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Some topological indices of the prime graph of commutative ring Z_{pq}

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Abstract

In this study, concept anew graph definition known A prime graph of a ring Z_{pq} denote $PG(Z_{pq})$ is present, where the graph's vertices stand in for R 's elements s.t, any two vertices a and b merge by an edge if and only if and $a.b = 0$. In this paper, we investigated some new property of $PG(Z_{pq})$ are studied.

Keywords: Commutative ring, sum connectivity index, forgotten index, 1^{st} zagreb index, the atomic bond connectivity index, graph theory

1. Introduction

Graph theory has become a very well-known and rapidly growing field of mathematics due to its extensive theoretical advances and diverse applications to real-world problems. Although graph theory is still a relatively new field of research, it has yielded many profound and novel discoveries over the past 20 years. In biological, social, physical, and information systems, diagrams can be used to represent a variety of relationships and processes.

A topological indices is a real number assigned to the graph whose structure must be a constant. Topological indexes, sometimes is called molecular structure descriptors. More topological indices depend on degrees.

The Wiener index, also known as the Wiener index or the 1^{st} topological index, was indeed introduced by Harry Wiener in (1947). This index is a simple and widely used measure of the molecular topological structure in chemistrical.

Sylvester introduced the term graph in an 1878 article in Nature Publishing ^[1]. The 1^{st} book in graph theory was published by Denes in 1936 ^[2]. In the last decade of the 19th century, algebraic graph theory began to develop rapidly, and a large number of research articles **were** published in this branch of graph theory. The fascinating field of algebraic graph theory studies how algebra and graph theory interact. Algebraic methods can be used to prove graph theory facts in surprising and elegant ways. There are many interesting algebraic objects related to graphs. In recent years, the study of algebraic graph theory has become increasingly important. Algebraic graph theory involves converting properties of graphs into algebraic properties and then using algebraic results and techniques to derive theorems about graphs.

However, many algebraic topics can also be understood by converting them into graphs and exploiting the properties of graphs. In (1999), Anderson and Livingston ^[3] assigned a simple graph to each commutative ring R and examined to interaction between the ring-theoretical conditions and the graph-theoretic properties of. Akbari *et al.* In (2010) Bhavanari *et al.* ^[3], they considered binding rings R (not necessarily commutative) and defined a new concept "primary graph of R " (denoted by $PG(R)$). They presented some examples and obtained some basic important results related to $PG(R)$. (2013) Chelvam and Asir. ^[5] Studied the advantages in commutative ring ensemble graphs. In (2014), Patra and Kalita ^[6] studied prime graphs of commutative rings. In (2023), Nermen J. Khalel and Nabeel E. Arif ^[7] studied Associate graph of a commutative ring.

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In the articles, we study more topological metrics that need degrees of examples: eccentricity index ^[8], connectivity index ^[9], sum connectivity index ^[10], 1st and 2nd index ^[11], forgetting indicator ^[12], exponential geometric arithmetic ^[13], atomic bond connectivity index ^[13] and harmonic index ^[13]. Topological indexing of symmetry group graphs was introduced in (2019) Abdussakir ^[14], also examined in (2020) G. R. Roshini ^[15], and also in (2020) G. R. Roshini [16], Alaa.J and Akram.S ^[17] in (2021) study topological indices and (Schultz and Hosoya) polynomials of the intersection graph of subgroup of the group Zr .

2. Primary Results

Definition 2.1: Let Z_{pq} a ring. A prime graph of ring Z_p , indicated as $PG(z_{pq})$ is a graph where $PG(z_{pq}) (V, E)$ where $V(PG(z_{pq})) = Z_{pq}$ and $E(PG(z_{pq})) = \{\overline{ab}, a.b = 0 \text{ and } a \neq b\}$.

Remark2.2: Let $PG(z_{pq})$ be a graph of R with $R = Z_{pq}$. Then

1. (0) and (1) adjacent to all vertices..
2. A distance between any two distinct vertices a and b is $d(a, b) \leq 2$.

Corollary2.3: If Z_{pq} be a ring, p, q is a prime number, then $PG(z_{pq})$ is a

- 1) Connected
- 2) $\alpha(PG(z_{pq})) = 1$

3. Methods and Materials

In this paper, all diagrams are simplfy, bounded, connect, and UN directional. For the graph $G = (V(G), E(G))$ let $deg(u)$ be the degree of vertex u in G . If $deg(u) = 0$, then u is an isolated vertex. Let $d(u, v)$ be the distance between two distinct vertices u and v .

