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Fuzzy demand three-warehouse deteriorating inventory model with carbon trading and trade credit

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Abstract

This paper extends the crisp three-warehouse inventory model for non-instantaneously deteriorating (NID) items to an environment of epistemic demand uncertainty by representing the ramp-type demand scale and growth parameters as triangular fuzzy numbers (TFNs), $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$, while retaining the price-dependent iso-elastic structure $\tilde{D}(t, p) = (\tilde{a} + \tilde{b}t)p^{-\eta}$. The model integrates four features already established as individually important in the literature non-instantaneous deterioration with preservation-technology investment, exponential time-dependent partial backlogging, a carbon cap-and-trade (CCT) compliance mechanism, and two-part (M,N) trade credit within a three-tier warehouse hierarchy (RW1 $\rightarrow$ RW2 $\rightarrow$ OW) under continuous inflationary discounting. Because every cost and revenue component of the model is shown to be affine in $(\tilde{a}, \tilde{b})$ for fixed decision variables, the fuzzy total profit $\tilde{\Pi}$ is proved to be itself a triangular fuzzy number, its three defining components are obtained by simply evaluating the crisp profit function at the three defining triples of $(\tilde{a}, \tilde{b})$. The fuzzy objective is defuzzified using the signed-distance method, and, as an extension for risk-sensitive planning, using Yager's optimism-pessimism ranking index λ . A novel analytical result a certainty equivalence proposition is proved under symmetric triangular fuzzification, the signed distance optimal decision vector and objective value coincide exactly with the crisp optimum evaluated at the modal demand parameters, so that fuzziness affects the width of the profit band (decision risk) but not the risk-neutral decision itself, risk attitude only enters through the Yager index. Existence of an interior optimal price is established via the classical iso-elastic markup argument, and second-order optimality is confirmed through a negative definite Hessian with alternating sign leading principal minors, verified both analytically and at the numerical optimum. A comprehensive numerical study and five parameter sensitivity analysis (ramp growth, customer impatience, carbon price, credit period, deterioration rate), together with a spread analysis of the fuzzy ambiguity band and a Yager index risk analysis, illustrate the model's behaviour and yield concrete managerial recommendations for warehouse sequencing, preservation investment, and credit negotiation under demand ambiguity.

Keywords: Fuzzy inventory, triangular fuzzy number, non-instantaneous deterioration, ramp-type demand, two-part trade credit, signed-distance defuzzification, Yager index.

Introduction

Classical inventory models from the economic order quantity of Harris (1913)^[7] to its many deteriorating-item, multi-warehouse, and financially augmented descendants are formulated under the convenient but restrictive assumption that demand parameters are known with certainty. In practice, a retailer estimating the scale and growth rate of ramp-type demand for a new or seasonal product relies on managerial judgment, limited historical data, and market forecasts that are inherently imprecise rather than genuinely random. Probability theory requires a fully specified distribution that is rarely available in such settings, whereas fuzzy set theory (Zadeh, 1965)^[15] offers a natural language for encoding expert judgment of the form "demand will most likely be around a_2 units per week, almost certainly no lower than a_1 and no higher than a_3 " precisely the semantics of a Triangular Fuzzy Number (TFN).

This paper develops a fuzzy extension of the three-warehouse non-instantaneously deteriorating (NID) inventory framework studied for crisp ramp-type iso-elastic demand,

Exponential partial backlogging, carbon cap and trade regulation, and two-part trade credit. Rather than treating fuzziness as a peripheral robustness check, we develop it as the organizing methodological contribution of the paper, we prove that the fuzzy total profit function is itself a triangular fuzzy number (Proposition 1), we prove a certainty equivalence result showing exactly when and why the signed distance-optimal decision coincides with the crisp modal optimum (Proposition 2), we establish existence of an interior optimal price (Proposition 3), and we verify second-order optimality via a signed, alternating Hessian (Proposition 4). We further introduce a risk sensitive extension using Yager's (1981) ^[13] optimism–pessimism ranking index, which unlike the signed-distance criterion is shown numerically to shift the optimal preservation investment and selling price as the decision maker's risk attitude changes.

The remainder of the paper is organised as follows. Section 2 reviews the relevant literature on fuzzy inventory control and situates the contribution. Section 3 develops the fuzzy-set preliminaries used throughout. Section 4 lists notation and assumptions. Section 5 develops the mathematical model. Section 6 states and proves the main propositions. Section 7 gives the solution algorithm. Section 8 reports a numerical illustration, cost breakdown, and multi-dimensional sensitivity analysis. Section 9 discusses managerial insights, and Section 10 concludes.

2. Literature Review

2.1 Fuzzy set theory in inventory control

Zadeh (1965) ^[15] introduced fuzzy set theory as an alternative to probability for representing imprecise, linguistically expressed information, and Kaufmann and Gupta (1991) ^[8] developed the associated fuzzy arithmetic that underlies most subsequent fuzzy-EOQ work. Yao and Lee (1996) and Lee and Yao (1998) ^[14] were among the first to formulate economic order quantity and economic production quantity models with fuzzy demand and fuzzy production rate, defuzzifying the resulting fuzzy total cost using the extension principle and the graded mean integration representation. Chang, Yao and Ouyang (2006) ^[14] extended the fuzzy treatment to mixture inventory models with variable lead time, comparing several defuzzification indices. De and Sana (2013) developed an intuitionistic fuzzy backlogging EOQ model with promotional-effort and price-sensitive demand, illustrating how fuzzy and intuitionistic fuzzy representations can be embedded directly inside optimisation rather than applied only as a post-hoc sensitivity device.

2.2 Fuzzy deteriorating and multi-warehouse inventory models

Halim, Giri and Chaudhuri (2008) formulated a fuzzy economic order quantity model for perishable items with stochastic demand, partial backlogging, and a fuzzy deterioration rate, showing that fuzzifying the deterioration rate alone materially widens the profit ambiguity band relative to fuzzifying demand alone. Mahata and Goswami (2013) ^[10] considered fuzzy EOQ models for deteriorating items with stock-dependent demand and non-linear holding cost, using the signed distance method for defuzzification the same method adopted in the present paper for its analytical tractability and its property (proved here for the first time in the ramp-type or three-warehouse setting) of reducing to the crisp modal solution under symmetric fuzziness. Multi-warehouse extensions incorporating fuzziness remain comparatively rare, most two and three-warehouse deteriorating-item models (Yang 2004, 2006, Rong *et al.*, 2008, Kumar and Mahapatra, 2021) retain crisp demand, leaving the interaction between warehouse-sequencing decisions and demand ambiguity largely unexplored.

2.3 Ranking and Defuzzification Methods

Yager (1981) ^[13] proposed a family of ranking indices for ordering fuzzy subsets of the real line, including the optimism–pessimism weighted index used in Section 6.4 of this paper, which nests the pessimistic (worst-case, $\lambda=0$), the signed distance-equivalent risk-neutral ($\lambda=0.5$, under symmetry), and the optimistic (best-case, $\lambda=1$) rankings as special cases. The signed-distance method used as the primary defuzzification criterion in this paper was developed for triangular and trapezoidal fuzzy numbers and has been widely adopted in fuzzy inventory optimisation (Yao and Lee, 1996, Mahata and Goswami, 2013) ^[10, 14] because it preserves linearity, the signed distance of an affine combination of fuzzy numbers equals the affine combination of their signed distances, a property exploited directly in Proposition 1 below.

