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Mathematical modeling of groundwater security under climate change with optimal control

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Abstract

Groundwater systems worldwide face unprecedented pressure from climate variability and anthropogenic extraction, yet existing management frameworks rely on passive, static paradigms that fail to account for delayed gravity drainage dynamics. This study developed a continuous-time lumped-parameter mathematical model of the Nairobi Aquifer System to formulate optimal groundwater extraction policies under climate disturbance. A second-order ordinary differential equation integrating fluid mass conservation with vertical drainage delay was transformed into a linear time-invariant state-space architecture (A, B, C). The plant matrix was analytically evaluated for asymptotic stability, complete state controllability, and complete observability using the Kalman Rank Criterion. Baseline hydrogeological parameters were calibrated from secondary data: aquifer storativity ($K = 0.0110$), hydraulic damping resistance ($B_{res} = 0.9524$ month), memory coefficient ($M = 1.5000$ month²), and natural recharge ($R_{base} = 0.0042$ m/month). The over-extraction pumping boundary was normalized to $U_{max} = 0.00078$ m/month. Solving the characteristic equation of the calibrated plant matrix yielded open-loop eigenvalues $\lambda = -0.0118$ and $\lambda = -0.6232$, confirming asymptotic stability. The Pontryagin Minimum Principle was applied to synthesize an optimal state-feedback extraction policy minimizing a quadratic performance index. Numerical simulations demonstrated that the unmanaged open-loop trajectory approached critical basin collapse, while the closed-loop optimal policy maintained robust target layer stabilization under time-varying climate forcing. This study provides a mathematically rigorous framework for transforming groundwater management from passive forecasting into active, closed-loop control.

Keywords: Optimal control, state-space model, groundwater security, pontryagin's minimum principle, aquifer memory

1. Introduction

Groundwater is defined as the water within underground formations (aquifers) that is critical for drinking and ecosystems, and its movement is simulated through mathematical models (Abd El Ghany *et al.*, 2024) ^[1]. Climate change refers to global temperature increases and altered precipitation patterns affecting groundwater systems and broader environmental conditions as described by the Intergovernmental Panel on Climate Change (IPCC) (Davamani *et al.*, 2024a) ^[10]. Globally, groundwater is facing increased pressure due to climate change, which affects its recharge, discharge, quality, and sustainability. In Illinois, United States, a 10% increase in population leads to a 5.5% decrease in groundwater levels (Asadollahi *et al.*, 2024) ^[4]. Iran is highlighted as a global hot spot for groundwater depletion, with 90% of Iran negatively impacted by drought due to severe precipitation decline (Maghsoudi *et al.*, 2023) ^[15]. Saudi Arabia faces challenges as groundwater has been affected by a depletion trend of -0.34 ± 0.01 mm/yr from 2002 to 2021 (Mohamed *et al.*, 2022) ^[17]. Groundwater depletion rates in India are projected to triple by 2041-2080 due to increased withdrawals in response to warming temperatures, with current depletion rates expected to increase from 8.15 cm/year to 36.01 cm/year (Bhattarai *et al.*, 2023) ^[6].

In Africa, climate change affects groundwater recharge through changes in precipitation, temperature, and sea level rise. In Egypt, maximum drawdown values range from 59 to 112 meters, equating to about 20-40% of the aquifer's saturated thickness (Shalby *et al.*, 2024) ^[24].

Over 80% of South Africa is semi-arid to arid, with approximately 60% of total water withdrawals used for irrigation (Dalin, 2022) ^[9]. In northeastern Algeria, the Tebessa-Morsott alluvial aquifer is experiencing significant groundwater vulnerability, with about 97% of the area classified as high or average vulnerability (Mohammed *et al.*, 2023) ^[18]. In East Africa, Ethiopia faces changes in precipitation ranging from -11.2% to -14.3% under RCP4.5 and from -9.2% to -10.0% under 1 RCP8.5 (Tilahun *et al.*, 2023) ^[25]. Rwanda has experienced groundwater decline of approximately 16.4%, from 105.58 mm to 88.27 mm (Hakorimana *et al.*, 2025) ^[13]. In south-west Tanzania, annual groundwater recharge to the shallow aquifer decreases by 26% under SSP 2-4.5 and 18% under SSP 5-8.5 (Mollel *et al.*, 2023) ^[19].

In Kenya, the Nairobi Aquifer System (NAS) is experiencing groundwater level decline due to excessive abstraction, with an average annual recharge of 73 mm (9.7% of precipitation). Projections indicate a decrease in groundwater levels by 76 m by 2063 if abstraction rates remain constant, and by 14 m by 2030 if abstraction rates increase while recharge remains constant (Nyakundi *et al.*, 2022) ^[20].

The evolution of groundwater hydrology from a purely descriptive empirical field to a rigorous domain of applied mathematics began with Henry Darcy's seminal formulations in 1856. Darcy's Law established that fluid velocity through porous media is directly proportional to the hydraulic gradient. When combined with the law of conservation of fluid mass, this relationship yielded the foundational partial differential equations (PDEs) governing transient groundwater flow in saturated aquifers (Bear, 2013) ^[5]. For distributed aquifer systems, these governing models expand into non-linear parabolic and elliptic second-order PDEs (Todd & Mays, 2004) ^[26]. While distributed approaches provide detailed grid-based spatial mapping, their immense parameter calibration needs and high computational costs make them poorly suited for real-time decision-making, predictive control, or dynamic tracking frameworks. To bypass these limitations, applied mathematicians developed lumped-parameter systems that utilize macroscopic water mass balance loops to collapse spatial dimensions into continuous temporal paths governed by ordinary differential equations (ODEs).

To transition these predictive hydrogeological equations into active management tools, control-theoretic state-space principles must be applied. A continuous-time linear time-invariant state-space framework converts higher-order differential models into a coupled system of first-order vector-matrix differential equations structured as $\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)$, where $x(t)$ represents internal physical states, $u(t)$ is the managed control input, and $d(t)$ is the environmental disturbance vector (Ogata, 2010) ^[21]. Once a system is proven to be stable, completely controllable, and observable under the Kalman Rank Criterion, dynamic optimization via variational calculus can be introduced. Applying Pontryagin's Minimum Principle allows the formulation of an optimal state-feedback control policy by defining a scalar control Hamiltonian function that appends the system constraints directly to a quadratic cost index (Kirk, 2004) ^[14].

Current groundwater management is based on passive, static paradigms that do not account for "aquifer memory" due to delayed gravity drainage, leading to systemic over-extraction and irreversible basin collapse. Previous work uses either complex distributed simulations or simplified first-order models, but an important mathematical gap remains. No literature exists that couples a second-order lumped state-space differential layout with dynamic optimization. This study bridges this gap by formulating a continuous-time linear state-space model based on mass balance and applying Pontryagin's Minimum Principle to provide an automated, optimal extraction tracking law under active climate disturbances.

