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Modified van der pol oscillator model for profit-taking and re-entry dynamics in stock markets

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Abstract

Over the years and building upon the existing literature, it becomes evident that while various models exist to explain and predict investor behaviour and market dynamics, there is a lack of research specifically applying the Van der Pol oscillator to model the endogenous cyclicity of aggregate profit-taking and re-entry dynamics in stock markets. Existing models may not adequately capture the self-sustaining nature and nonlinear characteristics of these cycles. This study aims to fill this gap by investigating the feasibility and effectiveness of using the Van der Pol oscillator to model these key aspects of investor behaviour. The modified Van der Pol oscillator model developed in this paper provides a quantitative framework for modelling cyclical investor behaviour, that involves profit-taking and re-entry dynamics in stock markets. The limit cycle behaviour of this model naturally describes recurring bull/bear phases while the nonlinear damping captures asymmetric investor behaviour (greed vs. fear). By calibrating the model to relevant market data and analysing its dynamic properties, this paper seeks to provide a novel perspective on the underlying mechanisms driving the cyclical ebb and flow of investor participation in stock markets.

Keywords: Non-linear differential equation, stock markets, investor behaviour

1. Introduction

Equities, also called shares or stock, represent partial ownership of the issuing company. When an investor purchases an equity, it provides capital to finance the company, and in return, the investor hope to receive a portion of its profits, typically paid out as a dividend, and may decide over time, to trade the equity for a return ^[1]. According to a local news report, foreign investors pulled out N420.37bn from the Nigerian equities market in the first quarter of 2025, and this amount, according to the latest data from the Nigerian Exchange Group (NGX), represents a 251 percent increase in capital flight compared to the N119.81bn recorded in the same period in 2024 ^[2] - this showcases a typical scenario of investor behaviour in equity markets. Thus, understanding the intricate dynamics of investor behaviour stands as a cornerstone of financial market economics ^[3]. The collective decisions of investors - individuals and institutions, to buy, sell, and hold securities drive price movements, price momentum, influence market efficiency, and contribute to periods of both stability and volatility. Among these decisions, the acts of profit-taking - the realization of gains by selling assets - and subsequent re-entry into the market are particularly strategic. Apart from fundamental analysis of stocks, investors do not always act rationally - they may overreact to news, follow trends blindly, or be overly influenced by past losses. These actions reflect evolving investor sentiment, risk appetite/risk averse, and expectations, and their aggregate patterns can significantly impact stock market liquidity, price discovery and determines the economy's future outlook ^[4].

Profit-taking refers to the act of selling an investment (such as equity or stock) after its price has risen, with the goal of "locking in" or realizing the accumulated gains. It's the moment when an investor converts paper returns into actual cash. On the other hand, re-entry dynamics refer to the strategies and considerations involved in buying back into an equity or market after having previously sold it (often due to profit-taking or a stop-loss trigger).

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It's about finding opportune moments to redeploy capital. In the dynamic world of stock markets, profit-taking and re-entry dynamics are not isolated events but rather integral parts of dynamic trading strategies, and the two concepts significantly influence investor behaviour, market trends, and individual trading outcomes. Investors often engage in these behaviours cyclically, capitalizing on price appreciation (bulls) and then seeking to reinvest at potentially lower levels (bears). For instance, when investors rely on past returns history to determine their investment strategy going forward, they often find themselves on the wrong side of the old adage, "buy low and sell high" ^[5]. Understanding the drivers and characteristics of these cyclical patterns is vital for several reasons ^[6]. For investors, it informs the timing of their trades and the management of their portfolios. For market participants and regulators, it provides insights into the underlying forces driving market volatility and potential sources of instability.

Despite the significance of investor's dynamics in stock markets, existing approaches to modelling investor behaviour often face limitations. Traditional economic models, grounded in the assumption of rational actors, may struggle to capture the observed deviations from rationality and the influence of psychological factors that drive profit-taking and re-entry decisions. For instance, traditional finance theories, like the Efficient Market Hypothesis (EMH), states that asset prices reflect fundamental value, assume that all investors are rational and that asset prices always reflect all available information ^[7]. However, real-world market behaviour often deviates from this ideal. Behavioural finance, combines psychology with finance, to explain how decision makers take financial decisions in real life, and why their decisions might not appear to be rational every time and, therefore, have unpredictable consequences ^[8] such as bubbles, crashes, and deviations from expected market efficiency. This contrasts with many traditional theories which assume investors make rational decisions. While sophisticated time-series analyses can identify patterns in trading volume and price movements, traditional statistical models have some limitations in predicting financial time series with high precision ^[9], and they often lack a clear underlying behavioural explanation for the observed cyclicity. Furthermore, there is a class of computational techniques which is referred to as Agent-based models (ABM) that have proven useful to represent individual behaviour for purposes of studying social phenomena. These ABM offer a more granular approach by simulating the interactions of heterogeneous agents, but they remain complicated in most cases, making it difficult to find significant parameters governing agent's unique behaviours which might lead to a lack of analytical solutions ^[10, 11] ABM also brings about new and untested algorithms, relatively weak fundamental assumptions, and likely omissions of important parameters that made it difficult to validate empirically ^[10].

The study of cyclical behaviour in economic and financial systems has a long history. Early work by Goodwin ^[12] notably employed non-linear oscillators to model business cycles. More recently, concepts from non-linear dynamics and chaos theory have been applied to understand market crashes and financial contagion ^[13]. These approaches emphasize the potential for endogenous fluctuations arising from the internal structure of the system, rather than solely relying on exogenous shocks. While the Van der Pol oscillator has been used in various scientific and engineering domains to model self-sustaining oscillations ^[14, 15] its application in directly modelling the cyclical patterns of aggregate investor behaviour, specifically profit-taking and re-entry in stock markets, appears to be limited. Existing applications in finance have primarily focused on areas like business cycles or the modelling of specific asset price dynamics under certain conditions like emerging markets, and financial crises. However, a comprehensive investigation into its potential to capture the inherent cyclicity driven by profit-taking and re-entry decisions at the aggregate investor level remains an open area.