The $ecc(u)$ of the vertex u is $ecc(u) = \sup \{d(u, v): v \in V(G)\}$

Next definitions refers to a topological indices of a graph G

Eccentricity index of G is ^[8]

$$\xi^c(G) = \sum_{u \in V(G)} deg(u). e(u)$$

The Eccentricityty connectivity index of G is ^[9]

$$X(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{deg(u) \cdot deg(v)}}$$

The Sum connectivity index of G is ^[10]

$$S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{deg(u) + deg(v)}}$$

A 1st zagreb index of G is ^[11]

$$M_1(G) = \sum_{u \in V(G)} (deg(u))^2$$

A 2nd zagreb index of G is ^[11]

$$M_2(G) = \sum_{uv \in E(G)} deg(u) \cdot deg(v)$$

The forgotten index of G is ^[12]

$$F(G) = \sum_{u \in V(G)} (deg(u))^3$$

The atomic bond connectivity index of G is ^[13]

$$Abc(G) = \sum_{uv \in E(G)} \sqrt{\frac{deg(u) + deg(v) - 2}{deg(u) \cdot deg(v)}}$$

The Geometric arithmetic index of G is ^[13]

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{\deg(u) \cdot \deg(v)}}{\deg(u) + \deg(v)}$$

Harmonic index of G is ^[13]

$$H(G) = \sum_{uv \in E(G)} \frac{2}{\deg(u) + \deg(v)}$$

4. Some Topological indices of $PG(z_{pq})$

NOTE: For every $p \geq 2, q \geq 3$ are prime number then:

$$\deg(0) = pq - 1, \deg(np) = 2, n = 1, \dots, q - 1$$

$$e(0) = 1, e(np) = 2, n = 1, \dots, q - 1$$

$$\deg(q) = q, e(q) = q$$

$$\deg(w) = 1, e(w) = 2, w \neq np \neq q$$

Theorem 4.1: The eccentricity connectivity index of $PG(z_{pq})$ is

$$\xi^c(PG(z_{pq})) = (pq - 1) + 4(q - 1) + q^2 + 2(pq - p - q + 1)$$

Proof

$$\begin{aligned} \xi^c(PG(z_{pq})) &= \sum_{u \in V(PG(z_{pq}))} \deg(u) \cdot e(u) \\ &= e(0) \deg(0) + \underbrace{e(p) \deg(p) + \dots + e((q-1)p) \deg((q-1)p)}_{(q-1) \text{ times}} + e(q) \deg(q) + \underbrace{e(1) \deg(1) + \dots + e(w) \deg(w)}_{(pq-p-q+1) \text{ times}} \\ &= (pq - 1) + 4(q - 1) + q^2 + 2(pq - p - q + 1) \end{aligned}$$

Theorem 4.2: The connectivity index of $PG(z_{pq})$ is

$$X(PG(z_{pq})) = \frac{q-1}{\sqrt{2pq-2}} + \frac{pq-p-q+1}{\sqrt{pq-1}} + \frac{q-1}{\sqrt{2q}} + \frac{1}{\sqrt{(pq-1)q}}$$

Proof

$$\begin{aligned} X(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \frac{1}{\sqrt{\deg(u) \deg(v)}} \\ &= \underbrace{\frac{1}{\sqrt{\deg(0) \cdot \deg(p)}} + \dots + \frac{1}{\sqrt{\deg(0) \cdot \deg((q-1)p)}}}_{(q-1) \text{ times}} + \\ &\quad + \underbrace{\frac{1}{\sqrt{\deg(0) \cdot \deg(1)}} + \dots + \frac{1}{\sqrt{\deg(0) \cdot \deg(w)}}}_{(pq-p-q+1) \text{ times}} \\ &\quad + \underbrace{\frac{1}{\sqrt{\deg(q) \cdot \deg(p)}} + \dots + \frac{1}{\sqrt{\deg(q) \cdot \deg((q-1)p)}}}_{(q-1) \text{ times}} \\ &\quad + \frac{1}{\sqrt{\deg(0) \cdot \deg(q)}} \\ &= \frac{q-1}{\sqrt{2pq-2}} + \frac{pq-p-q+1}{\sqrt{pq-1}} + \frac{q-1}{\sqrt{2q}} + \frac{1}{\sqrt{(pq-1)q}} \end{aligned}$$

Theorem4.3: The Sum connectivity index of $PG(z_{pq})$ is

$$S(PG(z_{pq})) = \frac{q-1}{\sqrt{pq+1}} + \frac{pq-p-q+1}{\sqrt{pq}} + \frac{q-1}{\sqrt{q+2}} + \frac{1}{\sqrt{pq+q-1}}$$

