter estimation

1. Introduction

The weighted distributions attained a significant popularity and dominance in distribution theory even classical distributions may not provide a proper fit to data. The idea of weighted distribution are applicable to adjust the probabilities of events and also biased data arise in theory of weighted distribution. The idea of weighted distribution was suggested by Fisher (1934) ^[8] to modify the ascertainment biases then after Rao (1965) ^[11] generalized it in a unified manner for problems were observations fall in non-experimental, non-replicated and non-random manner. The weighted distributions are important in statistics because it provides a comprehensive technique to extend the existing baseline distribution by incorporating extra parameter and this additional parameter accomplishes the existing distribution more capable. The weighted distribution is one of crucial and important factor in mathematics, probability and statistics because of great practicability and also it is very useful for efficient modeling of the statistical biased data arising from different fields of knowledge and prediction from data if standard distributions are not appropriate. It has been observed when an investigator records observation in the nature according to certain stochastic model, the distribution of recorded observation will not have original distribution until every observation is given an equal chance of being recorded. The weighted distributions occur in modeling the data sets have clustered sampling, heterogeneity and extraneous variation. The weighted distributions provide a unified access for problem of model specification and data interpretation. There are various authors who described and discussed significantly to introduce several important weighted probability distributions along with their application from various applied fields. Ganaie and Rajagopalan (2020) ^[9] derived and discussed weighted quasi gamma distribution with properties and applications. Bashir and Khan (2023) ^[6] studied the characterization of the weighted power function distribution by reliability functions and moments. Alzoubi (2024) ^[3] proposed length biased Loai distribution with statistical properties and application. Mudassir *et al.* (2018) ^[10] introduced weighted new weibull Pareto distribution with some characteristics and applications. Almakhaireez and Alzoubi (2024) ^[2] obtained the length biased Benrabia distribution with applications. Bashir and Rasul (2015) ^[5] derived some properties of the weighted Lindley distribution. Reshi *et al.* (2014) ^[13] studied characterizations and estimations of size biased generalized Rayleigh distribution. Shakhathreh (2012) ^[15] developed the two parameter of weighted exponential distributions.

Ahmad *et al.* (2022) ^[1] discussed weighted inverse Ailamujia distribution with applications. Saghir and Hamedani (2018) ^[14] derived the generalized weighted Maxwell distribution with properties and estimation. Andure (Yawale) and Ade (2021) ^[4] constructed the new length biased Hamza distribution with properties and applications.

A quasi Rama distribution is a newly introduced two parametric distribution developed by Shanker *et al.* (2023) ^[16] which is a particular case of Rama distribution. Some of its statistical properties like mean residual life function, hazard rate function, moments, mean deviations and stochastic ordering have been described. Its parameters are estimated based upon the method of moments and maximum likelihood estimation. Shanker (2017) ^[17] also discussed on Rama distribution which is a one parameter distribution discussed and derived its several mathematical and statistical properties. Its parameters are also estimated by using maximum likelihood estimation and method of moments.

2. Weighted Quasi Rama (WQR) Distribution

The probability density function of quasi Rama distribution is given by

$$f(x; \theta, \alpha) = \frac{\theta^3}{(\alpha\theta^2 + 6)} (\alpha + \theta x^3) e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \quad (1)$$

and the cumulative distribution function of quasi Rama distribution is given by

$$F(x; \theta, \alpha) = 1 - \left(1 + \frac{(\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x)}{(\alpha\theta^2 + 6)} \right) e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \quad (2)$$

Consider the random variable X constitutes the condition of non-negativity has probability density function $f(x)$. Assume $w(x)$ be its weight function that is also non-negative, then weighted random variable X_w with its probability density function is

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, x > 0$$

Here the weight function $w(x)$ that is non-negative then $E(w(x)) = \int w(x)f(x)dx < \infty$.

