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Tsk model of fuzzy inference system in relation to different membership functions with examples

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Abstract

This paper explores the possibility of TSK (Takagi-Sugeno-Kang) model within fuzzy inference systems, giving increasing attention on various types of membership functions and their comparisons between the Mamdani and TSK models. It also includes the membership functions such as triangular, trapezoidal, S-shaped, and Gaussian in addition to a set of general rules workable for scenario's involving two attributes arranged in rows and columns. The paper presents its findings through two distinct examples that enable the TSK model's relevance and potential applications in fields like machine learning and artificial intelligence.

Keywords: Fuzzy inference system, membership function, fuzzy logic and fuzzy sets theory, triangular and trapezoidal membership function

Introduction

Fuzzy logic begins with the concept of a fuzzy set, which differs from traditional sets by allowing elements to have degrees of membership rather than strict inclusion or exclusion. This flexibility enables fuzzy logic to handle uncertainties and ambiguities more effectively. A fuzzy inference system (FIS) is a method for making decisions based on fuzzy logic principles assembling input data according to a set of fuzzy rules in order to produce an output vector. There are two main types of fuzzy inference systems: Mamdani FIS and Sugeno FIS. The Mamdani FIS, pioneered by Ebrahim H. Mamdani (1974)^[1], was originally designed to control a steam engine and boiler using linguistic control rules provided by human operators. It defines the output of each rule as a fuzzy logic on the other hand, the Sugeno FIS, proposed by Takagi and Sugeno (1985)^[2] and later revised by Sugeno and Kang (1988)^[3] aims to systematically derive fuzzy rules from input-output datasets. In the Sugeno approach, the output is a crisp function of the input variables, denoted as $f(X, Y)$. In the Mamdani approach, the inference is typically expressed as "If x_1 is A_1 and x_2 is A_2 , then Y is B , where A_1 , A_2 , and B are fuzzy sets defined over their respective universes of discourse. In contrast, the

Sugeno model's output is computed using different defuzzification methods such as weighted arithmetic mean (WAM), weighted geometric mean (WGM), and weighted harmonic mean (WHM). These methods aggregate the outputs of each rule to produce a final crisp output. Overall, fuzzy inference systems provide a robust framework for decision-making in scenario's where traditional binary logic may fall short, allowing for flexible reasoning in uncertain environments.

Mamdani fuzzy inference systems are especially well-suited for applications where rules are derived from human expert knowledge, such as in medical diagnostics. In these systems, rules are formulated as intuitive if-then statements that describe relationships between inputs and outputs. Each rule generates a fuzzy set as its output, determined by the membership functions and implication methods used in the fuzzy inference system (FIS). These fuzzy sets are then aggregated into a single fuzzy set that represents the combined output. To obtain a final crisp output value, the aggregated fuzzy set undergoes defuzzification using methods like centroid or weighted average. In contrast, Sugeno fuzzy inference systems, also known as Takagi-Sugeno-Kang fuzzy inference, simplify the

output representation by using singleton output membership functions. These functions are either constant or linear functions of the input variables. This approach enhances computational efficiency compared to Mamdani systems because defuzzification involves calculating a weighted average or sum of a few data points rather than computing the centroid of a two-dimensional area.

The essence of fuzzy logic lies in mapping an input space to an output space using a set of rules structured as if-then statements. These rules operate independently of each other, allowing flexibility in their evaluation order. The key terms and descriptors used in the rules, such as "hot" or "tall," are defined through membership functions that assign degrees of membership between 0 and 1 to each point in the input space. For instance, the concept of "tall" in a fuzzy set considers a range of heights rather than a strict boundary, accommodating nuances that classical sets cannot handle.

Ultimately, fuzzy inference is a methodical approach to interpreting input values and determining corresponding output values based on a defined set of rules. It leverages the flexibility of fuzzy sets and membership functions to model complex systems where precise boundaries are impractical or insufficient.