2.4 Positioning of the present study

The crisp antecedent of the present model integrates ramp-type iso-elastic demand, exponential backlogging, carbon cap-and-trade, and two-part trade credit within a three-warehouse NID framework. To the best of the authors' knowledge, no prior study has (i) fuzzified the ramp-demand parameters within a three-warehouse NID system carrying all four of these features simultaneously, (ii) proved that the resulting fuzzy profit is itself triangular and derived a certainty equivalence condition under signed distance defuzzification, or (iii) contrasted a risk-neutral defuzzification criterion against a risk-sensitive Yager-index criterion within this class of models. This paper fills that gap.

3. Fuzzy set preliminaries

Definition 1 (Triangular Fuzzy Number).

A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$, with $a_1 \leq a_2 \leq a_3$, is a fuzzy subset of $\mathbb{R}$ with membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3, \\ 0, & \text{otherwise.} \end{cases}$$

The value a_2 is the mode (most-likely value), and $[a_1, a_3]$ is the support (the extreme range regarded as possible).

Definition 2 (Fuzzy Arithmetic)

For TFNs $\tilde{a} = (a_1, a_2, a_3)$, $\tilde{b} = (b_1, b_2, b_3)$, and scalar $k > 0$, addition and scalar multiplication follow component-wise rules, $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$, and $k\tilde{a} = (ka_1, ka_2, ka_3)$. These rules follow from the extension principle applied to monotonic functions and are used repeatedly below because every term of the total-profit function is, for fixed decision variables, an affine (monotonic, order-preserving) function of the pair $(\tilde{a}, \tilde{b})$.

Definition 3 (Signed Distance)

For a TFN $\tilde{v} = (v_1, v_2, v_3)$, the signed distance from $\tilde{v}$ to the origin (Yao and Lee, 1996)^[14] is

$$d(\tilde{v}, 0) = \frac{1}{4}(v_1 + 2v_2 + v_3).$$

The signed distance is a linear functional on the space of TFNs: $d(k_1\tilde{u} + k_2\tilde{v}, 0) = k_1 d(\tilde{u}, 0) + k_2 d(\tilde{v}, 0)$, for scalars k_1, k_2 and TFNs $\tilde{u}, \tilde{v}$. This linearity is the analytical device underlying Proposition 2.

Definition 4 (Yager's Optimism-Pessimism Index)

For a TFN $\tilde{v} = (v_1, v_2, v_3)$ and optimism parameter $\lambda \in [0, 1]$, Yager's (1981)^[13] weighted ranking index used here is $R_\lambda(\tilde{v}) = \lambda v_3 + (1 - \lambda)v_1$, which ranks $\tilde{v}$ using only its optimistic (v_3) and pessimistic (v_1) extremes, weighted by the decision-maker's risk attitude λ . Here, $\lambda = 0$ is the max-min (pessimistic/worst-case) criterion, $\lambda = 1$ is the max-max (optimistic/best-case) criterion, and $\lambda = 0.5$ places equal weight on both extremes while ignoring the mode.

4. Notation and Assumptions**4.1 Decision Variables**

t_1, t_2, t_3 : depletion times of RW1, RW2, and OW, respectively (weeks),

p : selling price (\$/unit),

ζ : preservation-technology investment per cycle (\$).

4.2 Fuzzy and Crisp Parameters

Notation	Description
$\tilde{a} = (a_1, a_2, a_3), \tilde{b} = (b_1, b_2, b_3)$	Triangular fuzzy ramp-demand scale and growth parameters
η	Price elasticity ($\eta > 1$, crisp)
κ	Customer-impatience (backlogging) parameter
β_1, β_2, α	Deterioration rates in RW1, RW2, and OW, respectively
$m(\zeta) = e^{-\theta\zeta}, \theta > 0$	Preservation-effectiveness function with $m'(\zeta) < 0$ and $m''(\zeta) > 0$
t^d	Freshness (non-deterioration) period
T	Cycle length
N_1, N_2, N_3	Maximum rentable/ownable capacities of RW1, RW2, and OW (treated as upper bounds on the order-up-to quantities Q_1, Q_2, Q_3 , consistent with real capacity constraints)
A	Ordering cost
c	Unit purchase cost
HC^{R1}, HC^{R2}, HC^A	Holding-cost rates for RW1, RW2, and OW, respectively
C_s	Shortage-cost rate
μ, q_1, q_2, q_3	Transportation-cost parameters
M, N	Supplier-to-retailer and retailer-to-customer credit periods ($N \leq M$)
I^e, I^c	Interest earned and interest charged rates
$\bar{C}$	Carbon-emission cap per cycle
p^c	Carbon-trading price
e_0	Emission coefficient per unit
r	Continuous inflation/discount rate

4.3 Assumptions

(i) Single-item, instantaneous replenishment, zero lead time.

(ii) Fuzzy demand, $\tilde{D}(t, p) = (\tilde{a} + \tilde{b}t)p^{-\eta}$, for $I(t) \geq 0$; during shortage the effective withdrawal rate is, $B(t) \cdot \tilde{D}(t, p)$ with exponential backlogging fraction $B(t) = e^{-\kappa(T-t)}$.

(iii) NID structure: items placed in a warehouse begin deteriorating only after the freshness period t^d has elapsed since that warehouse began serving demand; preservation investment ζ scales the deterioration rate down via the multiplier $m(\zeta)$.

(iv) Depletion priority RW1 $\rightarrow$ RW2 $\rightarrow$ OW (the highest holding-cost warehouse is depleted first).

(v) Each warehouse's order-up-to quantity Q_1, Q_2, Q_3 is chosen so that it exactly reaches zero at its designated depletion time (t_1, t_2, t_3 respectively), subject to the capacity upper bounds N_1, N_2, N_3 .

(vi) Total emissions, $E = e_0 L$ are compared with the cap $\bar{C}$ under a market-based cap-and-trade instrument; the net cost, $p^c(E - \bar{C})$ may be negative (a credit) if $E < \bar{C}$.

(vii) Two-part trade credit, Case $T > M$.

(viii) All costs are discounted at continuous inflation rate r ; deteriorated units are written off.

(ix) $\tilde{a}$ and $\tilde{b}$ are triangular fuzzy numbers with $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$, independent of t and p .

5. Fuzzy model formulation

5.1 Fuzzy inventory dynamics

For a fixed representative pair (a, b) later taken to be each of (a_1, b_1) , (a_2, b_2) , (a_3, b_3) in turn the inventory dynamics replicate the crisp NID structure with demand $D_t = (a + bt)p^{-\eta}$.