For several choices of weight function $w(x)$ obviously if $w(x) = x^c$, the executed distribution is termed as weighted distribution. In this paper by taking the weight function as $w(x) = x^c$ to get the weighted version of quasi Rama distribution and then its probability density function is defined as

$$f_w(x; \theta, \alpha, c) = \frac{x^c f(x; \theta, \alpha)}{E(x^c)} \quad (3)$$

$$\text{Here } E(x^c) = \int_0^{\infty} x^c f(x; \theta, \alpha) dx$$

$$E(x^c) = \int_0^{\infty} x^c \frac{\theta^3}{(\alpha\theta^2 + 6)} (\alpha + \theta x^3) e^{-\theta x} dx$$

After simplification, we obtain

$$E(x^c) = \frac{\alpha\theta^2\Gamma(c+1) + \Gamma(c+4)}{\theta^c(\alpha\theta^2 + 6)} \quad (4)$$

Now therefore by applying the equations (1) and (4) in equation (3), we will get the required probability density function of weighted quasi Rama distribution which is

$$f_w(x; \theta, \alpha, c) = \frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \quad (5)$$

The cumulative distribution function of weighted quasi Rama distribution can be determined as

$$F_w(x; \theta, \alpha, c) = \int_0^x f_w(x; \theta, \alpha, c) dx$$

$$F_w(x; \theta, \alpha, c) = \int_0^x \frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$F_w(x; \theta, \alpha, c) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \int_0^x x^c \theta^{c+3} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$F_w(x; \theta, \alpha, c) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^{c+3} \int_0^x x^c e^{-\theta x} dx + \theta^{c+4} \int_0^x x^{c+3} e^{-\theta x} dx \right) \quad (6)$$

$$\text{By putting } \theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$$

$$\text{Also } x = \frac{t}{\theta}$$

When $x \rightarrow \infty$, $t \rightarrow \infty$ and as $x \rightarrow 0$, $t \rightarrow 0$

After the simplification of equation (6), we will get the cumulative distribution function of weighted quasi Rama distribution which is

$$F_w(x; \theta, \alpha, c) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right) \quad (7)$$

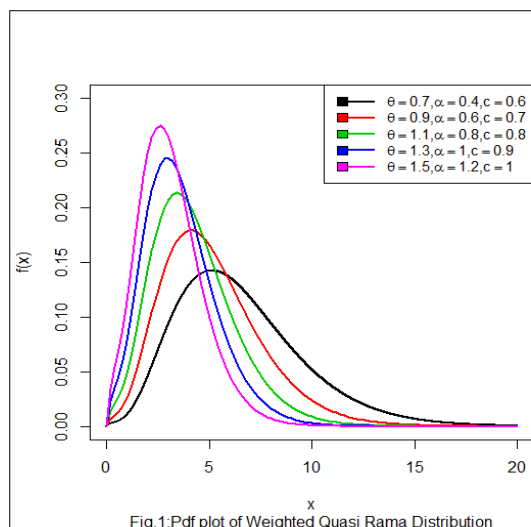


Fig.1: Pdf plot of Weighted Quasi Rama Distribution

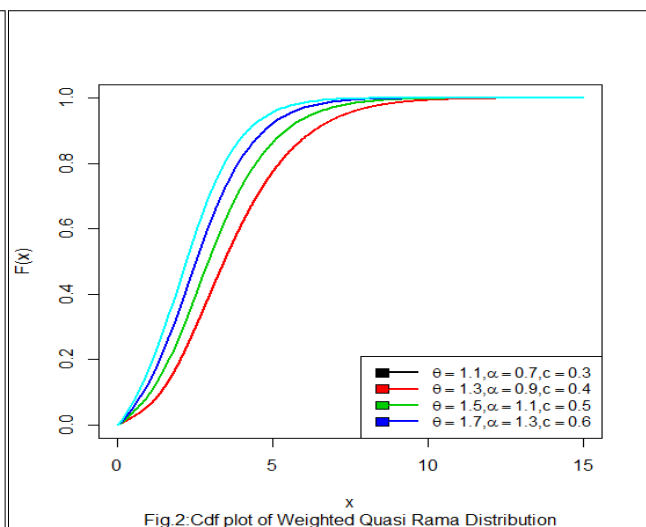


Fig.2: Cdf plot of Weighted Quasi Rama Distribution

3. Reliability Measures

In this portion, we will have derived some necessary reliability measures like reliability function, hazard function, reverse hazard function and Mills Ratio of weighted quasi Rama distribution.