2. Material and Methods

In Mamdani approach, if X_1 is A_1 and X_2 is A_2 then Y is B while in TSK, it is $f(X,Y)$, where $f(X,Y)$ is the crisp function of the FLC in the consequent, as proposed by Takagi, Sugeno and Kang. The overall aggregated value of the output will be determined by WAM, WGM and WHM defuzzification methods respectively.

The i^{th} rule of TSK model can be represented as:

If X_1 is A_1^i and X_2 is A_2^i and X_n is A_n^i , then $Y^i = \sum_{j=0}^n a_j^i (X_j)$ where $a_0, a_1, a_2, \dots, a_n$ are constants being $\times \text{phantom} 0 = 1$

The weight of the i^{th} rule can be represented as $W^i = \prod_{j=1}^n \mu_{A_j^i}(X_j)$

The combined control action can be determined using three methods i.e. WAM, WGM

and WHM respectively as (i) $X^* = (\sum_{i=1}^k w_i x_i) / (\sum_{i=1}^k w_i)$

(ii) $X^* = \text{Antilog} (\sum_{i=1}^k w_i \log x_i) / (\sum_{i=1}^k w_i)$

(iii) $X^* = (\sum_{i=1}^k w_i) / (\sum_{i=1}^k (\frac{w_i}{x_i}))$ where k represents the total number of rules.

Four rules of TSK model of FLC is given below:

Rule-I: If x and y are small then $z = -x + y + 3$

Rule-II: If x is small and y is large then $z = -y + 3$

Rule-III: If x is large and y is small then $z = -x + 3$

Rule-IV: If x and y are large then $z = x + y + 2$

2.1 Membership Functions of Various Types

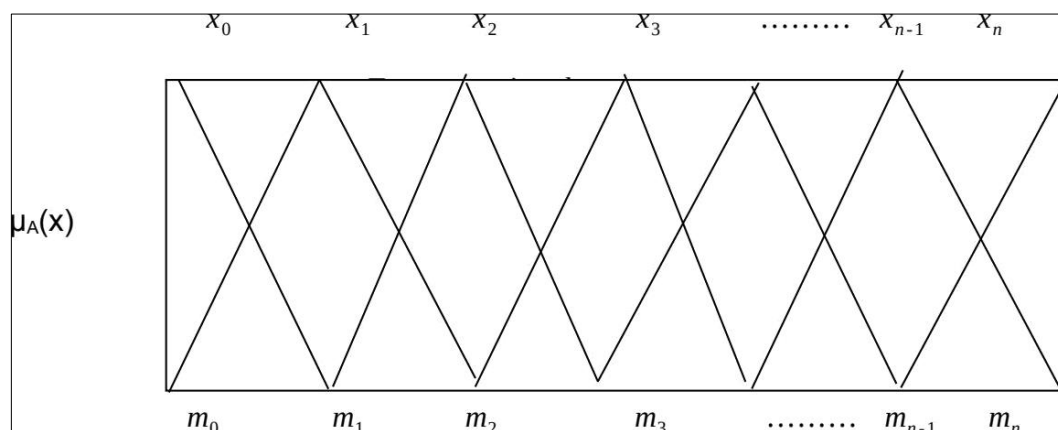
(i) Triangular Membership function:

Let $m_0, m_1, m_2, m_3, \dots, m_n$ be the coded values of amount of attribute corresponding to the linguistic input variable as $X_0, X_1, X_2, \dots, X_n$ respectively using triangular membership function in relation to the fuzzy set A . Also, let $m_0', m_1', m_2', m_3', \dots, m_{n-1}'$ be any points lying in between $(m_0, m_1), (m_1, m_2), (m_2, m_3), (m_3, m_4), (m_{n-1}, m_n)$ respectively, then using the principle of similarity of triangle, the degree of membership lying in between $(m_0, m_1), (m_1, m_2), (m_2, m_3), (m_3, m_4), \dots, (m_{n-1}, m_n)$ for the triangular membership function defined below is

$$\mu_A(x_0)(x) = \frac{m_1 - m_0'}{m_1 - m_0}, \frac{m_0' - m_0}{m_1 - m_0}, \text{ between } x_0 \text{ to } x_1$$

$$\mu_A(x_1)(x) = \frac{m_2 - m_1'}{m_2 - m_1}, \text{ and } \frac{m_1' - m_1}{m_2 - m_1}, \text{ between } x_1 \text{ to } x_2$$