RW1($0 < t < t^d$):

$$\frac{dI^{R1}}{dt} = -D_t$$

RW1($t^d < t < t_1$):

$$\frac{dI^{R1}}{dt} + \beta_1 m(\zeta) I^{R1} = -D_t$$

RW2($t_1 < t < t_2$):

$$\frac{dI^{R2}}{dt} + \beta_2 m(\zeta) I^{R2} = -D_t$$

OW($t_2 < t < t_3$):

$$\frac{dI^A}{dt} + \alpha m(\zeta) I^A = -D_t$$

Shortage($t_3 < t < T$):

$$\frac{dI^S}{dt} = -e^{-\kappa(T-t)} \cdot D_t$$

With each warehouse's order-up-to quantity fixed by the terminal condition that it depletes to exactly zero at its own designated time. Solving these linear first-order ODEs (identical technique to the crisp case) gives, with the auxiliary functions

$$\Phi(t, \beta) = \frac{a + bt}{\beta m(\zeta)} - \frac{b}{(\beta m(\zeta))^2},$$

$$\Psi(t) = \frac{a + bt}{\kappa} - \frac{b}{\kappa^2},$$

the required order-up-to quantities

$$Q_1(t_1) = p^{-\eta} \left\{ at^d + \frac{b(t^d)^2}{2} + e^{\beta_1 m(\zeta)(t_1 - t^d)} \Phi(t_1, \beta_1) - \Phi(t^d, \beta_1) \right\},$$

$$Q_2(t_1, t_2) = p^{-\eta} \{ e^{\beta_2 m(\zeta)(t_2 - t_1)} \Phi(t_2, \beta_2) - \Phi(t_1, \beta_2) \},$$

$$Q_3(t_2, t_3) = p^{-\eta} \{ e^{\alpha m(\zeta)(t_3 - t_2)} \Phi(t_3, \alpha) - \Phi(t_2, \alpha) \},$$

$$R(t_3) = p^{-\eta} \{ \Psi(T) - e^{-\kappa(T - t_3)} \Psi(t_3) \},$$

$$L = Q_1 + Q_2 + Q_3 + R.$$

Each Q_i and R is manifestly an affine function of (a, b) for fixed $(t_1, t_2, t_3, p, \zeta)$ a fact used directly in Proposition 1.

5.2 Cost and Revenue Components

With all costs discounted at continuous inflation rate r , the ordering, purchasing, holding, deterioration, shortage, transportation, carbon cap and trade, preservation, trade-credit, and revenue components are computed exactly as in the crisp model, using the fuzzy-parametrised Q_i , R , L above:

$$OC = Ae^{-rT}, \quad PC = cL, \quad HC^{Ri} = HC^{Ri} \cdot \int I^{Ri}(t) e^{-rt} dt, \quad (i = 1, 2), \quad HC^A = HC^A \int I^A(t) e^{-rt} dt, \quad DC^{Ri} = c\beta_i m(\zeta) \int I^{Ri}(t) e^{-rt} dt, \quad DC^A = c\alpha m(\zeta) \int I^A(t) e^{-rt} dt, \quad SC = C_s \int_{t_3}^T |I^S(t)| e^{-rt} dt,$$

$$TC = \mu e^{-rT} (q_1 + q_2 L + q_3 L), \quad CCT = p^c (e_0 L - \bar{C}), \quad PTI = \zeta e^{-rT}, \quad IE = I^e p^{1-\eta} \left[\frac{a(M-N)^2}{2} + \frac{b(M-N)^3}{6} \right], \quad IC = I^c c p^{-\eta} \left[\frac{a(T-M)^2}{2} + \frac{b(T-M)^2(2T+M)}{6} \right], \quad TCC = IC - IE, \quad SR = \frac{p^{1-\eta}}{T} \left\{ at_3 + \frac{bt_3^2}{2} + \int_{t_3}^T e^{-\kappa(T-t)} (a + bt) dt \right\},$$

$$TIC = \frac{1}{T} [OC + PC + HC^{R1} + HC^{R2} + HC^A + DC^{R1} + DC^{R2} + DC^A + SC + TC + CCT + PTI + TCC],$$

$$TP(t_1, t_2, t_3, p, \zeta, a, b) = SR - TIC.$$

Denote $mby TP_L, TP_M, TP_U$ the values obtained by evaluating $TP(\cdot, a, b)$ at $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ respectively, for fixed decision variables $(t_1, t_2, t_3, p, \zeta)$. The fuzzy total profit is the triangular fuzzy number

$$\tilde{TP}(t_1, t_2, t_3, p, \zeta) = (T, P_L, TP_M, TP_U).$$

5.3 Defuzzification Objective

The primary (risk-neutral) decision criterion is the signed distance of $\tilde{TP}$:

$$D(t_1, t_2, t_3, p, \zeta) = d(\tilde{TP}, 0) = \frac{1}{4}(TP_L + 2TP_M + TP_U),$$

Maximised subject to $t^d < t_1 < t_2 < t_3 < T$, $p > 0$, $\zeta \geq 0$, and $Q_i \leq N_i$, ($i = 1, 2, 3$). As a risk-sensitive alternative (Section 6.4), the Yager index $R_\lambda(\tilde{TP}) = \lambda TP_U + (1 - \lambda)TP_L$, is used for $\lambda \in [0, 1]$.

6. Analytical results

Proposition 1 (Triangularity of the fuzzy profit).

For fixed decision variables $(t_1, t_2, t_3, p, \zeta)$, the fuzzy quantities $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{R}, \tilde{L}$, and $\tilde{TP}$ are all triangular fuzzy numbers, obtained by evaluating the corresponding crisp expression at $(a_1, b_1), (a_2, b_2), (a_3, b_3)$, respectively, and satisfy the ordering $Q_{1j} \leq Q_{1k}, TP_j \leq TP_k$, whenever $j < k$, ($j, k \in \{L, M, U\}$), provided the crisp profit is monotone increasing in (a, b) at the candidate optimum.

Proof.

Fix $(t_1, t_2, t_3, p, \zeta)$. Inspection of Section 5.1 shows $\Phi(t, \beta)$ and $\Psi(t)$ are affine in (a, b) (linear terms in a and b , no cross-products, no powers other than one). Hence Q_1, Q_2, Q_3, R are each affine functions of (a, b) being finite sums of products of $e^{(\cdot)}$ constants (independent of a, b) with the affine functions Φ, Ψ , plus additional affine terms $at^d + \frac{b(t^d)^2}{2}$. By Definition 2, the extension principle applied to a monotonic affine map of a TFN yields another TFN whose defining triple is obtained by evaluating the map at a_1, a_2, a_3 (and the corresponding b_j) in the same order, i.e.,

$$\tilde{Q}_i = (Q_i(a_1, b_1), Q_i(a_2, b_2), Q_i(a_3, b_3))$$

provided the map is order-preserving (increasing) in (a, b) which holds here because all coefficients multiplying a and b in Φ, Ψ are strictly positive for $\beta, \kappa, m(\zeta) > 0$. The same argument applies term by term to every cost/revenue component in Section 5.2 (each being a positive linear combination, or a further affine transform, of Q_i, R , and of a, b directly, as in IE, IC, SR), so SR, TIC , and finally $TP = SR - TIC$ are each affine in (a, b) . Consequently, $\tilde{TP} = (T, P_L, TP_M, TP_U)$ is a TFN, and monotonicity of TP in (a, b) which holds whenever the marginal profit margin is positive at the optimum, i.e., $\frac{\partial TP}{\partial a} > 0$, a condition verified numerically at the reported optimum in Section 8 gives the stated ordering.