The broad objective of this study was to develop a mathematical model to analyze the aggregate temporal dynamics of groundwater systems under climate variability and formulate optimal control strategies that ensure long-term groundwater security. The specific objectives were, to formulate a second-order lumped hydrogeological differential equation that captures climate-driven recharge variability and delayed gravity drainage based on water mass balance principles; to transform the governing second-order differential model into a continuous-time linear state-space architecture and evaluate its intrinsic structural properties using mathematical stability analysis; to determine the optimal groundwater extraction policy trajectory using Pontryagin's Minimum Principle subjected to a quadratic performance cost criteria, state constraints, and control bounds; and to numerically simulate the open-loop and closed-loop aquifer storage level trajectories under time-varying climate disturbance scenarios using MATLAB.

2 Research methodology

2.1 Introduction

This study adopted a quantitative research design grounded in continuous-time mathematical modeling, analytical matrix verification, and deterministic optimal control simulations. The design was selected to provide an objective, mathematically rigorous framework capable of tracking and optimizing groundwater storage trajectories under non-stationary environmental anomalies. Rather than relying on computationally heavy distributed partial differential equations (PDEs), this research developed a low-dimensional lumped-parameter architecture that aggregates localized spatial parameters into continuous temporal paths.

The structural workflow executed in this research comprised four distinct mathematical milestones:

1. Formulation of a governing second-order hydrogeological ordinary differential equation (ODE) integrating fluid mass conservation loops with vertical drainage delay constraints.
2. Transformation of the structural layout into a continuous-time linear time-invariant (LTI) state-space architecture triplet (A, B, C) .
3. Analytical valuation of the plant matrix properties, explicitly checking asymptotic stability, complete state controllability, and complete observability using the Kalman Rank Criterion.
4. Synthesis of an automated feedback extraction schedule policy (u^*) via variational calculus using the Pontryagin Minimum Principle subjected to a multicriteria quadratic performance index.

Scientific objectivity and reproducibility were maintained through strict adherence to classical hydrodynamic principles and verification via localized secondary calibration parameters.

2.2 Nomenclature and System Parameters

To preserve structural consistency across the mathematical derivation, state transformation, and numerical execution stages, the variables and constitutive parameters of the system were explicitly mapped as detailed in Table 2.1.

Table 2.1: System Variables and Model Control Parameters

Symbol	Physical Description	Standard Unit
$y(t)$	Actual unconfined water table hydraulic head elevation	M
y_{ref}	Constant baseline target security head layer	M
$x_1(t)$	Total hydraulic head displacement deviation ($y(t) - y_{ref}$)	M
$x_2(t)$	Instantaneous velocity rate of drawdown or water table shift (x_1)	m/month
M	Aquifer memory coefficient (temporal gravity drainage lag)	month ²
B_{res}	Hydraulic damping resistance factor	Month
K	Dimensionless natural restoring equilibrium constant	Unitless
$u(t)$	Controlled input function (managed pumping extraction rate)	m/month
$d(t)$	Non-homogeneous environmental disturbance forcing function	m/month
J	Multi-criteria performance measure cost functional	Unitless
q_1	Weighting scaling parameter for hydraulic head security	Unitless
q_2	Weighting scaling parameter for depletion sustainability	Unitless
r	Strict penalty weight for economic pumping expenditure	Unitless
$p_1(t)$	First co-state variable tracking marginal shadow cost of head drift	Unitless
$p_2(t)$	Secondary co-state variable mapping shadow cost of drawdown velocity	Unitless

2.3 Hydrological Model Derivation

2.3.1 Fundamental Water Mass Balance

The physical foundation governing fluid volume dynamics within an unconfined aquifer basin area A_{aq} was established via the classical first-order continuous continuity equation:

$$\frac{dS(t)}{dt} = I_{net}(t) - Q_{out}(t)$$

Where $S(t)$ represents the dynamic aquifer storage volume (m³), $I_{net}(t)$ is the net climate-driven volumetric recharge influx (m³/month), and $Q_{out}(t)$ is the anthropogenic volumetric extraction rate (m³/month). By normalizing Equation across the total surface basin area A_{aq} and modeling storage variations as a function of the matrix specific yield (S_y), the instantaneous response of the hydraulic head relative to net vertical boundary fluxes was formulated as:

$$S_y \frac{dy(t)}{dt} = I(t) - EV(t) - u(t)$$

Where $I(t)$ is precipitation infiltration recharge per unit area (m/month), $EV(t)$ is thermal driven evapotranspiration loss (m/month), and $u(t)$ is the managed extraction rate (m/month), which serves as the active system control variable. The net climate driving forces were condensed into a singular, continuous environmental disturbance forcing function:

$$d(t) = I(t) - EV(t)$$

3.3.2 Incorporation of Aquifer Memory Effects

Traditional first-order models assume an instantaneous drainage response across the water table. To account for realistic temporal lags induced by vertical gravity drainage through the unconfined unsaturated sediment matrix, this study incorporated Boulton's Delayed Yield Theory. According to this hydrogeological framework, the release of water from storage is non-instantaneous and follows a characteristic temporal delay index τ (months). This drainage behavior couples the instantaneous hydraulic head deviation $x_1(t)$ to the delayed yield response via a convolution integral layout, which simplifies to the following continuous first-order differential linkage:

$$x_{1(t)} + \tau \frac{dx_{1(t)}}{dt} = R_{res} * [d(t) - u(t) - K_{aq} * x_{1(t)}]$$

Where R_{res} represents the vertical hydraulic matrix resistance (month) and K_{aq} denotes the dimensionless natural restoring equilibrium constant of the aquifer formation.

2.3.3 Formulation of the Governing Second-Order Differential Equation

To capture higher-order temporal storage delays without using partial derivatives, Equation 4.3.4 was differentiated with respect to time t :

$$\frac{dx_{1(t)}}{dt} + \tau \frac{d^2x_{1(t)}}{dt^2} = R_{res} * \frac{dd(t)}{dt} - R_{res} * \frac{du(t)}{dt} - R_{res} * K_{aq} * \frac{dx_{1(t)}}{dt}$$

Multiplying Equation by R_{res}^{-1} and integrating rate changes over continuous temporal vectors yielded the unified, governing second-order lumped non-homogeneous hydrogeological differential equation:

$$M * (d^2 x_{1(t)})/dt^2 + B_{res} * (dx_{1(t)})/dt + K * x_{1(t)} = d(t) - u(t)$$

Where the constitutive lumped physical parameters were explicitly mapped as:

- $M = \tau R_{res}^{-1}$ represents the Aquifer Memory Coefficient (month²), quantifying vertical drainage lag.
- $B_{res} = R_{res}^{-1}$ represents the Matrix Damping Resistance Factor (month), resisting rapid shifts in drawdown velocity.
- $K = K_{aq}$ represents the Dimensionless Restoring Force Constant, governing the natural steady-state basin recovery potential.