3.1 Reliability function

The reliability function of executed new distribution is

$$R(x) = 1 - F_w(x; \theta, \alpha, c)$$

$$R(x) = 1 - \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right)$$

3.2 Hazard function

The hazard function or failure rate of proposed distribution is

$$h(x) = \frac{f_w(x; \theta, \alpha, c)}{1 - F_w(x; \theta, \alpha, c)}$$

$$h(x) = \frac{x^c \theta^{c+3} (\alpha + \theta x^3) e^{-\theta x}}{(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)) - (\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x))}$$

3.3 Reverse hazard function

The proposed function is

$$h_r(x) = \frac{f_w(x; \theta, \alpha, c)}{F_w(x; \theta, \alpha, c)}$$

$$h_r(x) = \frac{x^c \theta^{c+3} (\alpha + \theta x^3) e^{-\theta x}}{(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x))}$$

3.4 Mills Ratio

$$M.R = \frac{1}{h_r(x)} = \frac{(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x))}{x^c \theta^{c+3} (\alpha + \theta x^3) e^{-\theta x}}$$

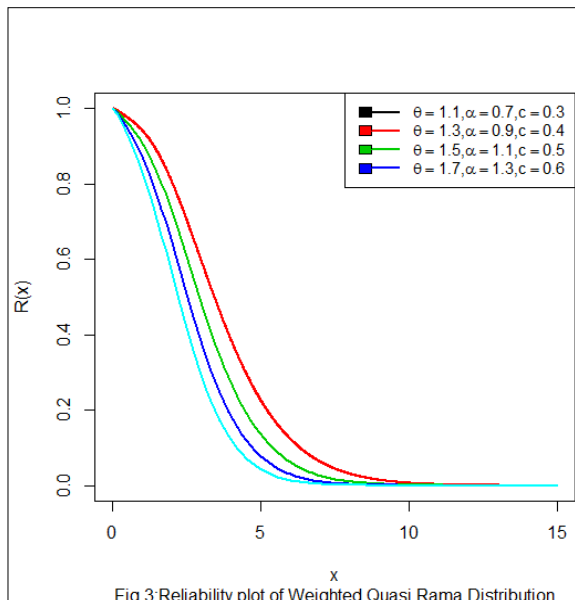


Fig.3: Reliability plot of Weighted Quasi Rama Distribution

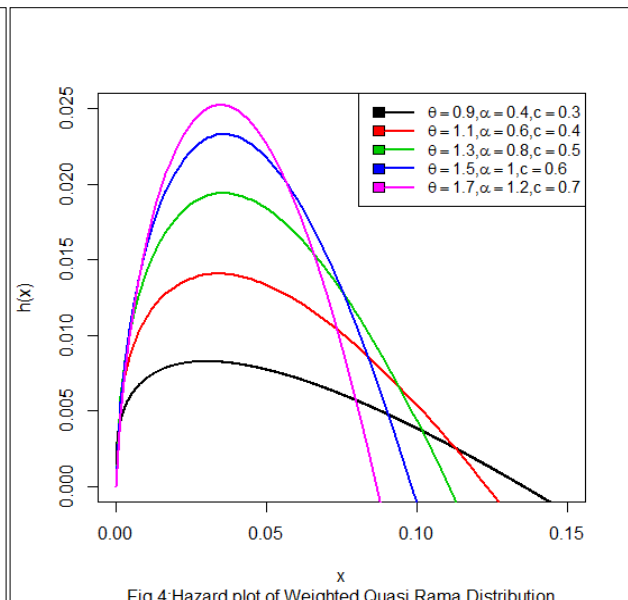


Fig.4: Hazard plot of Weighted Quasi Rama Distribution

4. Order Statistics

Order statistics is a significant concept in statistics which deals with arrangement of sample values in increasing order. Consider order statistics $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ of a random sample $X_1, X_2, \dots, X_n$ from continuous distribution has probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$, then its probability density function of r^{th} order statistics is