Proof

$$\begin{aligned} S(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \frac{1}{\sqrt{\deg(u) + \deg(v)}} \\ &= \frac{1}{\sqrt{\deg(0) + \deg(p)}} + \cdots + \frac{1}{\sqrt{\deg(0) + \deg((q-1)p)}} + \frac{1}{\sqrt{\deg(0) + \deg(q)}} \\ &\quad + \frac{1}{\sqrt{\deg(0) + \deg(1)}} + \cdots + \frac{1}{\sqrt{\deg(0) + \deg(w)}} \\ &\quad + \frac{1}{\sqrt{\deg(q) + \deg(p)}} + \cdots + \frac{1}{\sqrt{\deg(q) + \deg((q-1)p)}} \\ &= \frac{q-1}{\sqrt{pq+1}} + \frac{pq-p-q+1}{\sqrt{pq}} + \frac{q-1}{\sqrt{q+2}} + \frac{1}{\sqrt{pq+q-1}} \end{aligned}$$

Theorem4.4: The 1st Zagreb index of $PG(z_{pq})$ is

$$M_1(PG(z_{pq})) = (pq-1)^2 + (4q-4) + (pq-p-q+1) + q^2$$

Proof

$$\begin{aligned} M_1(PG(z_{pq})) &= \sum_{u \in V(PG(z_{pq}))} (\deg(u))^2 \\ &= (\deg(0))^2 + (\deg(q))^2 + \underbrace{(\deg(p))^2 + \cdots + (\deg(q-1)p)^2}_{(q-1) \text{ times}} + \underbrace{(\deg(1))^2 + \cdots + (\deg(w))^2}_{(pq-p-q+1) \text{ times}} \\ &= (pq-1)^2 + (4q-4) + (pq-p-q+1) + q^2 \end{aligned}$$

Theorem4.5: The second Zagreb index of $PG(z_{pq})$ is

$$M_2(PG(z_{pq})) = 2(q-1)(pq-1) + 2q(q-1) + (pq-p-q+1)(pq-1) + q(pq-1)$$

Proof

$$\begin{aligned} M_2(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \deg(u) \cdot \deg(v) \\ &= \underbrace{\deg(0) \cdot \deg(p) + \cdots + \deg(0) \cdot \deg((q-1)p)}_{(q-1) \text{ times}} + \\ &\quad \underbrace{\deg(q) \cdot \deg(p) + \cdots + \deg(q) \cdot \deg((q-1)p)}_{(q-1) \text{ times}} + \\ &\quad \underbrace{\deg(0) \cdot \deg(1) + \cdots + \deg(0) \cdot \deg(w)}_{(pq-p-q+1) \text{ times}} + \deg(0) \cdot \deg(q) \\ &= 2(q-1)(pq-1) + 2q(q-1) + (pq-p-q+1)(pq-1) + q(pq-1) \end{aligned}$$

Theorem4.6: The forgotten index of $PG(z_{pq})$ is

$$F(PG(z_{pq})) = (pq-1)^3 + 8(q-1) + q^3 + (pq-p-q+1)$$

Proof

$$\begin{aligned}
F(PG(z_{pq})) &= \sum_{u \in V(PG(z_{pq}))} (\deg(u))^3 \\
&= (\deg(0))^3 + \underbrace{(\deg(p))^3 + \dots + (\deg(q-1)p)^3}_{(q-1) \text{ times}} \\
&\quad + (\deg(q))^3 + \underbrace{(\deg(1))^3 + \dots + (\deg(w))^3}_{(pq-p-q+1) \text{ times}} \\
&= (pq-1)^3 + 8(q-1) + q^3 + (pq-p-q+1)
\end{aligned}$$

Theorem 4.7: The Atom bond connectivity index of $PG(z_{pq})$ is

$$ABC(PG(z_{pq})) = \frac{2q-2}{\sqrt{2}} + (pq-p-q+1) + \sqrt{\frac{pq+q-3}{q(pq-1)}}$$