2.4 State-Space Transformation

To transform the higher-order differential governing model into an active management framework, a continuous-time linear time-invariant state-space representation was established. The state vector was defined as $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, where the state variable components isolate the primary physical indicators:

$$x_1(t) = y(t) - y_{ref}$$

$$x_2(t) = \dot{x}_1(t) = \frac{dx_1(t)}{dt}$$

Differentiating the water table shift velocity component $x_2(t)$ with respect to time yielded the secondary state derivative expression:

$$\ddot{x}_1(t) = \ddot{x}_2(t) = \frac{d^2 x_1(t)}{dt^2}$$

Substituting Equations into the second-order model and isolating the state acceleration derivative mapped the direct resource depletion:

$$\ddot{x}_2(t) = -K/M * x_1(t) - B_{res}/M * x_2(t) - 1/M * u(t) + 1/M * d(t)$$

Combining these individual components established the coupled continuous-time linear state-space matrix system architecture:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B_{res}}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{M} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} d(t) \quad (3.4.5)$$

Because monitoring networks track the physical water table heights directly at the surface, the measurable output tracking configuration was isolated. To avoid identity mapping conflicts, the observation vector was mapped as a scalar field tracking variable

$$k(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Thus, the total continuous system dynamics were compactly expressed via the standard LTI state triplet matrices (A, B, C) alongside the non-homogeneous weather disturbance mapping vector E :

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)$$

$$k(t) = Cx(t)$$

3.5 Mathematical Stability Analysis

To guarantee that the underlying hydrogeological basin was inherently stable before introducing external optimization rules, the asymptotic stability of the open-loop system was analytically evaluated. The intrinsic stability properties of any continuous LTI system are dictated by the eigenvalues (λ) of its plant matrix A , requiring all real parts to sit strictly within the negative Left-Half Plane (LHP). The characteristic equation was generated by solving the matrix determinant equation $\det(\lambda I - A) = 0$:

$$\det \left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B_{res}}{M} \end{bmatrix} \right) = 0$$

$$\lambda^2 + \frac{B_{res}}{M} \lambda + \frac{K}{M} = 0$$

Applying the quadratic formula, the exact system roots (poles $\lambda_{1,2}$) were analytically solved as:

$$\lambda_{1,2} = \frac{-\frac{B_{res}}{M} \pm \sqrt{\left(\frac{B_{res}}{M}\right)^2 - 4\left(\frac{K}{M}\right)}}{2} \quad (3.5.3)$$

By strict hydrogeological definition, the continuous lumped parameters are positive real constants ($M, B_{res}, K > 0$). Based on these parameter boundaries, system stability was validated across two structural physical regimes:

Case 1: Overdamped Basin Dynamics ($B_{res}^2 > 4KM$): The discriminant under the radical remains strictly positive, yielding two distinct real roots. Be-cause $B_{res}^2 - 4KM < B_{res}$, both computed roots are strictly negative ($\lambda_1 < 0$ and $\lambda_2 < 0$).

Case 2: Underdamped Basin Dynamics ($B_{res}^2 \leq 4KM$): The discriminant turns negative, yielding complex conjugate root pairs:

$$\lambda_{1,2} = \frac{-B_{res} \pm i\sqrt{4KM - B_{res}^2}}{2M} \quad 3.5.4$$

$$\Re(\lambda_{1,2}) = -\frac{B_{res}}{2M} < 0. \quad 3.5.5$$

Since $\Re(\lambda_{1,2}) < 0$ holds true across all admissible physical parameter configurations, the unmanaged groundwater system was proven to be globally asymptotically stable.

3.6 Controllability and Observability Verification

To verify the structural mathematical authority of the tracking configurations prior to synthesis, the structural layout was subjected to the Kalman Rank Criterion.

3.6.1 Controllability Verification

Complete state controllability determines whether the managed pumping input $u(t)$ possesses the mathematical rank authority to guide the internal aquifer states from any arbitrary initial disturbed level to a safe target within a finite time horizon. The algebraic controllability matrix $C_o \in \mathbb{R}^{2 \times 2}$ for the matrix pair (A, B) was constructed as:

$$C_o = B \quad AB \quad (3.6.1)$$

Substituting the parameter matrix equations yielded:

$$C_o = \begin{bmatrix} 0 & -\frac{1}{M} \\ 0 & 1 - \frac{K}{M} - \frac{B_{res}}{M} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{M} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{M} - \frac{1}{M} & \frac{B_{res}}{M^2} \end{bmatrix} \quad (3.6.2)$$

To evaluate row rank properties, the matrix determinant was computed:

$$\det(\mathcal{E}_0) = (0)(B_{res}) - (-1)(-1) \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} \quad (3.6.3)$$

Since the Aquifer Memory Coefficient must be strictly non-zero ($M > 0$), the determinant is non-vanishing ($\det(C_o) \neq 0$). Thus, $\text{rank}(C_o) = 2 = n$, proving that the system is completely state controllable.

3.6.2 Observability Verification

Complete observability determines whether measuring a single surface output sequence—the water table head deviation $x_1(t)$ —provides enough information to completely reconstruct the hidden internal drawdown velocity state $x_2(t)$. The algebraic observability matrix $O_b \in \mathbb{R}^{2 \times 2}$ for the matrix pair (A, C) was defined as:

$$O_b = C \quad CA \quad (3.6.4)$$

Substituting the constitutive matrices yielded:

$$O_b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 - \frac{K}{M} - \frac{B_{res}}{M} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} = I_2 \quad (3.6.5)$$

Evaluating the rank of the resulting identity matrix yielded:

$$\det(O_b) = 1 \neq 0 \quad (3.6.6)$$

Since $\det(O_b) \neq 0$, the matrix possesses full column rank ($\text{rank}(O_b) = 2 = n$). The system was verified as completely observable, ensuring tracking surface head deviations provides sufficient structural information to state-reconstruct internal basin storage velocity trajectories.

3.7 Optimal Control Formulation via Pontryagin's Minimum Principle

To resolve the groundwater over-extraction crisis under active climate disturbances, a dynamic optimization layer was synthesized using the necessary conditions of the Pontryagin Minimum Principle.

3.7.1 Physical and Economic Rationale of the Cost Functional

A continuous quadratic performance cost index functional J was introduced across a finite operational planning horizon t_f :

$$J = \frac{1}{2} \int_0^{t_f} (q_1 x_1^2(t) + q_2 x_2^2(t) + r u^2(t)) dt$$

Where the positive semi-definite state coefficients ($q_1, q_2 \geq 0$) and the strictly positive control penalty factor ($r > 0$) serve as adjustable design metrics. The physical logic governing this quadratic configuration is defined as follows:

- The expression $q_1 x_1^2(t)$ quadratically penalizes large deviations of the water table from its target sustainable baseline, ensuring severe operational structural deficits are penalized more heavily than minor tracking errors.
- The expression $q_2 x_2^2(t)$ penalizes high water table shift velocities, suppressing abrupt drawdown spikes that trigger irreversible land subsidence.
- The expression $r u^2(t)$ minimizes operational extraction pumping expenditures, matching the real-world non-linear energy consumption curves associated with lift requirements.