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) (F_X(x))^{r-1} (1 - F_X(x))^{n-r} \quad (8)$$

Therefore by applying equation (5) and (7) in equation (8), we will get required probability density function of r^{th} order statistics of weighted quasi Rama distribution given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \right) \\ \times \left(\frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right) \right)^{r-1} \\ \times \left(1 - \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right) \right)^{n-r}$$

Now the probability density function of higher order statistic $X_{(n)}$ of weighted quasi Rama distribution is given by

$$f_{x(n)}(x) = \frac{n x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \\ \times \left(\frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right) \right)^{n-1}$$

Now therefore the probability density function of first order statistic $X_{(1)}$ of weighted quasi Rama distribution is given by

$$f_{x(1)}(x) = \frac{n x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \\ \times \left(1 - \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \theta^2 \gamma(c+1, \theta x) + \gamma(c+4, \theta x) \right) \right)^{n-1}$$

5. Entropy

Entropy is a scientific term which deals with state of randomness or uncertainty in a system. Some important measures of entropy are

5.1 Renyi Entropy

The Renyi entropy is an important aspect in fractal dimension estimation which formulates the basis of the concept of generalized dimensions. This concept is also important in ecology and statistics as index of diversity. The concept is defined as

$$e(\zeta) = \frac{1}{1-\zeta} \log \left(\int f_w^\zeta(x; \theta, \alpha, c) dx \right), \quad \text{Where } \zeta > 0 \text{ and } \zeta \neq 1$$

$$e(\zeta) = \frac{1}{1-\zeta} \log \int_0^{\infty} \left(\frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \right)^{\zeta} dx$$

$$e(\zeta) = \frac{1}{1-\zeta} \log \left(\left(\frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)^{\zeta} \int_0^{\infty} x^{\zeta c} \left(\alpha + \theta x^3 \right)^{\zeta} e^{-\theta \zeta x} dx \right) \quad (9)$$

After the simplification of equation (9), we obtain

$$e(\zeta) = \frac{1}{1-\zeta} \log \left(\left(\frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)^{\zeta} \sum_{i=0}^{\infty} \binom{\zeta}{i} \alpha^{\zeta-i} \theta^i \frac{\Gamma(\zeta c + 3i + 1)}{(\theta \zeta)^{\zeta c + 3i + 1}} \right)$$

Obviously, the Tsallis entropy of proposed distribution will be determined by applying the following expression

$$S_{\tau} = \frac{1}{\tau-1} \left(1 - \int_0^{\infty} f_w^{\tau}(x; \theta, \alpha, c) dx \right)$$

$$S_{\tau} = \frac{1}{\tau-1} \left(1 - \int_0^{\infty} \left(\frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \right)^{\tau} dx \right)$$

$$S_{\tau} = \frac{1}{\tau-1} \left(1 - \left(\frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)^{\tau} \int_0^{\infty} x^{\tau c} \left(\alpha + \theta x^3 \right)^{\tau} e^{-\theta \tau x} dx \right) \quad (10)$$

After the simplification of equation (10), we obtain

$$S_{\tau} = \frac{1}{\tau-1} \left(1 - \left(\frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)^{\tau} \sum_{k=0}^{\infty} \binom{\tau}{k} \alpha^{\tau-k} \theta^k \frac{\Gamma(\tau c + 3k + 1)}{(\theta \tau)^{\tau c + 3k + 1}} \right)$$

6. Statistical Properties

In this portion, some structural properties such as moments, harmonic mean and moment generating function of weighted quasi Rama distribution has been derived. Those are

6.1 Moments

Consider the random variable X which represents the weighted quasi Rama distribution with parameters θ , α and c , then r^{th} order moment of executed distribution will be determined by

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f_w(x; \theta, \alpha, c) dx$$

$$\begin{aligned}
\mu_r' &= \int_0^\infty x^r \frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \\
\mu_r' &= \int_0^\infty \frac{x^{c+r} \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \\
\mu_r' &= \frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \int_0^\infty x^{c+r} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \\
\mu_r' &= \frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \left(\alpha \int_0^\infty x^{(c+r+1)-1} e^{-\theta x} dx + \theta \int_0^\infty x^{(c+r+4)-1} e^{-\theta x} dx \right) \quad (11)
\end{aligned}$$