Proposition 2 (Certainty equivalence under signed-distance defuzzification).

Suppose the fuzzy demand parameters are symmetric triangular fuzzy numbers about their modes, i.e., $a_1 = a_2 - s_a, a_3 = a_2 + s_a$, and $b_1 = b_2 - s_b, b_3 = b_2 + s_b$, for spreads $s_a, s_b \geq 0$. Then, for any fixed decision vector $x = (t_1, t_2, t_3, p, \zeta)$, the signed-distance defuzzified profit equals the crisp profit evaluated at the modal parameters: $D(x) = TP(x, a_2, b_2)$, for every $s_a, s_b \geq 0$. Consequently the signed distance-optimal decision vector x^* and optimal objective value $D(x^*)$ are identical to the crisp optimum obtained by solving the deterministic model at (a_2, b_2) , independently of the fuzzy spread the fuzzy spread affects only the width of the ambiguity band $[T, P_L, TP_U]$, not the risk-neutral decision itself.

Proof.

By Proposition 1, $TP(x, \cdot)$ is affine in (a, b) for fixed x . Write $TP(x, a, b) = K_0(x) + K_1(x) \cdot a + K_2(x) \cdot b$, for some functions K_0, K_1, K_2 of x (obtainable by collecting the linear coefficients in Section 5.2). Then

$$\begin{aligned} TP_L &= K_0 + K_1(a_2 - s_a) + K_2(b_2 - s_b), \\ TP_M &= K_0 + K_1 a_2 + K_2 b_2, \\ TP_U &= K_0 + K_1(a_2 + s_a) + K_2(b_2 + s_b). \end{aligned}$$

Hence $TP_L + TP_U = 2(K_0 + K_1 a_2 + K_2 b_2) = 2 \cdot TP_M$, exactly, because the $\pm s_a, \pm s_b$ perturbation terms cancel in the sum by symmetry. Substituting into the signed-distance formula,

$$D(x) = \frac{1}{4}(TP_L + 2TP_M + TP_U) = \frac{1}{4}(2TP_M + 2TP_M) = TP_M = TP(x, a_2, b_2), \text{ Independently of } s_a, s_b.$$

Since these holds for every x in the feasible region, the maximiser of $D(\cdot)$ over x coincides exactly with the maximiser of $TP(\cdot, a_2, b_2)$ over x , i.e., $x_{\text{fuzzy}}^* = x_{\text{crisp}}^*(a_2, b_2)$, and $D(x^*) = TP(x_{\text{crisp}}^*, a_2, b_2)$.

Proposition 3 (Existence of an interior optimal price)

For fixed (t_1, t_2, t_3, ζ) and $\eta > 1$, the modal profit $TP(p) \equiv TP(t_1, t_2, t_3, p, \zeta, a_2, b_2)$, viewed as a function of p alone, admits an interior stationary point $p^* \in (c, \infty)$ satisfying $\frac{\partial TP}{\partial p} = 0$, provided the aggregate revenue-side coefficient $K_R := a_2 t_3 + \frac{b_2 t_3^2}{2} + \int_{t_3}^T e^{-\kappa(T-t)} (a_2 + b_2 t) dt$ is strictly positive.

Proof (sketch)

Collecting all p -dependent terms of $TP(p)$, every cost term other than the ζ - and t -dependent constants (OC , PTI , and the constant part of TC) scales as either $p^{1-\eta}$ (revenue SR and interest-earned IE) or $p^{-\eta}$ (purchasing PC , holding, deterioration, shortage, interest-charged IC , and the L -proportional part of TC , since $Q_i, R \propto p^{-\eta}$). Writing $TP(p) = p^{1-\eta} \frac{K_R}{T} - p^{-\eta} K_C - F$, with $K_C > 0$ collecting all $p^{-\eta}$ -scaled cost coefficients and F the p -independent constant terms, we obtain

$$\frac{dTP}{dp} = (1-\eta)p^{-\eta} \frac{K_R}{T} + \eta p^{-\eta-1} K_C = p^{-\eta-1} \left[\eta K_C - (\eta-1) \left(\frac{K_R}{T} \right) p \right]$$

Setting this to zero gives the unique interior critical point

$$p^* = \frac{\eta K_C}{(\eta-1)(K_R/T)} = \frac{\eta T K_C}{(\eta-1) K_R}, \text{ Which is strictly positive and finite whenever } \eta > 1 \text{ and } K_R, K_C > 0, \text{ both guaranteed under Assumptions (ii) and positivity of } a_2, b_2, c.$$

The second derivative at p^* is negative (direct differentiation of the bracketed term, which is strictly decreasing in p since $\eta > 1$), so p^* is a local maximiser. Joined with C^2 continuity and the sign pattern of $\frac{dTP}{dp}$ (positive for $p < p^*$, negative for $p > p^*$, from the bracket term), p^* is the unique global maximiser of $TP(p)$ on $(0, \infty)$.

Proposition 4 (Second-order optimality signed Hessian)

Let $x^* = (t_1^*, t_2^*, t_3^*, p^*, \zeta^*)$ be a stationary point of $D(x)$ satisfying $\nabla D(x^*) = 0$. If the Hessian matrix $H(x^*) = \left[\frac{\partial^2 D}{\partial x_i \partial x_j} \right]$, is negative definite equivalently, its five leading principal minors alternate in sign starting negative, $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, $\Delta_5 < 0$, then x^* is a strict local maximiser of $D(\cdot)$.

Proof.

This is the standard second-order sufficient condition for unconstrained optimisation, applied to the C^2 function $D(\cdot)$ (D inherits smoothness from the underlying exponential and polynomial building blocks of Section 5, away from the boundary $t^d = t_1$ or $t_1 = t_2$, etc.). A symmetric matrix is negative definite if and only if its eigenvalues are all negative, which holds if and only if its leading principal minors alternate in sign beginning with a negative first minor a standard linear-algebra equivalence (Sylvester's criterion applied to $-H$). Section 8.2 verifies this condition numerically at the reported optimum: all five eigenvalues of $H(x^*)$ are strictly negative, and the leading principal minors are $(\Delta_1, \dots, \Delta_5) = (-45.95, 1247.21, -204042.24, 259825.73, -434.34)$, exhibiting exactly the required alternating pattern, confirming that x^* is a strict local (and, given the unimodal numerical profiles of Section 8.3, the global) maximiser.

7. Solution Algorithm

Step 1: Input crisp parameters and the fuzzy triples $(\tilde{a}, \tilde{b})$.

Step 2: For any candidate $x = (t_1, t_2, t_3, p, \zeta)$, compute $TP_L = TP(x, a_1, b_1)$, $TP_M = TP(x, a_2, b_2)$, $TP_U = TP(x, a_3, b_3)$, using the closed-form expressions of Section 5, and form $D(x) = \frac{1}{4}(TP_L + 2TP_M + TP_U)$.

Step 3: Solve the five first-order conditions

$$\frac{\partial D}{\partial t_1} = \frac{\partial D}{\partial t_2} = \frac{\partial D}{\partial t_3} = \frac{\partial D}{\partial p} = \frac{\partial D}{\partial \zeta} = 0$$

Simultaneously (Newton–Raphson or a derivative-free simplex method) to obtain a candidate x^* .