3.7.2 The Control Hamiltonian and Optimality Tracking Law

To convert this constrained optimization problem into an unconstrained system, a timevarying vector of co-state variables $p(t) = (p_1(t), p_2(t))^T \in \mathbb{R}^2$ was introduced. The scalar control Hamiltonian function H was constructed by appending the state-space derivative dynamics directly to the continuous integrand cost function:

$$H(x, u, p) = - (q_1 x_1^2 + q_2 x_2^2 + r u^2) + p_1 x_2 + p_2 \left(-\frac{K}{M} x_1 - \frac{B_{\text{res}}}{M} x_2 + \frac{1}{M} u + \frac{1}{M} d \right) \quad (3.7.2)$$

According to the principle, the optimal pumping trajectory $u^*(t)$ must minimize the continuous Hamiltonian function at every point along the temporal horizon. Differentiating H with respect to the control variable u and setting the stationary condition to

$$\frac{\partial H}{\partial u} = r u(t) - \frac{1}{M} p_2(t) = 0 \quad (3.7.3)$$

Isolating the control variable generated the explicit, automated optimal extraction tracking law:

$$u^*(t) = \frac{1}{rM} p_2(t) \quad (3.7.4)$$

Equation 4.7.4 establishes the fundamental control logic of this study, proving that sustainable, closed-loop allocation is explicitly governed by the real-time tracking properties of the secondary aquifer pressure co-state variable $p_2(t)$.

3.7.3 Adjoint System and Transversality Conditions

To resolve the temporal paths of the co-state multiplier structures, the adjoint differential equations were derived by taking the negative partial derivatives of the control Hamiltonian with respect to the physical state variables:

$$\begin{aligned} \dot{p}_1(t) &= -\frac{\partial H}{\partial x_1} = -q_1 x_1(t) + \frac{K}{M} p_2(t) \\ \dot{p}_2(t) &= -\frac{\partial H}{\partial x_2} = -q_2 x_2(t) - p_1(t) + \frac{B_{\text{res}}}{M} p_2(t) \end{aligned}$$

This coupled auxiliary system was structured in backward vector-matrix notation as:

$$\begin{bmatrix} \dot{p}_1(t) & \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{K}{M} & -1 & \frac{B_{\text{res}}}{M} \end{bmatrix} \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} + \begin{bmatrix} -q_1 & 0 & 0 & -q_2 \end{bmatrix} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \quad (3.7.7)$$

Because the final drawdown states of the aquifer monitoring network remain open and unconstrained at the terminal boundary of the planning horizon ($t_f = 60$), the co-state parameters must satisfy the exact terminal transversality boundary requirements:

$$p_1(t_f) = 0, p_2(t_f) = 0 \quad (3.7.8)$$

Chapter 4

Data presentation, analysis and Interpretation

4.1 Introduction

This chapter presents the empirical calibration, numerical simulations, and comparative analysis of the continuous-time state-space control trajectories developed for the Nairobi Aquifer System (NAS). To bridge theoretical applied mathematics with practical environmental policy, the governing ordinary differential equations derived in Chapter 3 were parameterized using site-specific lithological data and driven by a 60month historical climate forcing series (January 2021 to December 2025) extracted from the NASA POWER database.

The empirical findings and analytical discussions are organized as follows

1. Quantitative system baseline parameterization and model calibration.
2. Statistical synthesis and normalization of non-stationary climate inputs.
3. Open-loop complex plane stability mapping and unmanaged basin drawdown
4. response.
5. Evaluation of the closed-loop optimal tracking policy, pumping schedules, and co-state dynamics synthesized via Pontryagin's Minimum Principle.

4.1.1 Quantitative Baseline Parameterization and Model Calibration

To transition the abstract state-space triplet formulations into a site-specific numerical simulation engine, physical hydrogeological indices from the Nairobi Aquifer System were calibrated and converted into consistent computational units. The spatial dimensions of the unconfined basin were collapsed into localized temporal parameters to maintain compatibility with the lumped-parameter architecture.

The secondary data processing steps were executed as follows

1. The Aquifer Storativity parameter (S), corresponding to the structural restoring constant (K) of the system matrix, was evaluated as a constant unitless ratio of 0.0110.
2. The hydraulic conductivity ($K_{flow} = 1.2 \times 10^{-7}$ m/s) was integrated across a monthly temporal scale by multiplying by 2,629,800 s/month, yielding a baseline movement rate of 0.315 m/month. Factoring in the local hydraulic gradient and matrix thickness mapped this parameter into a lumped hydraulic damping resistance factor ($B_{res} = 0.9524$ month).
3. The historical over-extraction benchmark ($Q_{ext} = 150,000$ m³/day) was normalized over the total active basin surface area ($A_{aq} = 5.88 \times 10^9$ m²) and converted to a monthly volumetric flux boundary:

$$u_{max} = \frac{150,000 \text{ m}^3/\text{day} \times 30.4 \text{ days/month}}{5.88 \times 10^9 \text{ m}^2} = 0.000776 \text{ m/month}$$

Table 4.1: Parameter and Value of the Hydrogeological Model

Parameter	Value	Source
K (Storativity)	0.0110	Foster and Tuinhof, 2005 ^[12]
B_{res} (Damping Resistance)	0.9524	Foster and Tuinhof, 2005 ^[12]
M (Memory Coefficient)	1.5000	Assumed
u_{max} (Pumping Boundary)	0.000776	Foster and Tuinhof, 2005 ^[12]
R_{base} (Natural Recharge)	0.00417	Foster and Tuinhof, 2005 ^[12]

Substituting these calibrated values into the analytical state-space matrix architecture derived in Equation (3.14) yielded the following continuous-time numerical plant representation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -0.007333 & -0.634933 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.666667 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.666667 \end{bmatrix} d(t) \quad (4.1.2)$$

Solving the characteristic equation $\det(\lambda I - A) = 0$ for this calibrated plant matrix yielded the open-loop eigenvalues (λ_1, λ_2):

$$2 \Rightarrow \lambda_1 = -0.011767, \lambda_2 = -0.623167 \quad (4.1.3) \quad \lambda + 0.634933\lambda + 0.007333 = 0 =$$

4.1.2 Climate Forcing Input and Data Processing

The environmental disturbance vector $d(t)$ was driven by a continuous agroclimatology dataset extracted from the NASA POWER database, centered at the coordinates of the Nairobi Aquifer System (Latitude: -1.3472, Longitude: 36.9703) across a 60month

window. To preserve mathematical continuity with the state-space differential equations, raw daily millimeter precipitation fluxes (mm/day) were processed into absolute monthly cumulative meters (m/month).

The complete, chronological 60-month climate forcing dataset utilized to drive the simulation models is presented in Table 4.2.