Type and level of attribute



$$\mu_A(x_2)(x) = \frac{m_3 - m_2'}{m_3 - m_2}, \text{ and } \frac{m_2' - m_2}{m_3 - m_2}, \text{ between } x_2 \text{ to } x_3$$

$$\mu_A(x_3)(x) = \frac{m_4 - m_3'}{m_4 - m_3}, \text{ and } \frac{m_3' - m_3}{m_4 - m_3}, \text{ between } x_3 \text{ to } x_4$$

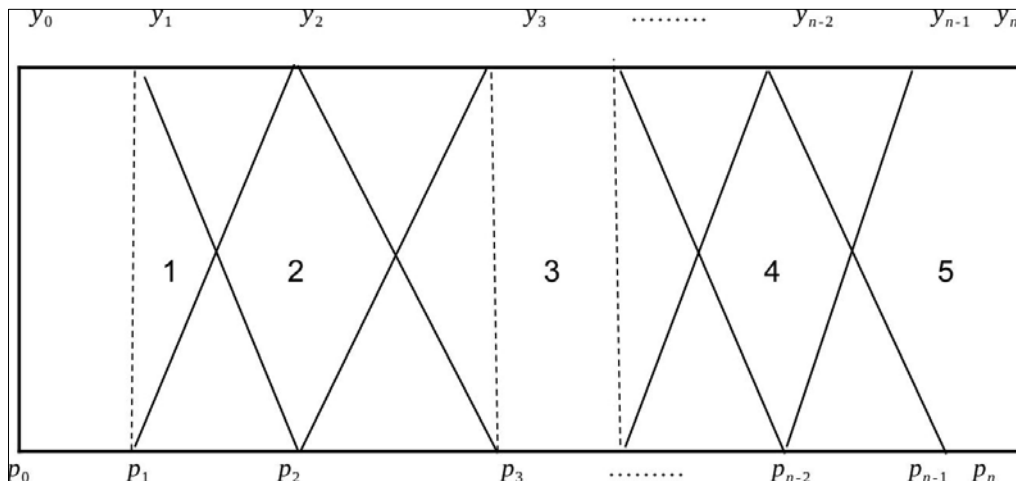
In general,

$$\mu_A(x_{n-1})(x) = \frac{m_n - m_{n-1}'}{m_n - m_{n-1}}, \text{ and } \frac{m_{n-1}' - m_{n-1}}{m_n - m_{n-1}}, \text{ between } x_{n-1} \text{ to } x_n$$

(ii) A mixture of triangle and trapezoidal membership function:

Similarly, let $p_0, p_1, p_2, p_3, \dots, p_n$ be the coded values of amount of another attribute corresponding to the linguistic input variable as $y_0, y_1, y_2, \dots, y_n$ respectively using triangular membership function in relation to the fuzzy set B. Also, let $p_1, p_2', p_3, \dots, p_{n-2}$ be any points lying in between $(p_1, p_2), (p_2, p_3), (p_3, p_4), (p_4, p_5), \dots, (p_{n-2}, p_{n-1})$ respectively, then using the principle of similarity of triangle, the degree of membership lying in between $(p_1, p_2), (p_2, p_3), (p_3, p_4), \dots, (p_{n-2}, p_{n-1})$ for the following triangular membership function is defined as

Type and level of another attribute:



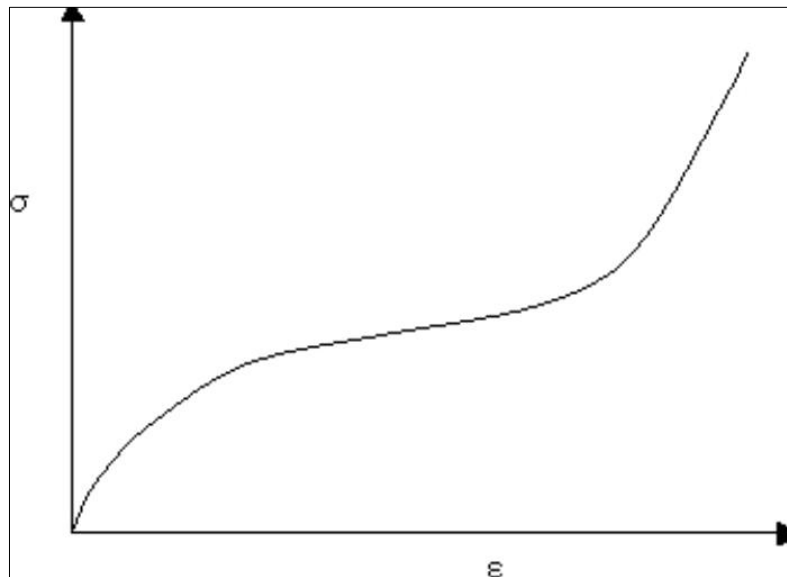
$$\mu_B(y_1)(y) = \frac{p_2 - p_1'}{p_2 - p_1}, \text{ and } \frac{p_1' - p_1}{p_2 - p_1}, \text{ between } y_1 \text{ to } y_2$$

$$\mu_B(y_2)(y) = \frac{p_3 - p_2'}{p_3 - p_2}, \text{ and}$$

$$= \begin{cases} 0 & , y \leq p_2 \\ \frac{p_2' - p_2}{p_3 - p_2} & , p_2 \leq y \leq p_3 \\ 1 & , p_3 \leq y \leq p_4 \\ \frac{p_5 - p_4'}{p_5 - p_4} & , p_4 \leq y \leq p_5 \\ 0 & , y \geq p_5 \end{cases}$$

Based on similar lines, the degree of membership function will be determined as and when required for the triangle or trapezoidal shape of the membership function.

(iii) A right and left S-shaped membership function:



$$(a+b)/2$$

$$f(x; a, b) = \begin{cases} 0, & \text{if } x \leq a \\ 2\left(\frac{x-a}{b-a}\right)^2, & \text{if } a \leq x \leq \frac{a+b}{2} \\ 1 - 2\left(\frac{x-b}{b-a}\right)^2, & \text{if } \frac{a+b}{2} \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases}$$

On the other hand, if the shape of the curve lies from the left side, then we have

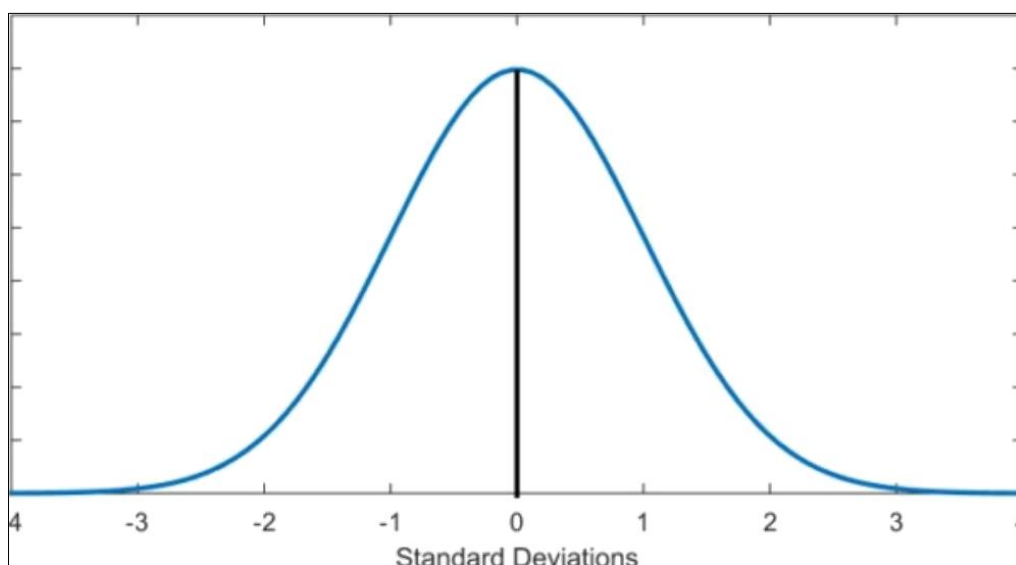
$$f(x; a, b) = \begin{cases} 1, & \text{if } x \leq a \\ 1 - 2\left(\frac{x-a}{b-a}\right)^2, & \text{if } a \leq x \leq \frac{a+b}{2} \\ 2\left(\frac{x-b}{b-a}\right)^2, & \text{if } \frac{a+b}{2} \leq x \leq b \\ 0, & \text{if } x \geq b \end{cases}$$