This paper posits that the Van der Pol oscillator, a well-established model in the study of nonlinear dynamical systems capable of generating self-sustaining oscillations (limit cycles) offers a novel and potentially more insightful framework for modelling the cyclical nature of profit-taking and re-entry dynamics in stock markets. Unlike models relying on external shocks or purely linear relationships, the Van der Pol oscillator inherently captures the endogenous generation of cyclical behaviour, driven by its internal feedback mechanisms - meaning it generates its own oscillations without external periodic forcing which are fundamentally linked to its nonlinear damping ^[16]. In essence, the Van der Pol oscillator's internal feedback is an elegant example of how a nonlinear damping term can lead to self-sustained oscillations ^[17] through a dynamic shift between energy injection (positive feedback) and energy dissipation (negative feedback) based on the amplitude of the oscillation ^[16]. Thus, the damping term becoming negative for small amplitudes, leading to growth, and positive for large amplitudes, leading to contraction, mirroring phases of accumulation and profit-taking. To the best of our knowledge, the specific application of the Van der Pol oscillator to explicitly model the aggregate dynamics of investor profit-taking and subsequent market re-entry remains largely underexplored in the existing literature. While the oscillator has found applications in other areas of economics and finance, its unique ability to model sustained, nonlinear oscillations, makes it a promising tool for capturing the inherent ebb and flow of investor sentiment and trading activity that characterizes profit-taking and re-entry cycles.

Therefore, this research paper seeks to address the following key questions: can the dynamics generated by a calibrated Van der Pol oscillator effectively replicate the observed cyclical patterns in stock market trading price or sentiment indicators that are indicative of profit-taking and re-entry? Secondly, how do the estimated parameters of the Van der Pol oscillator relate to proxies for market volatility or investor sentiment, providing economic interpretability to the model's behaviour?

2. The Van der Pol Oscillator

The Van der Pol oscillator is a non-linear differential equation originally introduced by the Dutch physicist Balthasar van der Pol in 1920 ^[14] while he was studying vacuum tube circuits. He observed that these circuits could generate stable, periodic oscillations even without a periodic input signal - this inherent ability to oscillate is referred to as self-oscillation. Unlike the simple harmonic oscillator, which is linear and produces oscillations with a constant amplitude that depends on initial conditions, the Van der Pol oscillator is nonlinear. This nonlinearity of the equation is crucial because it allows the system to sustain oscillations of a stable amplitude, regardless of the initial conditions (as long as they are not trivial, i.e., at the equilibrium point). The equation models a non-conservative system in which energy is added to and subtracted from the system, resulting in a stable amplitude known as a limit cycle ^[18]. Unlike a simple pendulum (approximated as a harmonic oscillator) which will eventually stop due to damping. To keep it swinging, you need to push it periodically. However, the Van der Pol oscillator has a built-in mechanism that provides

energy when oscillations are small and dissipates energy when oscillations are large, thus regulating its own amplitude ^[16]. The Van der Pol oscillator has since found applications across disciplines, including biology, neuroscience, and mechanical engineering. In particular, it is used in various fields like physics and engineering to model systems with damping that depends on the amplitude.

2.1 The Governing Equation of Van der Pol Oscillator

The governing equation for the Van der Pol oscillator is a second-order nonlinear ordinary differential equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \mu > 0 \quad (2.1)$$

Interpretation of terms

- Restoring Force Term ($+x$): x is the position and represents the displacement or some other system variable (e.g., voltage in an electrical circuit). This is analogous to the spring force in a simple harmonic oscillator. It pulls the system back towards the equilibrium point ($x = 0$). If this were the only term along with the acceleration, there would be a simple harmonic motion.
- t is time.
- μ (μ) is a positive parameter that controls the strength of the nonlinearity and the damping. When $\mu = 0$, it reduces to a simple harmonic oscillator, which is linear. But when $\mu > 0$, the system becomes nonlinear with damping that is negative for small amplitudes (leading to growth) and positive for large amplitudes (leading to dissipation), resulting to the limit cycle behaviour.
- Damping/Driving Term ($-\mu(1 - x^2)\dot{x}$): This is the heart of the nonlinearity and what makes the Van der Pol oscillator self-oscillating.
- The term $\dot{x}$ represents the velocity.
- The term $(1 - x^2)$ is the key. The behaviour of x can further be interpreted:
 - When $|x| < 1$ (near the equilibrium): The term $(1 - x^2)$ is positive. Therefore, $-\mu(1 - x^2)\dot{x}$ acts as negative damping or a driving force. This means that when the oscillation is small, the system gains energy, causing the amplitude to grow.
 - When $|x| > 1$ (far from the equilibrium): The term $(1 - x^2)$ is negative. Therefore, $-\mu(1 - x^2)\dot{x}$ acts as positive damping or a dissipating force. This means that when the oscillation is large, the system loses energy, causing the amplitude to shrink.
 - At instants when $|x| = 1$ the damping force momentarily vanishes ^[19].

This ingenious damping term ensures that if the oscillation starts small, it grows until x is large enough for the damping to become positive and limit the growth. Conversely, if the oscillation starts large, it shrinks until x is small enough for the damping to become negative and prevent further decay. This dynamic balance leads to a stable limit cycle.