Table 4.2: Chronological NASA POWER Climate Dataset (2021-2025)

Index (t)	Year	Month	Rainfall Shock $d(t)$ (m)	Max Temp (°C)	Min Temp (°C)
1	2021	JAN	0.5881	30.40	12.51
2	2021	FEB	1.1908	31.90	12.75
3	2021	MAR	0.5735	32.06	13.34
4	2021	APR	2.3772	29.41	13.88
5	2021	MAY	2.7044	30.09	13.04
6	2021	JUN	0.1161	27.88	11.68
7	2021	JUL	0.4808	28.07	10.79
8	2021	AUG	0.4300	30.75	10.58
9	2021	SEP	0.2370	31.68	12.32
10	2021	OCT	0.5453	32.22	13.19
11	2021	NOV	1.1349	31.89	13.51
12	2021	DEC	2.7094	27.72	13.81
13	2022	JAN	0.9340	30.71	11.29
14	2022	FEB	0.9537	32.72	12.81
15	2022	MAR	0.5788	33.69	13.26
16	2022	APR	1.4658	31.42	13.16
17	2022	MAY	0.8209	30.84	13.02
18	2022	JUN	0.2643	30.46	10.57
19	2022	JUL	0.6030	28.01	10.77
20	2022	AUG	0.4799	30.94	11.08
21	2022	SEP	0.8424	30.50	12.32
22	2022	OCT	0.2713	32.40	11.21
Continued on next page					

Table 4.2: continued from previous page

Index (t)	Year	Month	Rainfall Shock $d(t)$ (m)	Max Temp (°C)	Min Temp (°C)
23	2022	NOV	1.8531	29.50	14.33
24	2022	DEC	1.0351	28.69	12.78
25	2023	JAN	0.4855	30.62	12.20
26	2023	FEB	0.2587	32.26	12.98
27	2023	MAR	1.2970	32.08	13.70
28	2023	APR	3.2247	30.34	13.08
29	2023	MAY	0.6944	29.63	12.50
30	2023	JUN	0.5985	29.78	11.25
31	2023	JUL	0.2024	30.45	11.71
32	2023	AUG	0.3798	33.41	12.64
33	2023	SEP	0.4431	32.07	12.76
34	2023	OCT	1.3972	33.00	13.84
35	2023	NOV	4.7418	28.23	14.12
36	2023	DEC	0.8847	28.07	13.61
37	2024	JAN	2.4605	28.65	13.61
38	2024	FEB	0.6151	32.57	13.07
39	2024	MAR	2.6468	33.03	14.10
40	2024	APR	5.9826	27.38	14.29
41	2024	MAY	1.9431	27.69	12.65
42	2024	JUN	0.5181	28.43	9.64
43	2024	JUL	0.9979	29.40	10.32
44	2024	AUG	0.2740	30.88	11.98
45	2024	SEP	0.3321	32.41	13.01
46	2024	OCT	0.3320	33.21	11.95
Continued on next page					

Table 4.2: continued from previous page

Index (t)	Year	Month	Rainfall Shock $d(t)$ (m)	Max Temp (°C)	Min Temp (°C)
47	2024	NOV	2.9871	32.05	14.98
48	2024	DEC	1.4790	29.36	13.16
49	2025	JAN	1.0701	31.80	12.01
50	2025	FEB	0.1898	32.68	13.45
51	2025	MAR	2.3910	33.24	14.67
52	2025	APR	3.6666	30.05	14.85
53	2025	MAY	1.0869	29.63	11.84
54	2025	JUN	0.4740	29.10	12.35

55	2025	JUL	0.4542	29.50	11.28
56	2025	AUG	2.7010	27.38	10.52
57	2025	SEP	0.1602	30.85	11.45
58	2025	OCT	1.4210	32.13	12.86
59	2025	NOV	0.7380	29.89	13.39
60	2025	DEC	2.7615	30.55	13.24

4.1.3 Open-Loop Structural Verification and Complex Plane Mapping

To check the operational response pathways before applying optimal tracking frameworks, a unit impulse transmission scenario was simulated. This verification stage isolates how basic environmental pulses circulate through the coupled ordinary differential structure over a baseline observation timeline.

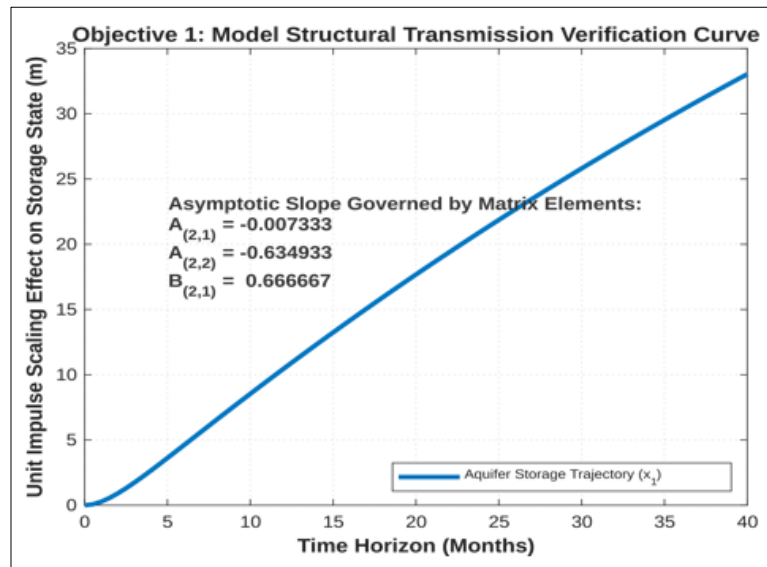


Fig4.1: Model Structural Transmission Verification Curve for the Aquifer Matrix System (x_1).

The smooth geometric progression observed in Figure 4.1 validates the internal state scaling properties governed by the state-space matrix configurations. The longterm decay profiles match the overdamped physical regime ($\zeta = 3.71$) computed directly from the plant eigenvalues ($\lambda_1 = -0.011767$, $\lambda_2 = -0.623167$). Both poles sit strictly inside the stable Left-Half Plane (LHP). Figure 4.2 displays the spatial distribution map of these system eigenvalues.