(iv) Gaussian membership function:

Gaussian membership function: The Gaussian membership function is represented as $\text{Gaussian}(x; c, s)$ where c represents the mean and s represents the standard deviation.

$$\mu_A(x; c, s, m) = e^{\frac{-1}{2} \left(\frac{x-c}{s} \right)^m}$$

where m represents fuzzification factor.



2.2 Setting of Rules for Controller Action

(i) Rule of one attribute (M) in relation to another type of attribute (P) if we have the levels of both attributes, equal.

Type of one attribute/Another attribute	m_1	m_2	m_3	m_4	 m_{n-1}	m_n
p_1	y_1	y_2	y_3	y_4	 y_{n-1}	y_n
p_2	y_{n+1}	y_{n+2}	y_{n+3}	y_{n+4}	 y_{2n-1}	y_{2n}
p_3	y_{2n+1}	y_{2n+2}	y_{2n+3}	y_{2n+4}	 y_{3n-1}	y_{3n}
p_4	y_{3n+1}	y_{3n+2}	y_{3n+3}	y_{3n+4}	 y_{4n-1}	y_{4n}
....						
p_{n-1}	$y_{(n-2)n+1}$	$y_{(n-2)n+2}$	$y_{(n-2)n+3}$	$y_{(n-2)n+4}$	y_{n-2}^2	y_{n-1}^2
p_n	$y_{(n-1)n+1}$	$y_{(n-1)n+2}$	$y_{(n-1)n+3}$	$y_{(n-1)n+4}$	$y_{(n-1)}^2$	y_n^2

In the above table, if rows and columns are being given weights as per the natural numbers, that is first row weighs 1, second row 2, ..., m rows as m and n rows as n, then the value of y lying in between m^{th} rows and n^{th} column in a $n \times n$ matrix is equal to $m \times$ value of Input 1 + $n \times$ value of Input 2. Similarly, other values of y can be determined.

(ii) Rule of one attribute (M) in relation to another type of attribute (P) if we have the levels of both attributes, not equal in the form of m rows and n columns.