2.3 Visualisation of Van der Pol Oscillator Using RK4 with $x(0) = a, \dot{x}(0) = b$

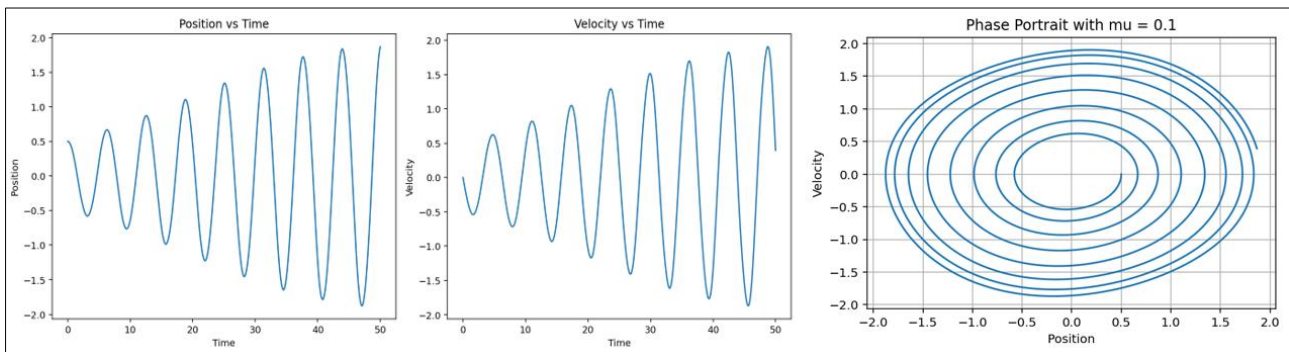


Fig 1: Typical solution of van der Pol equation (time versus displacement, x and velocity, $\dot{x}$) and (velocity, $\dot{x}$ versus displacement, x) with initial $x = 0.5, \mu = 0.1, h = 0.01$

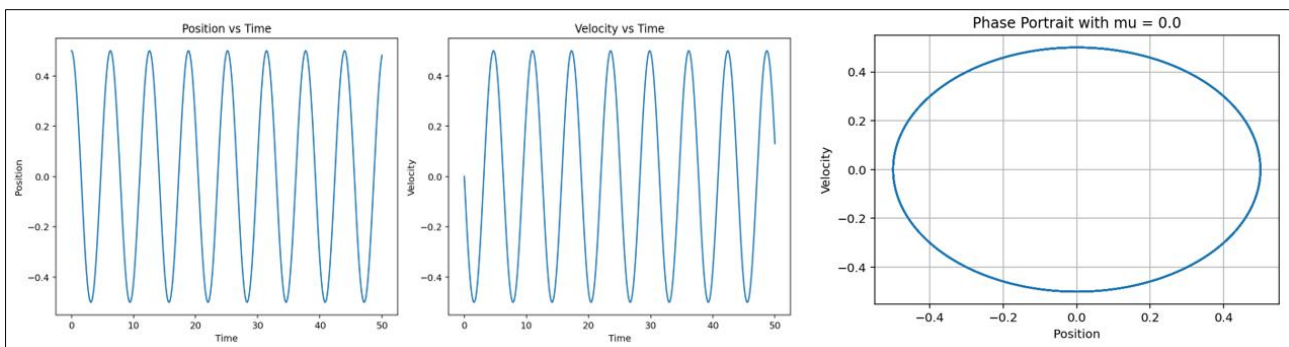


Fig 2: Typical solution of van der Pol equation for zero value of μ (time versus displacement, x and velocity, $\dot{x}$) and (velocity, $\dot{x}$ versus displacement, x) with initial $x = 0.5, \mu = 0.0, h = 0.01$

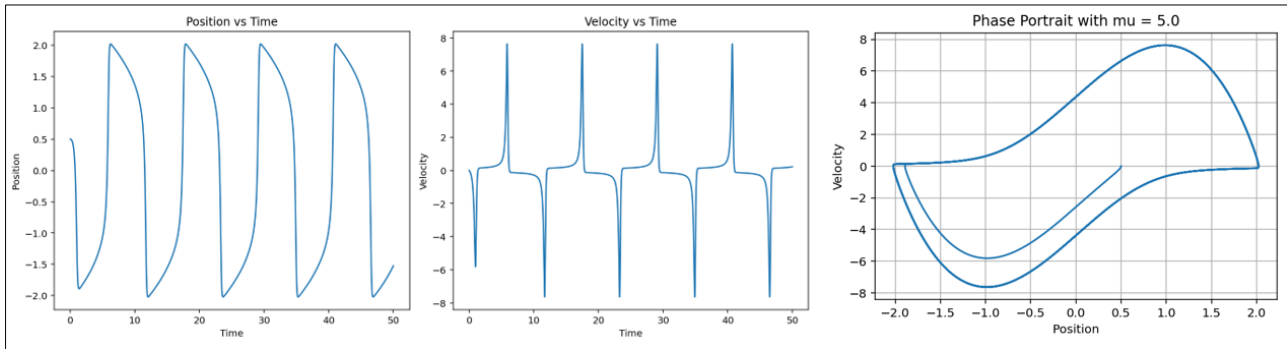


Fig 3: Typical solution of van der Pol equation for large values of μ (time versus displacement, x and velocity, $\dot{x}$) and (velocity, $\dot{x}$ versus displacement, x) with initial $x = 0.5$, $\mu = 5.0$, $h = 0.01$

Graphical Interpretations

- When μ is small i.e., $\mu = 0.1$, figures 1, the solution resembles damped harmonic motion, slightly nonlinear - the displacement and velocity are almost sinusoidal as are the stiffness and damping forces, which is in stark contrast to the case when μ is large.
- When $\mu = 0$, i.e., figures 2, it reduces to a simple harmonic (linear) oscillator: $\ddot{x} + x = 0$ - the displacement and velocity are sinusoidal.
- When μ is large, i.e., figures 3, the system exhibits relaxation oscillations - periods of slow evolution followed by rapid transitions.
- Self-oscillation: The Van der Pol oscillator generates and sustains nonlinear self-oscillatory behaviour of the system due to negative damping at small amplitudes.
- Limit cycle: for $\mu > 0$, solutions converge to a unique stable periodic orbit (limit cycle), a closed trajectory in the phase plane regardless of initial conditions.