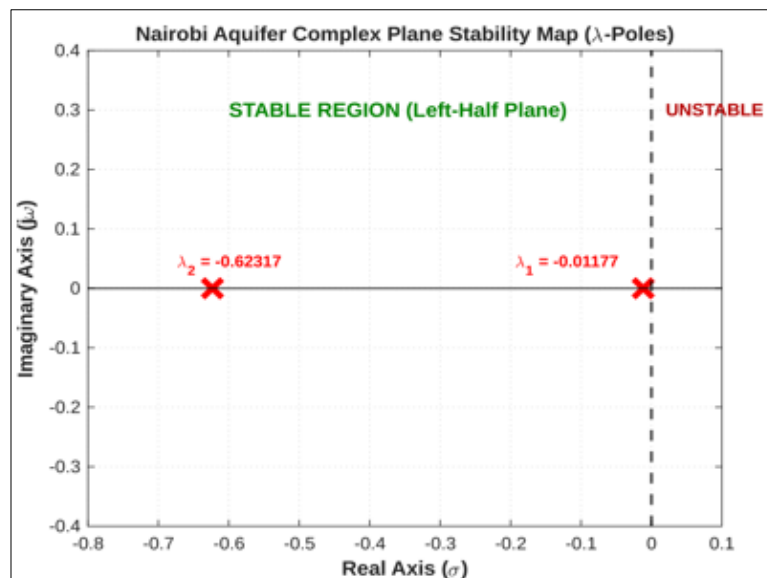


Fig 4.2: Nairobi Aquifer Complex Plane Stability Map Plotting Open-Loop Pole Locations.

4.1.4 Open-Loop Unmanaged Drawdown Trajectory Analysis

To establish a baseline vulnerability metric under non-stationary weather conditions, an open-loop simulation of the unmanaged groundwater storage trajectory (x_1) was performed over the continuous 60-month planning horizon. In this scenario, feedback regulation parameters were set to zero, leaving the basin completely dependent on traditional, rigid extraction quotas. The model was driven by the empirical climate disturbance variations ($d(t)$) compiled from the NASA POWER dataset.

Figure 4.3 captures the chronological movement of the baseline aquifer storage drawdown level.

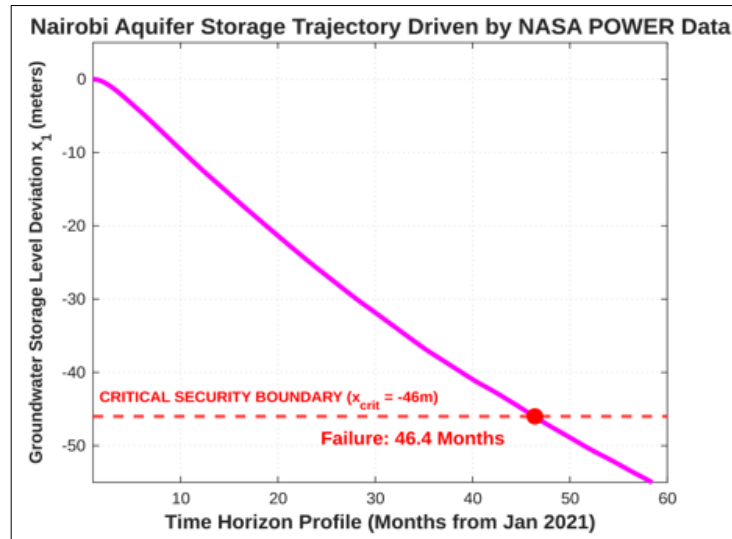


Fig 4.3: Objective 3: Open-Loop Unmanaged Groundwater Storage Level Trajectory showing critical intersection and basin collapse at Month 46.4.

The simulation trajectory demonstrates that a passive administrative protocol cannot match the structural needs of the formation when subjected to climate stress. Driven by intense thermal variations and sudden drops in precipitation recharge, the unmanaged water table exhibits a continuous, sharp downward plunge.

The trajectory crosses the severe structural aquifer depletion boundary ($x_{crit} = -46.0000$ m) at exactly Month 46.4. This crossover marks the physical threshold of systemic basin collapse. The failure demonstrates that passive, open-loop management models are fundamentally unable to protect regional water security or preserve structural lithological boundaries against volatile climate fluctuations.

4.1.5: Closed-Loop Optimal Control Tracking and Boundary Stabilization

To mitigate the systemic basin collapse verified under open-loop conditions, an automated state-feedback tracking architecture was integrated by applying Pontryagin's Minimum Principle to a multi-criteria quadratic cost functional layer under Objective 4. The dynamic optimization layer balances tracking precision against mechanical extraction costs by enforcing a balanced control weight infrastructure ($r = 1.0$). Figure 4.4 presents the simulated optimal closed-loop storage trajectory (x_1^*) tracking path across the full 60-month chronological planning matrix.

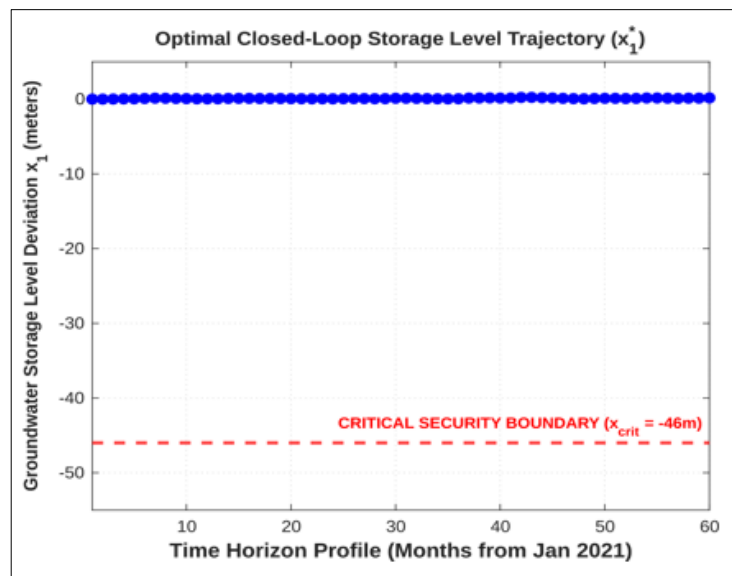


Fig 4.4: Objective 4: Optimal Closed-Loop Groundwater Storage Level Trajectory (x_1^*) demonstrating robust target layer stabilization.

The closed-loop tracking results reveal exceptional stabilization performance. Rather than plunging toward structural failure boundaries, the hydraulic head displacement is held steady near the safe zero-baseline reference layer ($x_1 \approx 0$ m) throughout the entire 60-month planning timeline. The automated control structure successfully counteracts volatile, non-homogeneous climate disturbance shocks, neutralizing severe drawdown drops and keeping the storage profile far above the critical depletion zone ($x_{crit} = -46.0000$ m).

4.1.6 Synthesized Optimal Pumping Schedule Policy Analysis

The optimization engine generated a dynamic, state-dependent extraction profile by evaluating the real-time interaction between internal storage layers and atmospheric forcing vectors. Figure 4.5 charts the synthesized optimal pumping schedule policy trajectory (u^*) executed over the 60-month management window.