After simplification, we obtain from equation (11)

$$\mu_r' = E(X^r) = \frac{\left(\alpha \theta^2 \Gamma(c+r+1) + \Gamma(c+r+4)\right)}{\theta^r \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \quad (12)$$

Therefore by substituting $r = 1, 2, 3$ and 4 in equation (12), we will get the mean and other three moments of weighted quasi Rama distribution

$$\mu_1' = \frac{\left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5)\right)}{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)}$$

$$\mu_2' = \frac{\left(\alpha \theta^2 \Gamma(c+3) + \Gamma(c+6)\right)}{\theta^2 \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)}$$

$$\mu_3' = \frac{\left(\alpha \theta^2 \Gamma(c+4) + \Gamma(c+7)\right)}{\theta^3 \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)}$$

$$\mu_4' = \frac{\left(\alpha \theta^2 \Gamma(c+5) + \Gamma(c+8)\right)}{\theta^4 \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)}$$

$$\text{Variance} = \frac{\left(\alpha \theta^2 \Gamma(c+3) + \Gamma(c+6)\right)}{\theta^2 \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} - \left(\frac{\left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5)\right)}{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4)\right)} \right)^2$$

$$S.D(\sigma) = \sqrt{\left(\frac{\left(\alpha \theta^2 \Gamma(c+3) + \Gamma(c+6) \right)}{\theta^2 \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} - \left(\frac{\left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)}{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)^2 \right)}$$

6.2 Harmonic mean

The harmonic mean of newly executed distribution will be determined by

$$H.M = E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} f_w(x; \theta, \alpha, c) dx$$

$$H.M = \int_0^{\infty} \frac{1}{x} \frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$H.M = \int_0^{\infty} \frac{x^{c-1} \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$H.M = \frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \int_0^{\infty} x^{c-1} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$H.M = \frac{\theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha \int_0^{\infty} x^{c-1} e^{-\theta x} dx + \theta \int_0^{\infty} x^{c+2} e^{-\theta x} dx \right) \quad (13)$$

After the simplification of equation (13), we obtain

$$H.M = \frac{\theta \left(\alpha \theta \Gamma(c+1) + \Gamma(c+3) \right)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)}$$

6.3 Moment generating function and characteristic function

Consider the random variable X which represents weighted quasi Rama distribution, then moment generating function of established distribution will be determined as

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f_w(x; \theta, \alpha, c) dx$$

By Taylor's series

$$M_X(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \dots \right) f_w(x; \theta, \alpha, c) dx$$

$$M_X(t) = \int_0^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f_w(x; \theta, \alpha, c) dx$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j'$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\frac{\left(\alpha \theta^2 \Gamma(j+c+1) + \Gamma(j+c+4) \right)}{\theta^j \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)$$

$$M_X(t) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \sum_{j=0}^{\infty} \frac{t^j}{j! \theta^j} \left(\alpha \theta^2 \Gamma(j+c+1) + \Gamma(j+c+4) \right)$$

Obviously, the characteristic function of weighted quasi Rama distribution is defined as

$$M_X(it) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \sum_{j=0}^{\infty} \frac{it^j}{j! \theta^j} \left(\alpha \theta^2 \Gamma(j+c+1) + \Gamma(j+c+4) \right)$$

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are known as classical curves which in earlier times were applied in economics but today it is a powerful tool and has several applications in different fields. The bonferroni and Lorenz curves should be defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f_w(x; \theta, \alpha, c) dx$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f_w(x; \theta, \alpha, c) dx$$

$$\text{Here } \mu_1' = \frac{\left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)}{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)}$$

$$B(p) = \frac{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)}{p \left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \int_0^q x \frac{x^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)}{p \left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \int_0^q \frac{x^{c+1} \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta^{c+4}}{p \left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \int_0^q x^{c+1} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta^{c+4}}{p \left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \left(\alpha \int_0^q x^{c+1} e^{-\theta x} dx + \theta \int_0^q x^{c+4} e^{-\theta x} dx \right) \quad (14)$$