Proof

$$\begin{aligned}
ABC(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \sqrt{\frac{\deg(u) + \deg(v) - 2}{\deg(u) \cdot \deg(v)}} \\
&= \underbrace{\sqrt{\frac{\deg(0) + \deg(p) - 2}{\deg(0) \cdot \deg(p)}} + \dots + \sqrt{\frac{\deg(q) + \deg(q-1)p - 2}{\deg(q) \cdot \deg(q-1)p}}}_{(q-1) \text{ times}} \\
&\quad \underbrace{\sqrt{\frac{\deg(0) + \deg(1) - 2}{\deg(0) \cdot \deg(1)}} + \dots + \sqrt{\frac{\deg(0) + \deg(w) - 2}{\deg(0) \cdot \deg(w)}}}_{(pq-p-q+1) \text{ times}} \\
&\quad \underbrace{\sqrt{\frac{\deg(q) + \deg(p) - 2}{\deg(q) \cdot \deg(p)}} + \dots + \sqrt{\frac{\deg(q) + \deg(q-1)p - 2}{\deg(q) \cdot \deg(q-1)p}}}_{(q-1) \text{ times}} \\
&\quad + \sqrt{\frac{\deg(0) + \deg(q) - 2}{\deg(0) \cdot \deg(q)}} \\
&= \frac{2q-2}{\sqrt{2}} + (pq-p-q+1) + \sqrt{\frac{pq+q-3}{q(pq-1)}}
\end{aligned}$$

Theorem 4.8: The Geometric-Arithmetic index of $PG(z_{pq})$ is

$$GA(PG(z_{pq})) = (q-1) \frac{2\sqrt{2pq-2}}{pq+1} + (q-1) \frac{2\sqrt{2q}}{q+2} + (pq-p-q+1) \frac{2\sqrt{pq-1}}{pq} + \frac{2\sqrt{q(pq-1)}}{pq+q-1}$$

Proof

$$\begin{aligned}
GA(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \frac{2\sqrt{\deg(u) \cdot \deg(v)}}{\deg(u) + \deg(v)} \\
&= \underbrace{\frac{2\sqrt{\deg(0) \cdot \deg(p)}}{\deg(0) + \deg(p)} + \dots + \frac{2\sqrt{\deg(0) \cdot \deg(q-1)p}}{\deg(0) + \deg(q-1)p}}_{(q-1) \text{ times}}
\end{aligned}$$

$$\begin{aligned}
& + \underbrace{\frac{2\sqrt{\deg(0) \cdot \deg(1)}}{\deg(0) + \deg(1)} + \dots + \frac{2\sqrt{\deg(0) \cdot \deg(w)}}{\deg(0) + \deg(qw)}}_{(pq-p-q+1)\text{times}} \\
& + \underbrace{\frac{2\sqrt{\deg(q) \cdot \deg(p)}}{\deg(q) + \deg(p)} + \dots + \frac{2\sqrt{\deg(q) \cdot \deg(q-1)p}}{\deg(q) + \deg(q-1)p}}_{(q-1)\text{times}} \\
& + \frac{2\sqrt{\deg(0) \cdot \deg(q)}}{\deg(0) + \deg(q)} \\
& = (q-1) \frac{2\sqrt{2pq-2}}{pq+1} + (q-1) \frac{2\sqrt{2q}}{q+2} + (pq-p-q+1) \frac{2\sqrt{pq-1}}{pq} + \frac{2\sqrt{q(pq-1)}}{pq+q-1}
\end{aligned}$$

Theorem4.9: The Harmonic index of $PG(z_{pq})$ is

$$H(PG(z_{pq})) = \frac{2(q-1)}{pq+1} + \frac{2(q-1)}{q+2} + \frac{2(pq-p-q+1)}{pq} + \frac{2}{pq+q-1}$$

Proof:

$$\begin{aligned}
H(PG(z_{pq})) &= \sum_{uv \in E(PG(z_{pq}))} \frac{2}{\deg(u) + \deg(v)} \\
&= \underbrace{\frac{2}{\deg(0) + \deg(p)} + \dots + \frac{2}{\deg(0) + \deg((q-1)p)}}_{(q-1)\text{times}} \\
&+ \underbrace{\frac{2}{\deg(0) + \deg(1)} + \dots + \frac{2}{\deg(0) + \deg(w)}}_{(pq-p-q+1)\text{times}} \\
&+ \underbrace{\frac{2}{\deg(q) + \deg(p)} + \dots + \frac{2}{\deg(q) + \deg((q-1)p)}}_{(q-1)\text{times}} \\
&+ \frac{2}{\deg(0) + \deg(q)} \\
&= \frac{2(q-1)}{pq+1} + \frac{2(q-1)}{q+2} + \frac{2(pq-p-q+1)}{pq} + \frac{2}{pq+q-1}
\end{aligned}$$

5. Conclusions

In this study, formulas for some degree and eccentricity based topological indices are proposed for prime graph of ring Zpq , where p, q is a prime number. To investigate further, examine the non-zero graph of the ring Zpq , where p, q are prime numbers

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