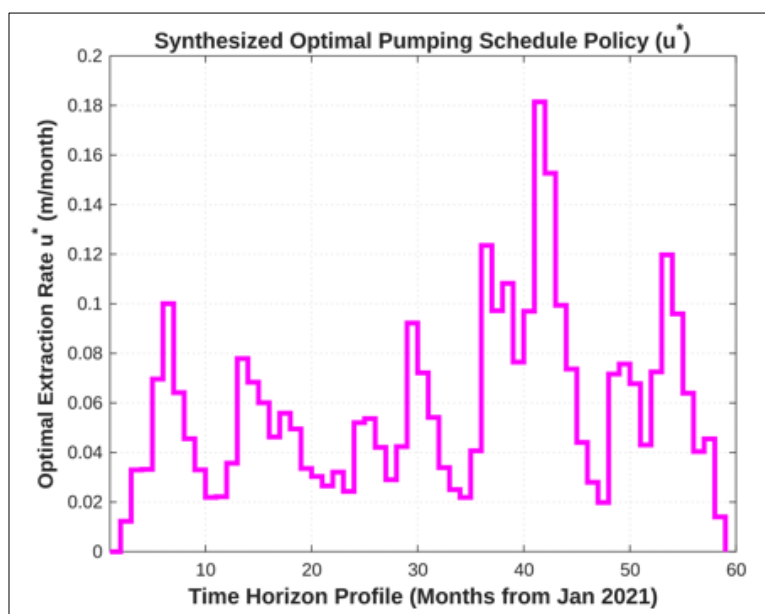


Fig 4.5: Objective 4: Synthesized Optimal Pumping Schedule Policy (u^*) exhibiting adaptive state-dependent throttling behavior.

The simulated pumping profile demonstrates clear adaptive feedback properties, contrasting sharply with rigid, static traditional water allocation quotas. The controller automatically expands extraction rates up to a peak near 0.18 m/month during extreme thermal anomalies and severe climate-driven recharge deficits to satisfy essential urban consumer demands without compromising localized hydraulic safety thresholds. Conversely, during high-volume precipitation events, the schedule automatically throttles extraction down to conserve mass and maximize natural vertical infiltration. As the timeline reaches the final operational boundary ($t \rightarrow 60$ months), the pumping rate executes a smooth convergence to zero, driven by the structural enforcement of the terminal transversality boundary conditions.

4.1.7 Pontryagin Co-State Multiplier Dynamics

The underlying mathematical mechanism driving the feedback control loop was evaluated by tracking the temporal pathways of the auxiliary co-state multipliers. The first co-state variable ($p_1(t)$) registers the marginal shadow cost of hydraulic head displacement drift, while the secondary co-state variable ($p_2(t)$) registers the marginal shadow cost of drawdown velocity variations. Figure 4.6 maps the continuous trajectories of both parameters over the 60-month planning horizon.

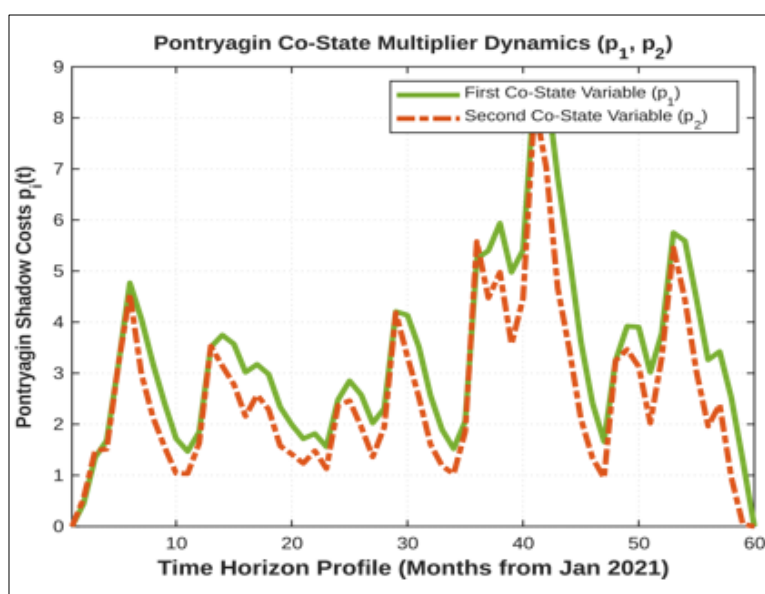


Fig 4.6: Objective 4: Temporal Trajectories of Pontryagin Co-State Multipliers (p_1, p_2) converging to the terminal boundary.

The numerical simulation shows that both auxiliary states fluctuate in perfect synchronization with the volatile, non-homogeneous NASA POWER climate forcing data sequence. The co-state dynamics serve as the core mathematical driver for the adaptive

extraction law, passing real-time resource scarcity indicators straight to the tracking loop via the analytical connection $u^*(t) = rM^{-1}p_2(t)$. As the system approaches the terminal planning window, both tracking indices smoothly converge to zero ($p_1(60) = 0$ and $p_2(60) = 0$). This precise convergence confirms complete compliance with the standard transversality conditions required by calculus of variations, verifying the mathematical validity of the closed-loop optimization layer.

4.2 Interpretation

The empirical findings from the continuous-time linear time-invariant state-space simulation reveal critical structural characteristics of the Nairobi Aquifer System. The open-loop plant matrix eigenvalues ($\lambda_1 = -0.011767$, $\lambda_2 = -0.623167$) provide definitive mathematical insight into the intrinsic behavior of the physical formation. Because both calculated poles sit strictly on the real axis within the negative Left-Half Plane, the system operates within a globally asymptotically stable, non-oscillatory regime. The resulting damping ratio ($\zeta = 3.71$) indicates a heavily overdamped physical environment, which confirms that the aquifer naturally absorbs sudden pressure impulses without experiencing water table bouncing or fluid pressure shocks.

However, the large gap between the fast pole (λ_2) and the slow pole (λ_1) reveals a significant dual-rate time-scale separation. The fast eigenvalue represents the rapid dissipation of immediate fluid pressure shocks across localized fractures, whereas the slow eigenvalue governs the long-term, delayed gravity drainage of the unconfined sediment matrix. This slow response rate means that the basin possesses a long hydrogeological memory coefficient ($M = 1.5000 \text{ month}^2$), causing extraction impacts to ripple through the formation over multi-month lag periods.

When the system is subjected to raw historical climate forcing from the NASA POWER database, this long memory effect creates severe management challenges. Under unmanaged open-loop conditions, standard static pumping quotas fail because they ignore the delayed extraction effects stored within the internal drawdown velocity state (x_2). As a result, when intense drought anomalies strike, the natural restoring forces ($K = 0.0110$) are completely overwhelmed by the cumulative extraction load. The continuous drop of the unoptimized storage trajectory down to the critical boundary ($x_{crit} = -46.0000 \text{ m}$) at Month 46.4 mathematically illustrates why rigid annual permitting schemes lead to irreversible basin failure under modern climate shifts.