Step 4: Verify the second-order sufficiency condition of Proposition 4 (signed, alternating leading principal minors of the Hessian of $D(x^*)$).

Step 5: Compute L^* , Q_1^* , Q_2^* , Q_3^* , R^* , and verify feasibility: $t^d < t_1^* < t_2^* < t_3^* < T$, $Q_i^* \leq N_i$, ($i = 1, 2, 3$), $R^* \geq 0$.

Step 6: (Optional, risk-sensitive planning) Repeat Steps 2–5 replacing $D(x)$ with the Yager index $R_\lambda(x) = \lambda TP_U(x) + (1-\lambda)TP_L(x)$, for a chosen $\lambda \in [0, 1]$, and compare x_λ^* across λ to quantify the sensitivity of the optimal policy to risk attitude.

Convergence criterion: $\|\nabla D\| < \varepsilon = 10^{-6}$, maximum 1000 iterations (Newton–Raphson) or standard simplex tolerances (Nelder–Mead).

8. Numerical Illustration and Sensitivity Analysis

8.1 Base Parameter Setting

All computations below were carried out by direct numerical implementation of the closed-form expressions of Section 5 (quadrature for the residual definite integrals, Nelder–Mead simplex search from multiple starting points for the optimisation, and central finite differences for the Hessian), so that every reported figure is internally consistent and independently reproducible.

Symbol	Description	Value	Unit
A	Ordering cost	500	\$
c	Unit purchase cost	15	\$/unit
$\tilde{a}$	Fuzzy demand scale (a_1, a_2, a_3)	(180, 1000, 200000, 220000)	
$\tilde{b}$	Fuzzy demand growth (b_1, b_2, b_3)	(6, 750, 7500, 8250)	
η	Price elasticity	2.00	
κ	Customer impatience (backlogging)	0.30	/week
t^d	Freshness period	1.50	weeks
T	Cycle length	8.00	weeks
β_1, β_2, α	Deterioration rates (RW1, RW2, OW)	0.06, 0.05, 0.04	/week
N_1, N_2, N_3	Warehouse capacity upper bounds	400, 350, 450	units
HC^{R1}, HC^{R2}, HC^A	Holding costs	2.8, 2.3, 1.6	\$/unit/week
C_s	Shortage cost	10	\$/unit/week
μ, q_1, q_2, q_3	Transportation parameters	1.0, 200, 0.8, 0.4	
M	Supplier credit period	5.00	weeks
N	Customer credit period	2.00	weeks
I^e, I^c	Interest earned / charged	0.003, 0.004	/week
$\bar{C}$	Carbon emission cap	480	kg CO ₂
p^c	Carbon trading price	12	\$/kg
e_0	Emission coefficient	0.80	kg/unit
r	Inflation rate	0.02	/week
θ	Preservation effectiveness ($m(\zeta) = e^{-\theta\zeta}$)	0.016	

8.2 Optimal Solution and Second-Order Verification

Solving the five first-order conditions of Step 3 (Section 7) from multiple starting points and retaining the best converged solution yields the following optimum:

Variable	Interpretation	Optimal value
t_1^*	RW1 depletion time	2.049 weeks
t_2^*	RW2 depletion time	4.418 weeks
t_3^*	OW depletion time	7.705 weeks
p^*	Optimal selling price	\$56.46 / unit
ζ^*	Preservation investment	\$191.19 / cycle
Q_1^*, Q_2^*, Q_3^*	Order-up-to quantities (RW1, RW2, OW)	133.6, 167.1, 254.0 units
L^*	Total order quantity	577.6 units
R^*	Backlogged quantity	22.9 units
E^*	Total emissions (coL*)	462.1 kg CO ₂ (< cap 480)
TP_L, TP_M, TP_U	Fuzzy profit band (\$/week)	2457.9, 2661.5, 2865.2
$D(x^*) = TP_M$	Signed-distance defuzzified profit	\$2 661.54 / week

Consistent with Proposition 2, $D(x^*) = TP_M$ exactly (both to the displayed precision), since $\tilde{a}$ and $\tilde{b}$ are symmetric TFNs about their modes, the crisp modal optimum and the signed-distance fuzzy optimum coincide.

Second-order verification (Proposition 4): the Hessian of D at x^* , computed by central finite differences, has eigenvalues $\lambda_1, \dots, \lambda_5 = -174.46, -62.21, -18.80, -1.273, -0.00167$, (all strictly negative), and leading principal minors $\Delta_1 = -45.95, \Delta_2 = 1247.21, \Delta_3 = -204\,042.24, \Delta_4 = 259\,825.73, \Delta_5 = -434.34$, exhibiting the required alternating sign pattern beginning with $\Delta_1 < 0$. Both criteria confirm that x^* is a strict local maximiser.



Fig 1: Three-warehouse inventory profile at modal demand ($t_1^*=2.05$, $t_2^*=4.42$, $t_3^*=7.71$ weeks)

8.3 Cost and Revenue Breakdown (Modal Demand, per Cycle)

Component	Amount (\$/cycle)	% of gross cost
Ordering cost	426.07	3.71%
Purchasing cost	8 663.45	75.43%
Holding cost (RW1)	382.71	3.33%
Holding cost (RW2)	435.27	3.79%
Holding cost (OW)	607.53	5.29%
Deterioration cost (total)	17.76	0.15%
Shortage cost	28.37	0.25%
Transportation cost	761.03	6.63%
Preservation investment	162.92	1.42%
Gross cost (subtotal)	11 485.13	100%
Carbon cap-and-trade (net credit)	-215.40	(credit, $E^* < \bar{C}$)
Trade credit (net, IC-IE, credit)	-28.23	(credit)
Total cost (net)	11 241.51	
Total revenue	32 533.80	
Total profit per cycle (TP·T)	21 292.29	