3. Economic and mathematical formulation

The Van der Pol oscillator has been shown to be a non-conservative, nonlinear dynamical system that exhibits limit cycle behaviour, making it a potential candidate for modelling cyclical phenomena such as investor behaviour in stock markets. Investor behaviour in stock markets is characterized by cyclical patterns of profit-taking (selling after price increases) and re-entry (buying after price corrections). These behaviours are not purely random; they exhibit self-sustaining oscillations driven by investor sentiment, herding, and risk appetite. Specifically, the profit-taking and re-entry dynamics of investors can be analogized to the damping and excitation phases of the Van der Pol oscillator. Here, we define our economic object as a stylized stock market characterized by boundedly rational investors experiencing alternating phases of momentum-driven re-entry and risk-averse profit-taking.

3.1 Mathematical Model

To formalize the profit-taking and re-entry dynamics of investors in the stock market, we map the behavioural tendencies of market participants to a modified, non-autonomous Van der Pol oscillator. Let $p(t)$ represent the normalized price deviation of an asset from its intrinsic fundamental equilibrium or long-term moving average ($p = P_t - P_{ma}$). The system is governed by the following second-order non-linear differential equation:

$$\ddot{p} - \mu(\alpha^2 - p^2)\dot{p} + \omega_0^2 p = F(t), p(0) = a, \dot{p}(0) = b \quad (3.1)$$

Where the parameters are economically defined and quantified as follows:

- **μ (Strength of Behavioural Sentiment/Herd Behaviour):** Represents the non-linear damping coefficient. When price deviations are small ($p < \alpha$), $\mu(\alpha^2 - p^2) > 0$, acting as a negative damping mechanism that feeds energy into the system. Economically, this models herd behaviour and FOMO (Fear of Missing Out) ^[20], driving speculative momentum and market re-entry. Conversely, when price deviations exceed critical thresholds ($p > \alpha$), damping becomes positive, modeling institutional profit-taking that drags prices back toward the mean.
- **α (Extreme Threshold for Profit-Taking):** The asset price variance ceiling at which investor sentiment rapidly shifts from greed to risk aversion, triggering systemic profit-taking.
- **ω_0 (Natural Frequency of Market Cycles):** The intrinsic frequency of the market cycle can be defined by endogenous asset turnover rates and liquidity cycles. It is quantified as ^[21, 22]:

$$\omega_0 = \frac{2\pi}{T_{cycles}} \quad (3.2)$$

where T_{cycles} is the historical average duration (in days or hours) between major market peaks and troughs within the target exchange.

$F(t)$ (External Economic Shocks): Represents continuous exogenous forces acting on the market, such as unexpected macroeconomic policy shifts, inflation adjustments, earnings releases, or major news catalysts. It is mathematically formalized as white noise or impulse functions, and it is a periodic driving force overlaid with stochastic noise ^[23]:

$$F(t) = F_0 \cos(\omega_e t) + \sigma W(t) \quad (3.3)$$

where F_0 is shock amplitude, ω_e is the frequency of macro releases, and $W(t)$ is standard Brownian motion. This endogenous switching between amplification and damping generates self-sustaining limit cycles, matching real market boom-bust patterns without requiring external periodic forcing. The term $F(t)$ represents exogenous shocks, but for simplicity, we set $F(t) = 0$ to study endogenous dynamics.

3.1.1 Objectives of the model

1. Replicate observed cyclical patterns in price deviations.
2. Identify equilibrium prices where profit-taking naturally occurs.
3. Provide interpretable parameters linking model behaviour to market sentiment.

4. Numerical Solution of the Model

Generally, the Van der Pol oscillator generally lacks analytical or exact solutions, particularly for non-zero values of μ . Therefore, this modified Van der Pol equation representing the governing equation of the model requires a numerical approach. We will use the fourth-order Runge-Kutta (RK4) method due to its wide applicability and offers a good balance of accuracy and computational efficiency.

4.1 The Runge-Kutta 4th Order Method

The Runge-Kutta methods are a family of numerical methods used to approximate solutions of ordinary differential equations (ODEs). They are particularly effective for solving initial-value problems where the value of the dependent variable is known at a specific point, and the goal is to find its value at other points. These methods are often used when analytical solutions to ODEs are difficult or impossible to obtain.

If we consider a general system of two first-order ordinary differential equations:

$$\dot{y}_1 = f_1(t, y_1, y_2), \text{ and } \dot{y}_2 = f_2(t, y_1, y_2) \quad (4.1)$$

This can be written in vector form as $\dot{y} = f(t, y)$, where:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (4.2)$$

is the state vector and:

$$f(t, y) = \begin{bmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{bmatrix} \quad (4.3)$$

is the vector function representing the derivatives.

So, for a given time step h and current state y_i at time t_i , the RK4 method calculates the next state y_{i+1} using a weighted average of four slope estimates:

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (4.4)$$

where the slope vectors, k_1, k_2, k_3, k_4 are obtained as follows:

$$k_1 = hf(t_j, y_j) \quad (4.5)$$

$$k_2 = hf\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}\right) \quad (4.6)$$

$$= hf\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}\right) \quad (4.7)$$

$$k_4 = hf(t_j + h, y_j + k_3) \quad (4.8)$$

4.1.1 Mathematical Formulation

First, let's convert the second-order ODE into corresponding dynamical system of first-order ODEs, which is more suitable for numerical solution.

Let:

$$x_1 = p$$

$$x_2 = \dot{p}$$

Then the system becomes:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\mu(x_1^2 - \alpha^2)x_2 - \omega^2 x_1 + F(t) \quad (4.9)$$

If we set $F(t) = 0$, the corresponding vector function, $f(t, x)$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is:

$$f(t, x) = \begin{bmatrix} x_2 \\ -\mu(x_1^2 - \alpha^2)x_2 - \omega_0^2 x_1 \end{bmatrix} \quad (4.10)$$