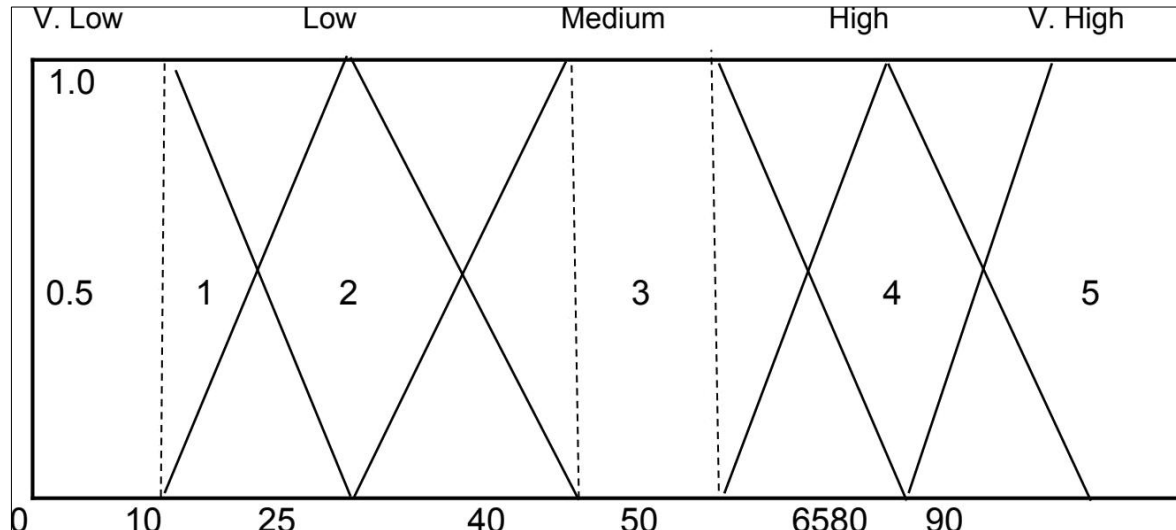
Type of dirt/Type of grease	r_1	r_2	r_3	r_4	 r_{n-1}	r_n
s_1	y_1	y_2	y_3	y_4	 y_{n-1}	y_n
s_2	y_{n+1}	y_{n+2}	y_{n+3}	y_{n+4}	 y_{2n-1}	y_{2n}
s_3	y_{2n+1}	y_{2n+2}	y_{2n+3}	y_{2n+4}	 y_{3n-1}	y_{3n}
s_4	y_{3n+1}	y_{3n+2}	y_{3n+3}	y_{3n+4}	 y_{4n-1}	y_{4n}
....						
s_{m-1}	$y_{(m-2)n+1}$	$y_{(m-2)n+2}$	$y_{(m-2)n+3}$	$y_{(m-2)n+4}$	y_{mn-2}	y_{mn-1}
s_m	$y_{(n-1)n+1}$	$y_{(n-1)n+2}$	$y_{(n-1)n+3}$	$y_{(n-1)n+4}$	$y_{(mn-1)}$	y_{mn}

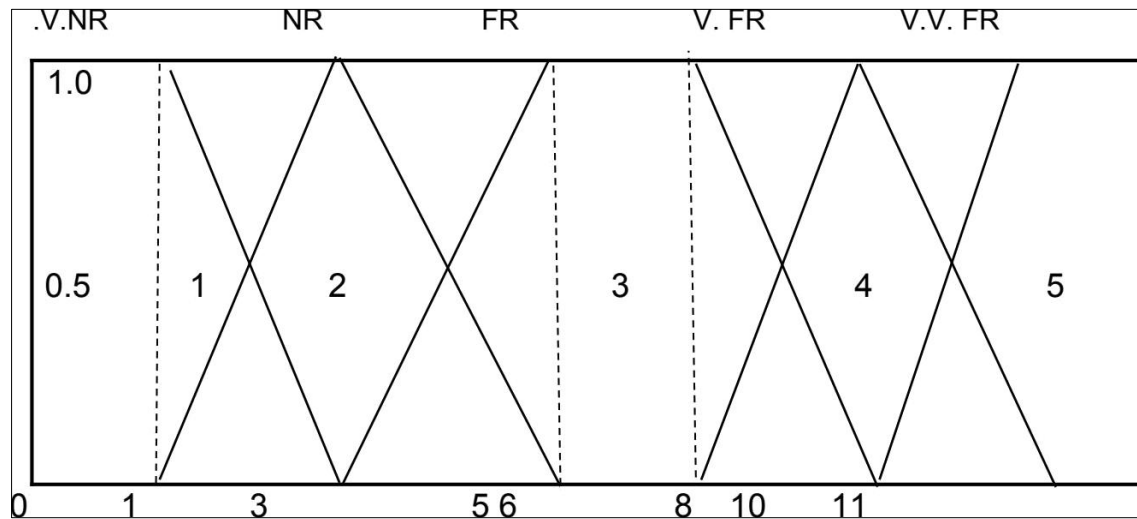
In the same way, if rows and columns are being given weights as per the natural numbers, that is first row weighs 1, second row 2, ..., m rows as m and n rows as n, then the value of y lying in between m^{th} rows and n^{th} column in a $m \times n$ matrix is equal to $m \times$ value of Input 1 + $n \times$ value of Input 2. Similarly, other values of y can be determined.