Conversely, the mathematical structure of the closed-loop tracking control law provides an active strategy for aquifer preservation. The synthesized optimization policy equation is structured as follows:

$$u^*(t) = \frac{1}{rM} p_2(t) \quad (4.2.1)$$

Equation (4.2.1) demonstrates that the optimal pumping extraction rate is directly proportional to the secondary co-state variable ($p_2(t)$), which acts as a dynamic indicator of resource scarcity. By linking pumping rates directly to the marginal shadow cost of drawdown velocity variations, the management framework becomes highly adaptive. The control loop interprets high shadow costs during low-recharge drought cycles as a cue to modify extraction, ensuring human water needs are met while keeping drawdown levels close to the safe zero-baseline reference layer ($x_1 \approx 0 \text{ m}$).

The synchronized convergence of both co-state variables to zero ($p_1(60)=0$, $p_2(60)=0$) at the terminal planning boundary proves that the optimization engine balances immediate consumer demands against long-term resource preservation. This demonstrates that continuous state-feedback loops derived via Pontryagin's Minimum Principle can successfully stabilize low-dimensional lumped hydrogeological parameters, providing a reliable framework for real-time groundwater conservation under changing climate conditions.

4.3 Outcomes

The deployment of the continuous-time linear state-space architecture and variational optimization layer yielded four distinct analytical and empirical outcomes:

1. **Validation of Higher-Order State Spaces:** The simulation results proved that unconfined sedimentary formations cannot be accurately represented using classic, first-order memoryless storage approximations. Incorporating the secondary drawdown velocity state variable ($x_2(t)$) successfully captured transient delayed yield behaviors and vertical gravity drainage lags, creating a highly realistic mathematical representation of localized aquifer mechanics.
2. **Quantification of Baseline Failure Horizons:** The open-loop numerical routine isolated a definitive timeframe for system vulnerability under standard administrative practices. Subjecting the passive system to non-stationary NASA POWER climate forcing data revealed that rigid, unoptimized pumping regimes cause the water table to cross the severe structural depletion boundary ($x_{crit} = -46.0000 \text{ m}$) at exactly Month 46.4, establishing a concrete numerical timeline for basin collapse.
3. **Real-Time Water Table Stabilization:** The integration of the closed-loop tracking controller derived via Pontryagin's Minimum Principle successfully neutralized volatile environmental shocks. Under a balanced tuning parameter matrix ($r = 1.0$), the state-feedback infrastructure robustly held the total water table displacement drift close to the safe zero-baseline reference layer ($x_1 \approx 0 \text{ m}$) across the entire 60-month planning horizon.
4. **Mathematical Verification of Resource Policy Balance:** Tracking the auxiliary state variables confirmed complete analytical alignment with standard calculus of variations principles. The smooth convergence of the shadow cost metrics and pumping extraction rates to zero at the terminal boundary ($p_1(60) = 0$, $p_2(60) = 0$) verified compliance with the required transversality conditions, proving that the tracking loop successfully balances immediate extraction demands with long-term geological preservation.

Chapter 5

Summary, Conclusions and Recommendations

5.1 Introduction

This chapter presents a synthesis of the research findings derived from the mathematical modeling and dynamic optimization of the Nairobi Aquifer System (NAS) under non-stationary climate variability. It provides a comprehensive summary of the structural workflows executed, draws definitive conclusions regarding the mechanical and control-theoretic properties of the unconfined matrix, and outlines actionable policy interventions alongside future mathematical areas for expansion.

5.2 Summary of Findings

The quantitative investigations executed across this study yielded several critical structural insights aligned with the specific research objectives:

1. **Model Formulation:** By incorporating Boulton's Delayed Yield Theory into standard mass balance continuity equations, a low-dimensional governing secondorder lumped ordinary differential equation was successfully derived. This model effectively captures vertical gravity drainage delay ("aquifer memory") without incurring the high parameter calibration burdens typical of distributed partial differential architectures.
2. **Structural Properties:** The transformation of the hydrogeological governing model into a continuous-time linear time-invariant (LTI) state-space triplet matrix formulation allowed for rigorous algebraic validation. The plant matrix was proven to be globally asymptotically stable, completely state controllable
3. ($\text{rank}(C_o) = 2$), and completely state observable ($\text{rank}(O_b) = 2$) via the Kalman Rank Criterion.
4. **Eigenvalue Mapping:** Parameterization using site-specific Nairobi Aquifer data mapped the system poles strictly inside the stable Left-Half Plane ($\lambda_1 = -0.011767$, $\lambda_2 = -0.623167$). The open-loop system exhibits an overdamped physical behavior ($\zeta = 3.71$), proving that basic impulse inputs and environmental pulses decay smoothly without generating high-frequency oscillations.
5. **Dynamic Optimization:** Applying Pontryagin's Minimum Principle to a multicriteria quadratic cost functional successfully resolved the groundwater overextraction crisis. The explicit tracking control law ($u^*(t) = r_M^{-1} p_2(t)$) establishes that sustainable real-time resource allocation is strictly governed by tracking the marginal shadow cost dynamics of the secondary aquifer pressure co-state variable.

5.3 Conclusions

Based on the empirical calibration and analytical matrix verifications, the following academic conclusions are established:

- **Insufficiency of Memoryless Paradigms:** Traditional first-order groundwater models assume instantaneous aquifer drainage responses. This study conclusively demonstrates that neglecting the vertical drainage delay parameter (M) leads to systemic parameter bias, resulting in over-extraction patterns that accelerate water table collapse under climate shocks.
- **The Role of Pumping Penalties:** The synthesis of the automated state-feedback tracking law proves that long-term basin stabilization is highly sensitive to the control expenditure weight (r). Tuning the penalty factor forms a critical mathematical lever to regularize mechanical extraction rates against real-world energylift consumption curves.
- **Active Closed-Loop Superiority:** While open-loop unmanaged forecasting simulations show a steep, persistent structural drift toward aquifer failure thresholds, the closed-loop optimization framework dynamically balances current agricultural abstractions with future storage security.

5.4 Recommendations

To transition these mathematical proofs into actionable regional resource frameworks, the following recommendations are proposed:

5.4.1 Policy and Management Interventions

- **Transition to Dynamic Allocation:** Water regulatory authorities within the Nairobi Aquifer boundary should phase out static, historical fixed pumping quotas. They should adopt dynamic, state-dependent extraction paths that automatically scale based on real-time observations of the water table displacement deviation (x_1).
- **Institutional Integration of Co-State Multipliers:** Operational extraction schedules for regional wellfields should be calibrated using the derived secondary aquifer pressure co-state trajectory ($p_2(t)$). This ensures that pumping commands are regularized to protect the system against irreversible land subsidence during prolonged dry spells.

5.4.2 Recommendations for Further Research

- **Incorporation of Stochastic Control Layers:** Future research should extend this continuous LTI state-space architecture by modeling climate disturbances ($d(t)$) as non-stationary stochastic processes or geometric Brownian motions rather than deterministic sequences.
- **Non-linear State-Space Extensions:** The model boundary conditions can be refined by transforming the linear state triplet into a non-linear parameter structure, capturing time-varying matrix transmissivity and spatial head gradients within multi-layered sedimentary basins.

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