By putting $\theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$

Also $x = \frac{t}{\theta}$

After the simplification of equation (14), we obtain

$$B(p) = \frac{1}{p \left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \left(\alpha \theta^2 \gamma(c+2, \theta q) + \gamma(c+5, \theta q) \right)$$

$$L(p) = \frac{1}{\left(\alpha \theta^2 \Gamma(c+2) + \Gamma(c+5) \right)} \left(\alpha \theta^2 \gamma(c+2, \theta q) + \gamma(c+5, \theta q) \right)$$

8. Parameter Estimation and Fisher's Information Matrix

In this portion, the parameters of weighted quasi Rama distribution have been estimated by applying the technique of maximum likelihood estimation. Consider the random sample $X_1, X_2, \dots, X_n$ of size n from weighted quasi Rama distribution, then likelihood function should be derived as

$$L(x) = \prod_{i=1}^n f_w(x; \theta, \alpha, c)$$

$$L(x) = \prod_{i=1}^n \left(\frac{x_i^c \theta^{c+3}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \left(\alpha + \theta x_i^3 \right) e^{-\theta x_i} \right)$$

$$L(x) = \frac{\theta^{n(c+3)}}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)^n} \prod_{i=1}^n \left(x_i^c \left(\alpha + \theta x_i^3 \right) e^{-\theta x_i} \right)$$

The log likelihood function should be defined as

$$\begin{aligned} \log L &= n(c+3) \log \theta - n \log \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right) + c \sum_{i=1}^n \log x_i + \\ &+ \sum_{i=1}^n \log \left(\alpha + \theta x_i^3 \right) - \theta \sum_{i=1}^n x_i \end{aligned} \quad (15)$$

Now therefore by differentiating the log likelihood equation (15) with respect to parameters θ , α and c . The following normal equations must be established

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+3)}{\theta} - n \left(\frac{2\alpha\theta\Gamma(c+1)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right) + \sum_{i=1}^n \left(\frac{x_i^3}{\left(\alpha + \theta x_i^3 \right)} \right) - \sum_{i=1}^n x_i = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -n \left(\frac{\theta^2 \Gamma(c+1)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right) + \sum_{i=1}^n \left(\frac{1}{\left(\alpha + \theta x_i^3 \right)} \right) = 0$$

$$\frac{\partial \log L}{\partial c} = n \log \theta - n \psi \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right) + \sum_{i=1}^n \log x_i = 0$$

Here $\psi(\cdot)$ is the digamma function

Now it is quite important to describe here that above likelihood equations are complicated to solve through algebraically. Therefore, we use R and wolfram Mathematica for determining the required parameters.

In order to apply the asymptotic normality results for determining the confidence interval. We state if $\hat{\gamma} = (\hat{\theta}, \hat{\alpha}, \hat{c})$ which represents the MLE of $\gamma = (\theta, \alpha, c)$. The results can be analyzed

$$\sqrt{n}(\hat{\gamma} - \gamma) \rightarrow N_3(0, I^{-1}(\gamma))$$

Here $I^{-1}(\gamma)$ is Fisher's information matrix, i.e

$$I(\gamma) = -\frac{1}{n} \begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n(c+3)}{\theta^2} - n \left(\frac{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right) (2\alpha \Gamma(c+1)) - (2\alpha \theta \Gamma(c+1))^2}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)^2} \right) \\ - \sum_{i=1}^n \left(\frac{\left(x_i^3 \right)^2}{\left(\alpha + \theta x_i^3 \right)^2} \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n \left(\frac{\left(\theta^2 \Gamma(c+1) \right)^2}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)^2} \right) - \sum_{i=1}^n \left(\frac{1}{\left(\alpha + \theta x_i^3 \right)^2} \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi' \left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n}{\theta} - n\psi' \left(\frac{2\alpha \theta \Gamma(c+1)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) = -n\psi' \left(\frac{\left(\theta^2 \Gamma(c+1) \right)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)} \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = -n \left(\frac{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right) \left(2\theta \Gamma(c+1) - (2\alpha \theta \Gamma(c+1)) \left(\theta^2 \Gamma(c+1) \right) \right)}{\left(\alpha \theta^2 \Gamma(c+1) + \Gamma(c+4) \right)^2} \right)$$

$$- \sum_{i=1}^n \left(\frac{\left(x_i^3 \right)}{\left(\alpha + \theta x_i^3 \right)^2} \right)$$

Here $\psi(\cdot)$ is the first order derivative of digamma function.