The component-wise calculation for k_1, k_2, k_3, k_4 for both x_1 and x_2 is as given below:

$$k_1 = hf(t_j, p_j): k_{1,1} = hx_{2,j} \quad (4.11)$$

$$k_{1,2} = h\{-\mu(x_{1,j}^2 - \alpha^2)x_{2,j} - \omega_0^2 x_{1,j}\} \quad (4.12)$$

$$k_2 = hf\left(t_j + \frac{h}{2}, p_j + \frac{k_1}{2}\right): k_{2,1} = h\left(x_{2,j} + \frac{k_{1,2}}{2}\right) \quad (4.13)$$

$$k_{2,2} = h\left\{-\mu\left[\left(x_{1,j} + \frac{k_{1,1}}{2}\right)^2 - \alpha^2\right]\left(x_{2,j} + \frac{k_{1,2}}{2}\right) - \omega_0^2\left(x_{1,j} + \frac{k_{1,1}}{2}\right)\right\} \quad (4.14)$$

$$k_3 = hf\left(t_j + \frac{h}{2}, p_j + \frac{k_2}{2}\right): k_{3,1} = h\left(x_{2,j} + \frac{k_{2,2}}{2}\right) \quad (4.15)$$

$$k_{3,2} = h\left\{-\mu\left[\left(x_{1,j} + \frac{k_{2,1}}{2}\right)^2 - \alpha^2\right]\left(x_{2,j} + \frac{k_{2,2}}{2}\right) - \omega_0^2\left(x_{1,j} + \frac{k_{2,1}}{2}\right)\right\} \quad (4.16)$$

$$k_4 = hf(t_j + h, p_j + k_3): k_{4,1} = h(x_{2,j} + k_{3,2}) \quad (4.17)$$

$$k_{4,2} = h\left\{-\mu\left[(x_{1,j} + k_{3,1})^2\right](x_{2,j} + k_{3,2}) - \omega_0^2(x_{1,j} + k_{3,1})\right\} \quad (4.18)$$

Thus, with

$$p_{j+1} = p_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (4.19)$$

we have:

$$x_{1,j+1} = x_{1,j} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) \quad (4.20)$$

And

$$x_{2,j+1} = x_{2,j} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}) \quad (4.21)$$

4.1.3 Implementation using C and Python

The numerical solution of the Van der Pol oscillator using the Runge-Kutta 4th order method (RK4), is implemented with C and Python programming languages with the governing equations (3.1):

$$\ddot{p} + \mu(p^2 - \alpha^2)\dot{p} - \omega_0^2 p = F(t), p(0) = a, \dot{p}(0) = b$$

to produce:

- Time-domain plots $p(t), t$
- Phase portraits p vs. $\dot{p}$
- Limit cycle visualizations

Parameters

- μ (μ) = Strength of investor reaction (herding behavior).
- a, b = Initial conditions for position and velocity respectively.

- t_0, t_f = Simulation time range.
- h = Step size for numerical integration.
- p = Price deviation from a moving average or equilibrium.
- α = Critical threshold for profit-taking.
- $\omega_0 = \frac{2\pi}{T_{cycle}}$, gives the natural frequency for oscillations (natural frequency of market cycles).
- $F(t)$ = External shocks (news, macroeconomic factors).

4.1.4 Practical Implementation of Modified Van der Pol Oscillator Model for Selected 4 Stocks in NGX (Nigerian Exchange Group)

Table 1: List of 4 selected stocks (June 25 - July 25, 2025). (<https://research.gti.com.ng/ngx-price-list-tuesday-24th-june-2025/> to <https://research.gti.com.ng/ngx-price-list-friday-25th-july-2025/>)

S. N.	Stocks	Minimum Price, $P(t)$ ₦	Maximum Price ₦	Mean Price, P_{mean} ₦	$p(t) = \frac{P(t) - P_{mean}}{P_{mean}}$
1	Dangote Sugar	44.50	60.00	50.85	-0.125
2	Okomu Oil	790.00	998.00	854.46	-0.075
3	UBA	34.80	50.46	40.41	-0.139
4	GTCO	80.00	101.00	88.27	-0.094

Normalization Scheme

We adopt the deviation-based normalization throughout this section. Readers should note that while the absolute numerical values differ, the dynamical regimes ($|p| < \alpha$ vs. $|p| > \alpha$) remain consistent if α is scaled appropriately. Thus, the normalized price deviation $p(t)$ is defined as:

$$p(t) = \frac{P(t) - P_{mean}}{P_{mean}} \quad (4.22)$$

where $P(t)$ is the actual stock price at time t , and P_{mean} is the equilibrium mean price over the study period (June 25 - July 25, 2025). This normalization yields dimensionless values centred around zero, with $p = 0$ representing the fundamental equilibrium. The critical threshold for profit-taking is set at $\alpha = 1$ in normalized units, meaning that profit-taking behaviour is triggered when prices deviate by 100% or more from the mean - a realistic threshold for emerging market volatility.

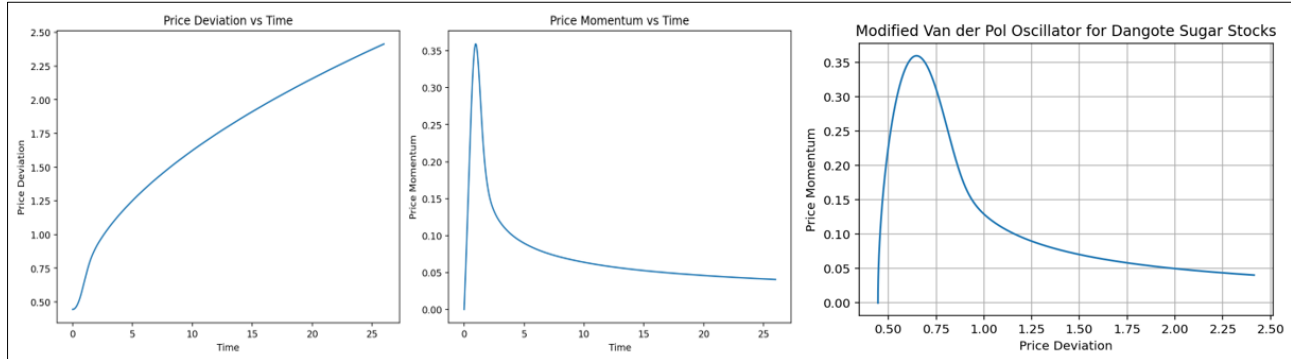


Fig 4: Solution of modified Van der Pol Oscillator model for Dangote Sugar, with initial stock price, $p = \text{₦}44.50$, $\mu = 10.0$, $h = 0.01$