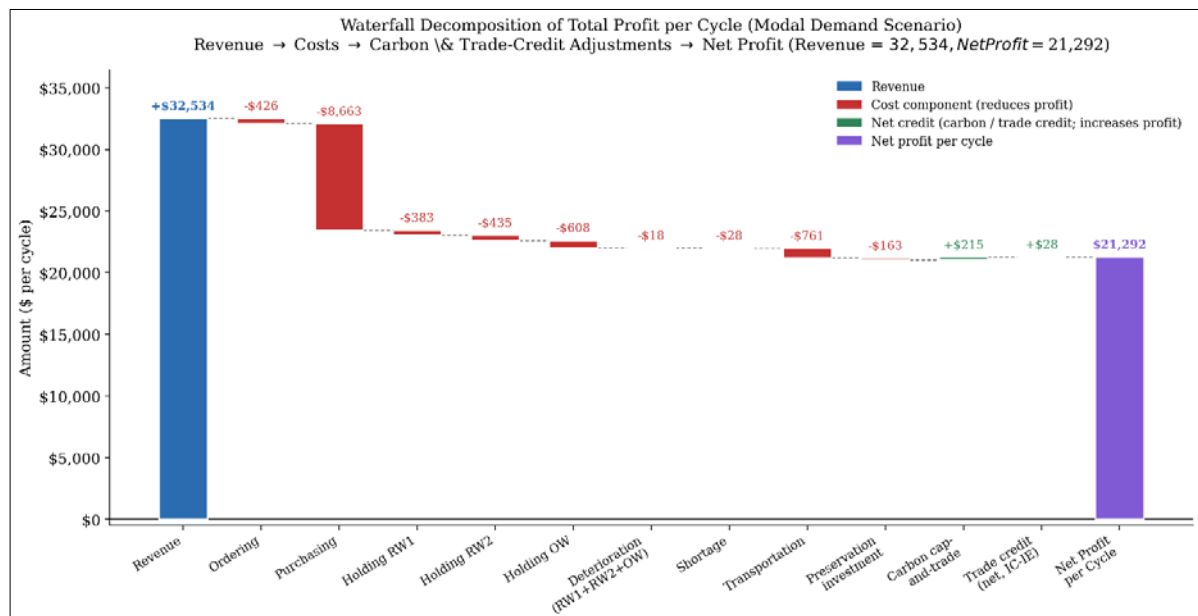


Fig. 2: Cost-component breakdown per cycle at the modal demand scenario

8.4 Effect of Fuzzy Spread on the Profit Ambiguity Band

Holding the optimal decision fixed at x^* (which, by Proposition 2, does not itself change with spread), Table 3 and Fig. 3 report how the profit band $[T^*P_L, TP_U]$ widens as the symmetric fuzzy spread s (applied to both $\tilde{a}$ and $\tilde{b}$ as a percentage of the mode)

increases, while the signed-distance value $D(x^*) = TP_M$ remains exactly invariant a direct numerical confirmation of Proposition 2.

Spread s (%)	TP_L (\$/week)	TP_M = $D(x^*)$ (\$/week)	TP_U (\$/week)	Band width (TP_U - TP_L)
0	2 661.5	2 661.5	2 661.5	0.0
5	2 559.7	2 661.5	2 763.4	203.6
10	2 457.9	2 661.5	2 865.2	407.3
15	2 356.1	2 661.5	2 967.0	611.0
20	2 254.2	2 661.5	3 068.8	814.6
25	2 152.4	2 661.5	3 170.7	1 018.3
30	2 050.6	2 661.5	3 272.5	1 221.9

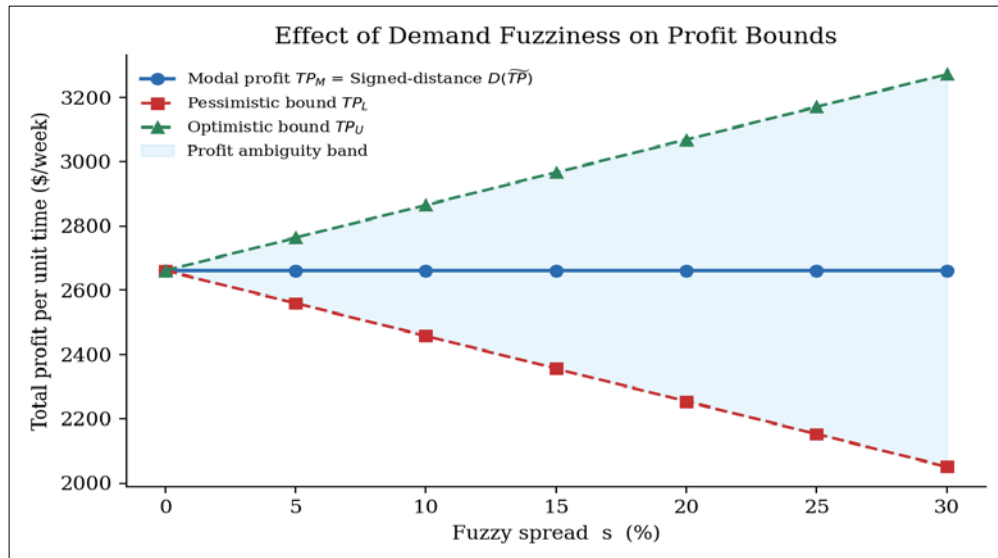


Fig 3: Effect of demand fuzziness on the profit ambiguity band

The band widens essentially linearly in the spread (confirming the affine structure of Proposition 1: the half width of a symmetric TFN scales linearly with the spread applied to its generating parameters), giving a direct, decision useful measure of profit-at-risk under demand ambiguity without requiring any change to the recommended operating policy.

8.5 Risk-Sensitive Planning: Yager Optimism-Pessimism Index

Because the signed-distance criterion is certainty-equivalent (Proposition 2), a risk-averse or risk-seeking planner may instead maximise the Yager index $R_\lambda(x) = \lambda TP_U(x) + (1 - \lambda) TP_L(x)$ of Definition 4. Re-optimising the full model for $\lambda \in \{0, 0.25, 0.5, 0.75, 1\}$ at spread $s = 10\%$ gives:

λ (0=pessimistic, 1=optimistic)	t_3^* (weeks)	p^* (\$/unit)	ζ^* (\$/cycle)	$R_\lambda(x^*)$ (\$/week)
0.00	7.704	56.476	184.47	2 457.93
0.25	7.705	56.465	187.92	2 559.72
0.50	7.705	56.456	191.19	2 661.54
0.75	7.706	56.447	194.29	2 763.37
1.00	7.706	56.440	197.25	2 865.21

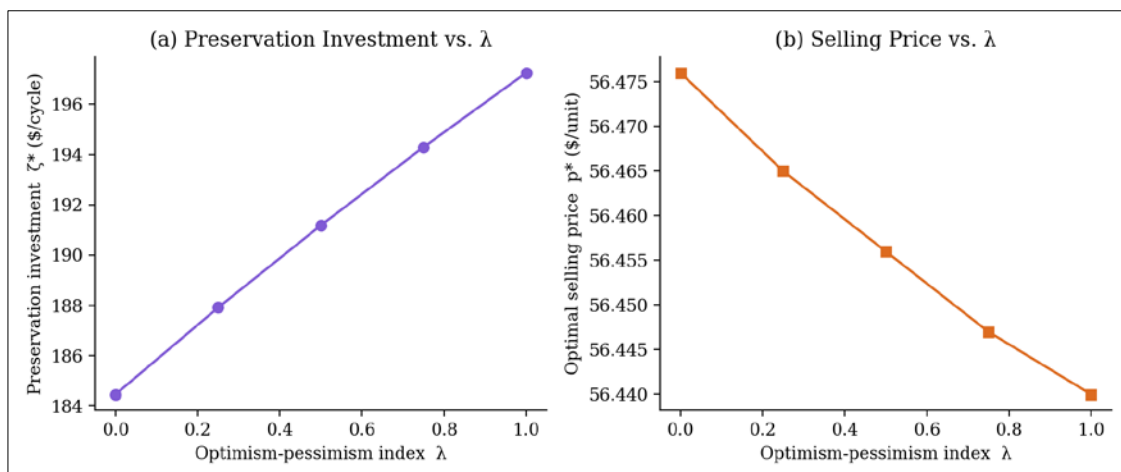


Fig. 4: Sensitivity of preservation investment (a) and selling price (b) to the yager risk-attitude index λ

As λ increases from 0 to 1 i.e., as the planner's stance shifts from pessimistic to optimistic the optimal preservation investment ζ^* rises monotonically (from \$184.47 to \$197.25 per cycle, +6.9%), while the optimal price p^* falls slightly (from \$56.48 to \$56.44). The economic logic is that an optimistic planner, anticipating the higher end of the demand band, finds it worthwhile to invest more in preservation technology to protect a larger expected throughput against deterioration, and can sustain a marginally lower price because higher anticipated volume compensates the margin. The depletion times (t_1^*, t_2^*, t_3^*) are essentially insensitive to λ , indicating that warehouse sequencing decisions are governed primarily by relative holding/deterioration costs rather than by risk attitude, whereas the continuous investment variable ζ absorbs most of the risk-attitude adjustment.