Hence γ is not known, so $I^{-1}(\gamma)$ should be estimated by $I^{-1}(\hat{\gamma})$ and it should be applied to attain asymptotic confidence intervals for θ, α and c .

9.

Application

In this portion, we have determined and examined the goodness of fit of weighted quasi Rama distribution by analyzed and inspected a real lifetime data set in order to reveal that weighted quasi Rama distribution provides a quite satisfactory fit as compared over quasi Rama, two parameter Rama, Rama, exponential and Lindley distributions. The real data set is The following real data set reported by Bader and Priest (1982) which constitutes the failure stresses (in GPa) of 65 single carbon fibers of lengths 50 mm respectively and the real data set is given as under in table 1

Table 1: Data represents the failure stresses (in GPa) of 65 single carbon fibers reported by Bader and Priest (1982)

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.764	1.807
1.812	1.84	1.852	1.852	1.862	1.864	1.931	1.952	1.974	2.019
2.051	2.055	2.058	2.088	2.125	2.162	2.171	2.172	2.18	2.194
2.211	2.27	2.272	2.28	2.299	2.308	2.335	2.349	2.356	2.386
2.39	2.41	2.43	2.431	2.458	2.471	2.497	2.514	2.558	2.577
2.593	2.601	2.604	2.62	2.633	2.67	2.682	2.699	2.705	2.735
2.785	3.02	3.042	3.116	3.174					

In order to determine the model comparison criterions, the unknown parameters are also estimated thoroughly by applying the R software. To execute the performance of weighted quasi Rama distribution in comparison over quasi Rama, two parameter Rama, Rama, exponential and Lindley distributions, the considered criterions *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected) and $-2\log L$ have been applied. The distribution is good if it has smaller criterion values of *AIC*, *BIC*, *AICC* and $-2\log L$ as compared over other well known distributions. The criterion values are computed by applying the formulae as follows

$$AIC = 2k - 2 \log L, \quad BIC = k \log n - 2 \log L \quad \text{and} \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Here $-2\log L$ is the maximized value of log-likelihood function under considered model, n is the sample size and k is the number of parameters in statistical model.

Table 2: Shows Performance, Interpretation and Analysis of Fitted Distribution

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Weighted Quasi Rama	$\hat{\alpha} = 57.741001$ $\hat{\theta} = 13.551018$ $\hat{c} = 27.299467$	$\hat{\alpha} = 254.558607$ $\hat{\theta} = 2.534938$ $\hat{c} = 7.193256$	69.99966	75.99966	82.52282	76.3931
Quasi Rama	$\hat{\alpha} = 0.0010000$ $\hat{\theta} = 1.7819955$	$\hat{\alpha} = 0.0011000$ $\hat{\theta} = 0.0754906$	144.0618	148.0618	152.4106	148.2553
Two Parameter Rama	$\hat{\alpha} = 0.0010000$ $\hat{\theta} = 1.7812366$	$\hat{\alpha} = 0.0101000$ $\hat{\theta} = 0.0608739$	144.1092	148.1092	152.458	148.3027

Rama	$\hat{\theta} = 1.37708448$	$\hat{\theta} = 0.07644915$	193.4714	195.4714	197.6458	195.5348
Exponential	$\hat{\theta} = 0.44562451$	$\hat{\theta} = 0.05527264$	235.0765	237.0765	239.2509	237.1399
Lindley	$\hat{\theta} = 0.70672316$	$\hat{\theta} = 0.06482461$	213.9985	215.9985	218.1729	216.0619

It is quite clear and obvious from result given above in table 2 that the weighted quasi Rama distribution has smaller criterion values of *AIC*, *BIC*, *AICC* and $-2\log L$ in comparison over quasi Rama, two parameter Rama, Rama, exponential and Lindley distributions which illustrates and indicates that the weighted quasi Rama distribution provides a better fit over quasi Rama, two parameter Rama, Rama, exponential and Lindley distributions.