3.0 Examples

Example 3.1

Solve for Input 1 as 13.0 and Input 2 as 2.





Input: I_1 of 13.0 may be called very low or low.

Input: I_2 of 2.5 may be called very NR or NR

Using principle of similar triangle $\mu_w = (25-13)/(25-10) = (12)/(15) = 0.8$

Similarly, $\mu_L = (13-10)/(25-10) = \frac{[2]}{(15)} = 0.2$

In the same way, $\mu_{VNR} = (3-2.5)/\frac{[2, 1]}{[3]} = (0.5)/[3] = 0.25$

$$\mu_{NR} = \frac{2.5 - 1}{3 - 1} = \frac{1.5}{2} = 0.75$$

That is I_1 belongs to the set of very low or low and. I_2 belongs to the set of very NR or NR. Then, the fired set of rules can be constructed as

I_1 is very low and I_2 is very NR

I_1 is very low and I_2 is NR

I_1 is low and I_2 is very NR

I_1 is low and I_2 is NR

Then their corresponding weights can be computed as

$$W_1 = \mu_{VL} \times \mu_{VNR} = 0.8 \times 0.25 = .20$$

$$W_2 = \mu_{VL} \times \mu_{NR} = 0.8 \times 0.75 = 0.60$$

$$W_3 = \mu_L \times \mu_{VNR} = 0.20 \times 0.25 = 0.05$$

$$W_4 = \mu_L \times \mu_{NR} = 0.20 \times 0.75 = 0.15,$$

Given $I_1 = 13.0$ and $I_2 = 2.5$,

Then the values of y can be computed as

$$Y_1 = I_1 + 2I_2 = 13.0 + 2 \times 2.5 = 18.0$$

$$Y_2 = I_1 + 3I_2 = 13.0 + 3 \times 2.5 = 20.5$$

$$Y_3 = 2I_1 + 2I_2 = 2 \times 13.0 + 2 \times 2.5 = 31.0$$

$$Y_4 = 2I_1 + 3I_2 = 2 \times 13.0 + 3 \times 2.5 = 33.5$$

i	w_i	y_i
1.	0.20	18.0
2.	0.60	20.5
3.	0.05	31.0
4.	0.15	33.5

The crisp value of the output is then determined by calculating the weighted mean of the weights and values of y , as shown below:

$$y^* = \frac{w_1y_1 + w_2y_2 + w_3y_3 + w_4y_4}{w_1 + w_2 + w_3 + w_4}$$

$$y^* = \frac{0.20 \times 18.0 + 0.60 \times 20.5 + 0.05 \times 31.0 + 0.15 \times 33.5}{0.20 + 0.60 + 0.05 + 0.15}$$

$$y^* = \frac{22.475}{1.00} = 22.475$$

The crisp value of the output can also be determined using the formula of WGM and WHM respectively which are mentioned in Section 2 of Material and Methods as given below.

- (ii) $Y^* = \text{Antilog}(\sum_{i=1}^k w_i \log x_i) / (\sum_{i=1}^k w_i)$
- (iii) $Y^* = (\sum_{i=1}^k w_i) / (\sum_{i=1}^k (w_i / x_i))$ where k represents the total number of rules.