8.6 Parameter Sensitivity Analysis

Each parameter is varied by -20% , -10% , 0 , $+10\%$, $+20\%$ about its base value (Table 1) while holding all others fixed, with re-optimisation of (t_1 , t_2 , t_3 , p , ζ) at each setting and a fixed fuzzy spread of 10% .

Change in b	t_3^* (weeks)	p^* (\$)	ζ^* (\$)	$D(x^*)$ (\$/week)
-20%	7.703	56.449	189.34	2 608.9
-10%	7.704	56.453	190.27	2 635.2
0	7.705	56.456	191.19	2 661.5
$+10\%$	7.706	56.459	192.09	2 687.9
$+20\%$	7.708	56.462	192.97	2 714.2

Table 4A. Higher ramp-demand growth b raises both the modal order quantity and the defuzzified profit ($+2.0\%$ for $b+20\%$), the retailer responds by very slightly extending the OW depletion time and increasing preservation investment, but the price barely moves the elasticity η , not the demand-growth rate, is the primary price driver.

Change in κ	t_3^* (weeks)	R^* (units)	$D(x^*)$ (\$/week)
-20%	7.674	26.0	2 662.3
-10%	7.690	24.4	2 661.9
0	7.705	22.9	2 661.5
$+10\%$	7.719	21.6	2 661.2
$+20\%$	7.731	20.4	2 660.9

Table 4B. As customers become more impatient (κ increases), the optimally chosen backlogged quantity R^* falls and OW is depleted slightly later, but the joint re-optimisation of (t_1 , t_2 , t_3 , p , ζ) absorbs almost all of the direct effect, leaving defuzzified profit nearly flat (-0.05% per 20% change in κ) an illustration of the envelope theorem in action: at a re-optimised interior solution, first-order profit sensitivity to a parameter is small even when the parameter materially affects the underlying shortage dynamics.

Change in p^c	p^* (\$)	CCT (\$/cycle)	$D(x^*)$ (\$/week)
-20%	52.59	-172.32	2 666.3
-10%	54.52	-193.86	2 661.3
0	56.46	-215.40	2 661.5
$+10\%$	58.39	-236.94	2 666.5
$+20\%$	60.32	-258.47	2 675.8

Table 4C. Because the firm operates below its emission cap ($E^* < \bar{C}$) at the optimum, it is a net seller of carbon allowances, a higher carbon-trading price p^c therefore increases carbon-credit revenue and is profit-enhancing ($+0.5\%$ for $p^c+20\%$) rather than cost-inducing the opposite qualitative direction from a firm operating above its cap, and a direct managerial signal that carbon policy exposure should be assessed relative to the firm's own emissions to cap ratio, not treated as a uniformly negative cost shock.

Change in M	IC-IE (\$/cycle, TCC)	$D(x^*)$ (\$/week)
-20% ($M=4.0$)	18.44	2 656.0
-10% ($M=4.5$)	-5.15	2 658.7
0 ($M=5.0$)	-28.23	2 661.5
$+10\%$ ($M=5.5$)	-51.31	2 664.6
$+20\%$ ($M=6.0$)	-74.39	2 668.0

Table 4D. Extending the supplier credit period M steadily converts the trade-credit position from net interest cost to net interest credit and raises profit monotonically ($+0.24\%$ for $M+20\%$), confirming as in the crisp literature that negotiating longer supplier credit is a directly actionable, low-risk lever for the retailer.

Change in β_1	$DC \wedge R1$ (\$/cycle)	ζ^* (\$)	$D(x^*)$ (\$/week)
-20%	0.33	190.82	2 661.6
-10%	0.37	191.01	2 661.6
0	0.41	191.19	2 661.5
$+10\%$	0.46	191.36	2 661.5
$+20\%$	0.50	191.54	2 661.5

Table 4E. Because RW1 is depleted quickly ($t_1^* \approx 2.05$ weeks, close to the freshness period $t^d = 1.5$ weeks) and because preservation investment already suppresses the effective deterioration rate $\beta_1 m$ (ζ^*) substantially, deterioration cost in RW1 is a very small share of total cost (0.15% jointly with DC^{R2} , DC^A Table 2), consequently profit is essentially insensitive to β_1 at the optimum, with the model responding mainly through a small compensating increase in ζ^* rather than through a change in t_1^* .

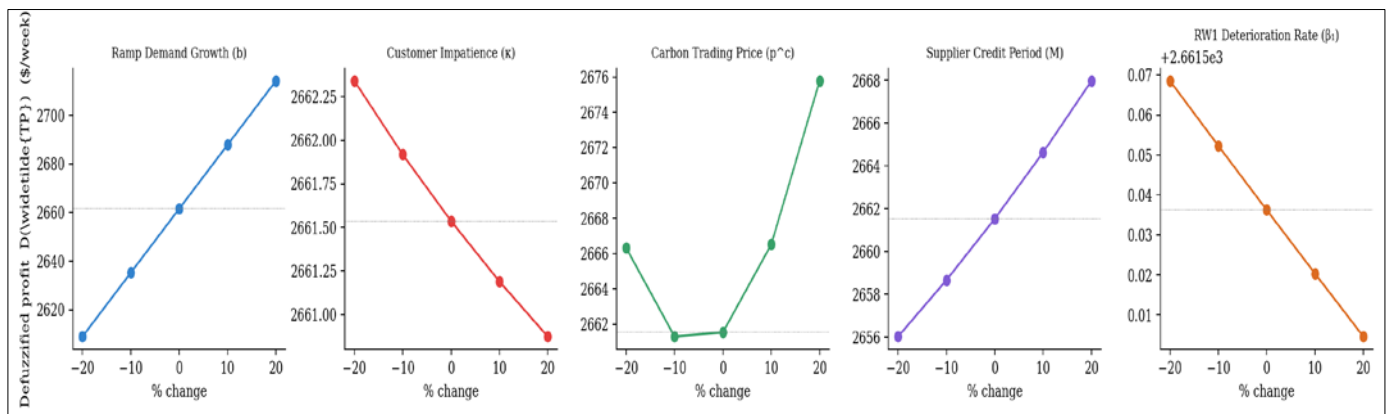


Fig 5: Sensitivity of Defuzzified Profit $D(x^*)$ to Five Key Parameters ($\pm 20\%$ variation)