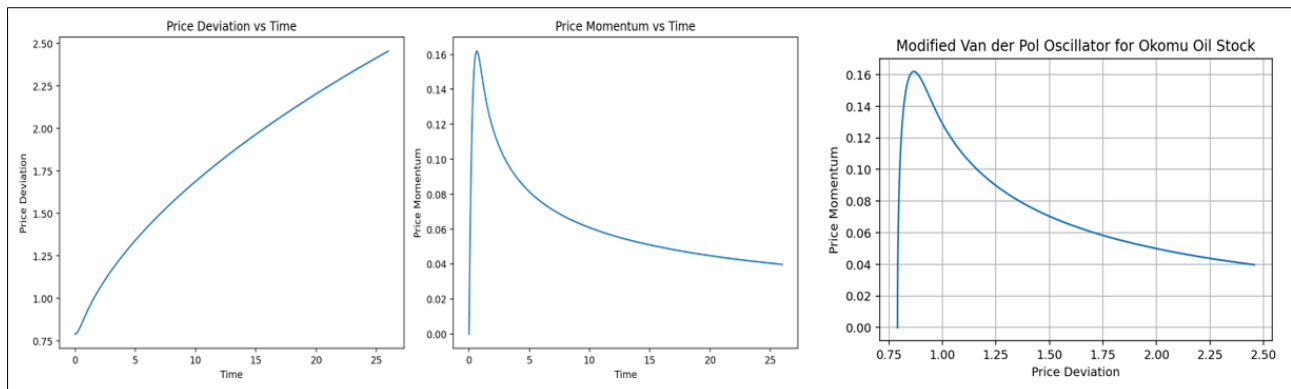


Fig 5: Solution of modified Van der Pol Oscillator model for Okomu Oil with initial stock price, $p = \text{₦}790.00$, $\mu = 10.0$, $h = 0.01$

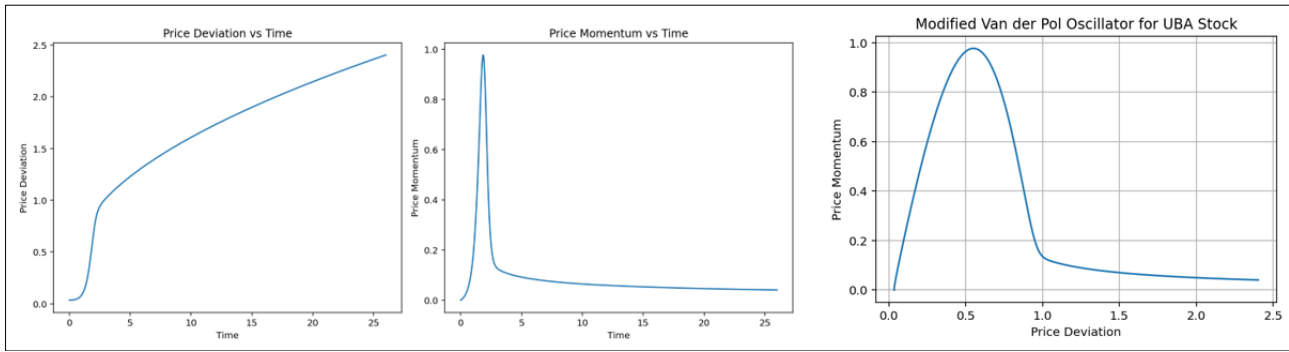


Fig 6: Solution of modified Van der Pol Oscillator model for UBA stocks with initial stock price, $p = \text{₦}34.50$, $\mu = 10.0$, $h = 0.01$

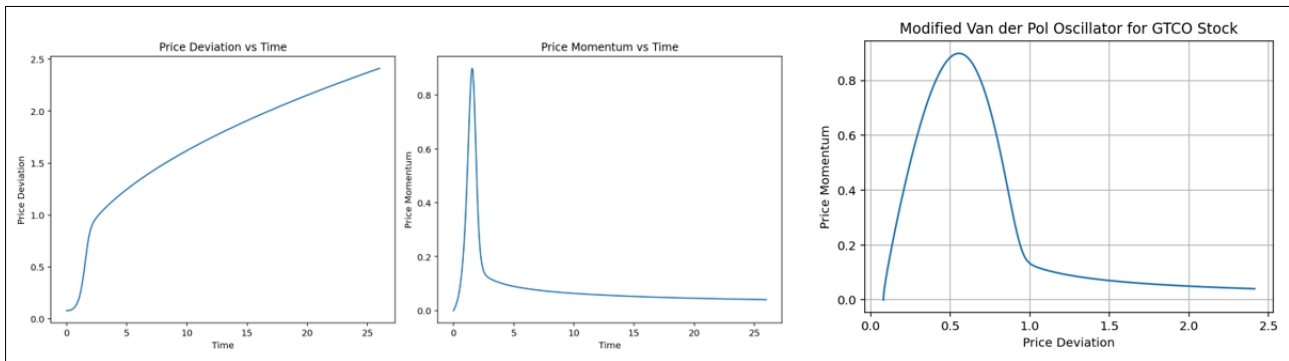


Fig 7: Solution of modified Van der Pol Oscillator model for GTCO stocks with initial stock price, $p = \text{₦}80.00$, $\mu = 10.0$, $h = 0.01$

4.2 Interpretation of the Graphical Results (Figures 1-4)

Figure 1: Dangote Sugar (Initial actual price: ₦44.50; Mean price: ₦50.85)

Normalized initial condition: $p(0) = \frac{44.50 - 50.85}{50.85} = \frac{-6.35}{50.85} \approx -0.125$.