Using the data of y_i and W_i given above, a fresh table has been prepared for the computation of crisp value of y^{star} through WGM and WHM.

i	w_i	y_i	$\log y_i$	$w_i \log y_i$	w_i / y_i
1.	0.20	18.0	1.255	0.251	0.011
2.	0.60	20.5	1.312	0.787	0.029
3.	0.05	31.0	1.491	0.075	0.002
4.	0.15	33.5	1.525	0.229	0.004
Total				1.342	0.046

Using WGM $y^{\text{star}} = \text{anti log}(1.342 / 1.00) = 21.979$ and using WHM $y^{\text{star}} = 1 / 0.046 = 21.739$.
All three estimated values of y^{star} from different methods are found to be almost the same.

Example 3.2

Consider a two-input, single-output Sugeno model with four rules as follows::

Rule 1: If both X_1 and X_2 are small, then $y_1 = -x_1 + x_2 + 1$

Rule 2: If X_1 is small and X_2 is large, then $y_2 = -x_2 + 8$

Rule 3: If X_1 is large and X_2 is small, then $y_3 = -x_1 + 5$

Rule 4: If both X_1 and X_2 are large, then $y_4 = x_1 + x_2 + 1$

Find the output when $x_1 = 3.5$ and $x_2 = 5.5$ on the right and left S shaped curve as per the four rules given above.

Here, $x_1 = 3.5$ and $x_2 = 5.5$

As per the four rules, the weights being in this model become

$$W_1 = \min(0.617, 0.195) = 0.195$$

$$W_2 = \min(0.617, 0.805) = 0.617$$

$$W_3 = \min(0.383, 0.195) = 0.195$$

$$W_4 = \min(0.383, 0.805) = 0.383$$

i	w_i	y_i
1.	0.195	3.0
2.	0.617	2.5
3.	0.195	1.5
4.	0.383	10

The crisp value of the output is then determined by calculating the weighted mean of the weights and values of y , as shown below:

$$y^* = \frac{w_1y_1 + w_2y_2 + w_3y_3 + w_4y_4}{w_1 + w_2 + w_3 + w_4}$$

$$y^* = \frac{0.195 \times 3.0 + 0.617 \times 2.5 + 0.195 \times 1.5 + 0.383 \times 10}{0.195 + 0.617 + 0.195 + 0.383}$$

$$y^* = \frac{5.016}{1.39} = 3.609$$

Again, using the data of y_i and w_i given above, a fresh table has been prepared for the computation of crisp value of y^{star} through WGM and WHM.

i	w_i	y_i	$\log y_i$	$w_i \log y_i$	w_i/y_i
1.	0.195	3.0	0.477	0.093	0.065
2.	0.617	2.5	0.398	0.246	0.247
3.	0.195	1.5	0.176	0.034	0.130
4.	0.383	10	1.000	0.383	0.038
Total	1.39			0.756	0.480

Using WGM $y^{star} = \text{anti log}(0.324 / 1.39) = 3.499$ and using WHM $y^{star} = 1.39 / 0.480 = 2.896$

Two estimated values of y^{star} from weighted mean and WGM are almost the same but WHM provides the lower estimate of the crisp value rather than that of other two methods (almost 20% less).

4.0 Conclusions

Despite the various membership functions discussed and applied earlier, such as Generalized Bell-shaped, Sigmoidal, Pi-shaped, and Open Right, these functions can effectively describe the inherent variability in control problems, especially within the dynamic and non-linear systems of the TSK fuzzy model.

The Mamdani approach remains widely favored for capturing expert knowledge, despite imposing significant computational demands. In contrast, the TSK model is noted for its computational efficiency. This model integrates seamlessly with optimization and adaptive techniques, making it exceptionally suitable for control problems in fuzzy inference systems. Moreover, it circumvents the need for time consuming defuzzification methods required by the Mamdani approach.

Furthermore, a combination of different membership functions can be employed to address and solve diverse control problems across various fuzzy models. This versatility extends to applications in fields like machine learning and artificial intelligence, where fuzzy logic plays a crucial role in handling uncertainty and imprecision effectively.

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