10. Conclusion

In this article, we have introduced and formulated a new distribution called as weighted quasi Rama distribution which has been established by applying the weighted technique to classical quasi Rama distribution. The newly developed distribution has been redressed with several statistical properties such as moments, shape of behavior of pdf and cdf, reliability function, hazard function, reverse hazard function, moment generating function, harmonic mean, order statistics, bonferroni and lorenz curves, Renyi and Tsallis entropy are presented and explored. The parameters of newly executed distribution are estimated by applying the maximum likelihood estimation. Furthermore, the applicability and flexibility of a newly proposed distribution has been examined and investigated by using a real data set and hence it is found from result that weighted quasi Rama distribution provides a quite better fit over quasi Rama, two parameter Rama, Rama, exponential and Lindley distributions.

References

1. Ahmad A, Ain SQ, Ahmad A, Tripathi R. The weighted inverse Ailamujia distribution with applications to real life data. Pak J Statist. 2022;38(4):451-471.
2. Almahhareez S, Alzoubi L. Length-biased Benrabia distribution with applications. Electronic Journal of Applied Statistical Analysis. 2024;17(2):255-277.
3. Alzoubi L. Length-biased Loai distribution: Statistical properties and application. Electronic Journal of Applied Statistical Analysis. 2024;17(2):278-301.
4. Andure (Yawale) NW, Ade RB. The new length biased Hamza distribution: Statistical properties and applications. International Journal of Statistics and Reliability Engineering. 2021;8(1):63-68.
5. Bashir S, Rasul M. Some properties of the weighted Lindley distribution. EPRA International Journal of Economic and Business Review. 2015;3(8):11-17.
6. Bashir S, Khan H. Characterization of the weighted power function distribution by reliability functions and moments. Research in Mathematics. 2023;10(1):2202023.
7. Badar MG, Priest AM. Statistical aspects of fiber and bundle strength in hybrid composites. In: Hayashi T, Kawata K, Umekawa S, editors. Progress in Science and Engineering Composites: Proceedings of ICCM-IV; Tokyo. Tokyo: ICCM-IV; 1982. p. 1129-1136.
8. Fisher RA. The effects of methods of ascertainment upon the estimation of frequencies. Annals of Eugenics. 1934;6:13-25.
9. Ganaie RA, Rajagopalan V. A new generalization of quasi gamma distribution with properties and applications. International Journal of Engineering and Advanced Technology. 2020;9(3):3855-3861.
10. Mudasir S, Bhat BA, Bhat AA, Ahmad SP. The weighted new Weibull Pareto distribution: Some characteristics and applications. International Journal of Scientific Research in Mathematical and Statistical Sciences. 2018;5(5):153-163.
11. Rao CR. On discrete distributions arising out of methods of ascertainment. In: Patil GP, editor. Classical and Contagious Discrete Distributions. Calcutta: Statistical Publishing Society; 1965. p. 320-332.
12. R Core Team. R: A language and environment for statistical computing. Version 3.5.3. Vienna (Austria): R Foundation for Statistical Computing; 2019. Available from: <https://www.R-project.org/>
13. Reshi JA, Ahmed A, Mir KA. Characterizations and estimations of size biased generalized Rayleigh distribution. Mathematical Theory and Modeling. 2014;4(6):87-101.
14. Saghir A, Hamedani GG. A generalized weighted Maxwell distribution: Properties and estimation. Pak J Statist. 2018;34(3):195-209.
15. Shakhathreh MK. A two-parameter weighted exponential distribution. Statistics and Probability Letters. 2012;82:252-261.
16. Shanker R, Das H, Shukla KK. A quasi Rama distribution. Model Assisted Statistics and Applications. 2023;18:267-277.
17. Shanker R. Rama distribution and its application. International Journal of Statistics and Applications. 2017;7(1):26-35.