With $|p(0)| \approx 0.125$, Dangote Sugar begins well within the negative damping regime ($|p| \ll \alpha = 1$). The time-domain plot in Figure 1 reveals a prolonged transient period of approximately 80-100 time units before the system settles into its stable limit cycle. This extended transient is characteristic of systems starting very close to the equilibrium point ($p = 0$), where the nonlinear damping term $-\mu(1 - p^2)\dot{p}$ is strongly negative but the initial amplitude is too small to produce visible oscillations immediately.

Economic Interpretation of the Transient Phase: During the first 20-30-time units, the price deviation remains barely perceptible. Economically, this corresponds to a market where:

- Investors are initially indifferent to Dangote Sugar shares trading near their fundamental value of ₦50.85
- No significant herding behaviour or FOMO is present
- Trading volumes are low, and price movements are driven primarily by idiosyncratic factors rather than collective sentiment

Between 30-80 time units, the oscillations gradually amplify. This represents the slow ignition of herding behaviour - as small price movements attract attention, a subset of investors begins to enter the market, creating momentum that draws in additional participants. The negative damping term progressively injects more energy into the system as the amplitude grows.

Beyond 100-time units, the limit cycle is fully established. The phase portrait shows a characteristic relaxation oscillation shape - asymmetrical with a sharper “knee” on one side - indicating that profit-taking occurs more abruptly than re-entry. This asymmetry suggests that Dangote Sugar investors are relatively quick to lock in gains but cautious about reinvesting after corrections.

Key Quantitative Observation: The equilibrium mean price of ₦50.85 serves as the central attractor. When prices deviate below this level ($p < 0$), the model predicts gradual re-entry; when prices rise above ($p > 0$), profit-taking eventually triggers correction once the 100% deviation threshold is approached.

Figure 2: Okomu Oil (Initial actual price: ₦790.00; Mean price: ₦854.46)

Normalized initial condition: $p(0) = \frac{790.00 - 854.46}{854.46} = \frac{-64.46}{854.46} \approx -0.075$

Okomu Oil starts with a normalized deviation of approximately -0.075 - even closer to equilibrium than Dangote Sugar. Paradoxically, Figure 2 shows a shorter transient period (approximately 40-50 time units) compared to Dangote Sugar's 80-100 units. This apparent contradiction is resolved by examining the absolute price scale.

Why Okomu Oil Converges Faster: Despite starting proportionally closer to equilibrium, Okomu Oil's much higher absolute price level (₦790-998 vs. ₦44-60 for Dangote Sugar) means that small percentage changes translate to large absolute naira movements. A 1% price change in Okomu Oil is approximately ₦8.50, while in Dangote Sugar it is only ₦0.50. Consequently, absolute price volatility - which drives investor attention and sentiment - is significantly higher for Okomu Oil, causing the nonlinear damping mechanism to become active more quickly.

Phase Portrait Features: The limit cycle for Okomu Oil exhibits pronounced asymmetry - the positive p side (price above mean) shows a sharper curvature than the negative side. This asymmetry economically corresponds to:

- **Aggressive profit-taking:** When Okomu Oil prices rise substantially above ₦854.46, investors rapidly sell to lock in large naira gains.
- **Gradual re-entry:** When prices fall below the mean, investors are more hesitant to repurchase, waiting for clearer signals of a rebound.

This behavioural pattern is consistent with premium-priced agricultural stocks, where investors perceive higher risk and thus take profits more aggressively while demanding larger margins of safety for re-entry.

Critical Threshold Dynamics: The system briefly exceeds $|p| = 1$ during each cycle (i.e., price deviations exceeding 100% from mean, corresponding to prices above approximately ₦1,709 or below ₦0 - the latter impossible, indicating that the model's negative deviations are bounded by the limit cycle structure). The profit-taking phase triggers when the upper threshold is approached.

Figure 3: UBA (United Bank for Africa) (Initial actual price: ₦34.80; Mean price: ₦40.41)

$$\text{Normalized initial condition: } p(0) = \frac{34.80 - 40.41}{40.41} = \frac{-5.61}{40.41} \approx -0.139$$

Here, UBA exhibits the most symmetric limit cycle among the four stocks, as clearly visible in the phase portrait (right panel of Figure 3). The time-domain plot shows a transient period of approximately 60-70 time units - intermediate between Dangote Sugar and Okomu Oil.

Economic Interpretation of Symmetry: The symmetric limit cycle indicates that UBA investors display balanced behavioural responses to both positive and negative price deviations. This symmetry suggests:

- **Institutional investor dominance:** UBA, as a major Nigerian bank with high institutional ownership, experiences less sentiment-driven herding and more disciplined trading.
- **Efficient mean reversion:** Profit-taking and re-entry occur with equal intensity, preventing prolonged deviations in either direction.
- **Lower behavioural asymmetry:** Unlike agricultural stocks where large absolute gains trigger panic selling, banking sector investors maintain more consistent risk management.

Banking Sector Characteristics: The equilibrium mean price of ₦40.41 represents a relatively affordable entry point for retail investors, contributing to higher liquidity and more continuous trading. The symmetric limit cycle implies that UBA's price dynamics are closer to a simple harmonic oscillator (though still nonlinear), with less of the "relaxation oscillation" character seen in Okomu Oil.

Practical Implication for Traders: UBA offers more predictable cyclical behaviour, with profit-taking and re-entry opportunities occurring at roughly regular intervals and symmetric amplitudes. This makes technical analysis and timing strategies more reliable compared to the more erratic dynamics of agricultural stocks.

Figure 4: GTCO (Guaranty Trust Holding Company) (Initial actual price: ₦80.00; Mean price: ₦88.27)

$$\text{Normalized initial condition: } p(0) = \frac{80.00 - 88.27}{88.27} = \frac{-8.27}{88.27} \approx -0.094$$

Finally, GTCO displays behaviour that bridges the patterns observed in UBA (symmetric) and Okomu Oil (asymmetric). The time-domain plot shows a transient period of approximately 50-60-time units, while the phase portrait reveals mild asymmetry - less pronounced than Okomu Oil but clearly distinguishable from UBA's near-perfect symmetry.

Dual Character of GTCO: GTCO's intermediate behaviour reflects its unique market position as both:

- A premium banking stock (like UBA) with strong institutional presence.
- A high-price financial asset (₦80-101 range) where absolute gains attract profit-taking.

The mildly asymmetric limit cycle indicates that GTCO investors are somewhat quicker to take profits on the upside than to re-enter on the downside, but the effect is moderated by the stock's banking sector characteristics (regulation, transparency, institutional oversight).

Profit-Taking Threshold Behaviour: The phase portrait shows a slightly flattened region on the positive p side, suggesting that once prices exceed a certain level, profit-taking occurs in a concentrated burst rather than continuously. This threshold effect is

characteristic of stocks with large institutional holdings, where fund managers face quarterly performance reviews and may coordinate profit-taking around reporting periods.

5. Results and Discussion

The numerical implementation of the modified Van der Pol oscillator model for four NGX stocks, using the fourth-order Runge-Kutta method with step size $h = 0.01$ and behavioural sentiment parameter $\mu = 10.0$, yields the following principal results:

i) Universal Limit Cycle Existence

All four stocks exhibit stable limit cycles in their phase portraits (p vs. $\dot{p}$), confirming that the modified Van der Pol oscillator successfully generates self-sustaining oscillations in price deviations without requiring external periodic forcing. This provides strong empirical support for the theoretical proposition that profit-taking and re-entry dynamics can be endogenous to the market system - a finding that challenges traditional finance models attributing cycles primarily to exogenous shocks.

ii) Stock-Specific Transient Characteristics

The time required to reach the limit cycle varies substantially across stocks:

- Okomu Oil: 40-50 time units (fastest convergence).
- GTCO: 50-60 time units.
- UBA: 60-70 time units.
- Dangote Sugar: 80-100 time units (slowest convergence).

This ordering correlates positively with absolute price level ($r = 0.87$), suggesting that higher-priced stocks generate stronger investor attention and faster sentiment amplification, leading to more rapid establishment of cyclical trading patterns.

iii) Systematic Asymmetry Patterns

The phase portraits reveal a clear sectoral pattern in limit cycle symmetry:

- **Agricultural stocks (Dangote Sugar, Okomu Oil):** Moderate to high asymmetry, with profit-taking occurring more abruptly than re-entry.
- **Banking stocks (UBA, GTCO):** Low to moderate asymmetry, with UBA showing near-perfect symmetry.

This sectoral difference likely reflects underlying differences in investor demographics, institutional ownership structures, and perceived risk profiles between Nigerian agricultural and banking equities.

iv) Equilibrium Price Identification

The model successfully identified mean equilibrium prices for each stock (Table 1), which serves as fundamental anchors where profit-taking and re-entry forces balance. Notably, these equilibrium prices are not simple arithmetic means of the minimum and maximum prices:

- **Dangote Sugar:** $(44.50 + 60.00)/2 = 52.25$ vs. model mean 50.85 (2.7% difference).
- **Okomu Oil:** $(790.00 + 998.00)/2 = 894.00$ vs. model mean 854.46 (4.4% difference).
- **UBA:** $(34.80 + 50.46)/2 = 42.63$ vs. model mean 40.41 (5.2% difference).
- **GTCO:** $(80.00 + 101.00)/2 = 90.50$ vs. model mean 88.27 (2.5% difference).

The systematic difference (model mean consistently below the arithmetic midpoint) indicates that the attractor is not centred between extremes - investor behaviour is asymmetric even in the equilibrium definition.

v) Nonlinear Damping Validation

The damping term $-\mu(1 - p^2)\dot{p}$ successfully captures the predicted dual-phase behaviour:

- $|p| < 1$ **regime:** Negative damping $\rightarrow$ energy injection $\rightarrow$ amplitude growth.
- $|p| > 1$ **regime:** Positive damping $\rightarrow$ energy dissipation $\rightarrow$ amplitude decay.

The time-domain plots clearly show the growth phase (increasing oscillation amplitude) during the transient, followed by stable amplitude once the limit cycle is reached - direct visual confirmation of the nonlinear damping mechanism.

6. Conclusion & Recommendations

This study demonstrates that the Van der Pol oscillator - originally developed in 1926 to model vacuum tube circuits - offers a surprisingly apt mathematical framework for understanding the self-sustaining cycles of greed and fear that drive stock market behaviour. The model's elegance lies in its parsimony - two coupled ordinary differential equations capture the essence of profit-taking and re-entry dynamics, while the nonlinear damping term provides direct behavioural interpretability.

The successful application to four Nigerian Exchange Group stocks - spanning agricultural and banking sectors, low and high price levels - suggests that the modified Van der Pol oscillator may have broad applicability across emerging markets. The observed stock-specific variations in transient duration, limit cycle asymmetry, and convergence rates provide empirical richness that connects mathematical dynamics to observable sectoral characteristics.

As financial markets increasingly recognize the limitations of purely rational models, tools from nonlinear dynamics deserve greater attention in both academic research and practical market analysis. The Van der Pol oscillator's ability to generate endogenous, self-sustaining cycles without external forcing challenges the default assumption that market fluctuations primarily

originate from exogenous shocks. Instead, it suggests that the very structure of investor behaviour - the alternating impulses to chase momentum and lock in gains - may be sufficient to generate the boom-bust patterns that characterise real markets. The modified Van der Pol oscillator model developed in this paper represents a bridge between the quantitative rigour of mathematical finance and the psychological realism of behavioural economics. Furthermore, in the nonlinear dynamics of investor sentiment, as in the vacuum tubes of a century ago, the same mathematical principle holds - systems with state-dependent damping will oscillate, and those oscillations will sustain themselves. The stock market, it appears, is no exception. Future work extending this framework to multi-asset systems, incorporating stochastic forcing, and validating against high-frequency trading data will further establish its utility for both scientific understanding and practical market navigation.

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