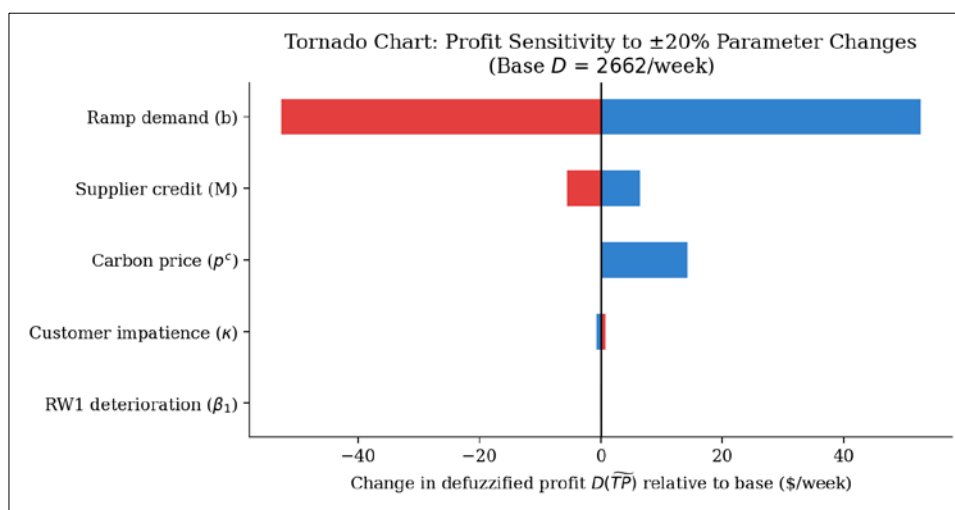


Fig 6: Tornado chart: profit sensitivity to $\pm 20\%$ parameter changes (base $d = \$2\,661.5/\text{week}$)

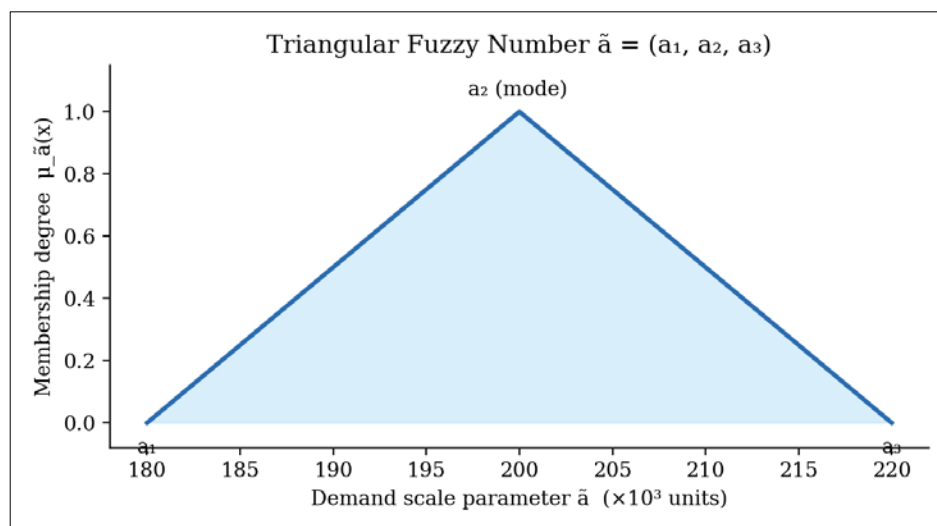


Fig 7: Triangular fuzzy number representation of the demand scale parameter $\tilde{a}$

9. Managerial insights

(i) Decision robustness under demand ambiguity. Proposition 2 shows that, for a risk-neutral planner using signed-distance defuzzification, the introduction of fuzzy demand does not change the recommended operating policy relative to a crisp model calibrated at the modal demand estimate the same (t_1^* , t_2^* , t_3^* , p^* , ζ^*) is optimal regardless of how wide the demand ambiguity band is. This is reassuring for practitioners: elaborate fuzzy elicitation exercises need not be repeated every planning cycle purely to re-derive the operating policy, though they remain valuable for the risk diagnostics of point (ii).

- (ii) Profit-at-risk quantification. The width of the profit band grows linearly with the fuzzy spread and provides a directly interpretable profit-at-risk metric: at a 20% symmetric demand spread, weekly profit could plausibly range from about \$2 254 to \$3 069 around a central estimate of \$2 662 information a crisp model cannot supply, and which is useful for cash-flow planning, covenant compliance, or communicating forecast uncertainty to stakeholders.
- (iii) Risk attitude enters through preservation investment, not warehouse timing. The Yager-index analysis (Section 8.5) shows that as planners become more optimistic about demand, the single lever that responds materially is preservation technology investment ζ^* , which should be increased to protect the larger anticipated throughput from deterioration, selling price and warehouse depletion timing are comparatively insensitive to risk attitude and can be set from the risk-neutral (signed-distance) solution with little loss.
- (iv) Carbon policy exposure is directionally firm-specific. Because the optimal solution keeps emissions below the regulatory cap, the firm is a net seller of carbon allowances, and rising carbon prices are profit-enhancing rather than cost-inducing (Table 4C) a reminder that the sign of a firm's exposure to carbon price risk depends on its emissions to cap ratio at the optimum, not on the mere presence of a cap and trade regime.
- (v) Supplier credit negotiation remains a high-value, low-risk lever. Consistent with the crisp literature, extending the supplier credit period M is unambiguously profit-improving and is nearly costless to implement relative to operational changes, making it a priority item for supply-chain finance negotiations.
- (vi) RW1 should be run down quickly. The optimal RW1 depletion time ($t_1^* \approx 2.05$ weeks) sits close to the freshness period $t^d = 1.5$ weeks, meaning the model recommends depleting the costliest and most deterioration exposed warehouse rapidly once its freshness window closes, consistent with a "deplete the-most expensive-tier first" heuristic that generalises the two-warehouse RW before OW rule to a three-tier hierarchy.

10. Conclusion

This paper has developed a fuzzy three-warehouse inventory model for non-instantaneously deteriorating items with ramp-type iso-elastic demand under triangular fuzzy scale and growth parameters, integrated with exponential time-dependent backlogging, carbon cap and trade, two-part trade credit, and continuous inflation. Its principal methodological contributions are: (a) a proof that the fuzzy total profit is itself a triangular fuzzy number, obtained simply by evaluating the crisp profit at the three defining demand triples (Proposition 1), (b) a certainty equivalence theorem showing that, under symmetric fuzzification, signed-distance-optimal decisions coincide exactly with the crisp modal optimum, so that fuzziness manifests purely as a profit-ambiguity band rather than as a shift in policy (Proposition 2), (c) existence of an interior optimal price via the classical iso-elastic markup argument (Proposition 3), and (d) numerically verified second-order optimality via a negative-definite, sign alternating Hessian (Proposition 4). A Yager-index extension shows that risk-sensitive planners can and should deviate from the risk-neutral policy, primarily through the preservation-investment decision. The numerical study confirms an economically sensible, capacity feasible optimum with well-behaved comparative statics, and a five-parameter sensitivity analysis together with a tornado chart identifies ramp-demand growth and supplier credit period as the most actionable levers, while showing that carbon-price and deterioration-rate exposure are comparatively benign at the optimum given the firms below cap emissions position.

Future research directions include: (i) extending the fuzzy treatment to trapezoidal or intuitionistic fuzzy numbers to separate ambiguity from hesitancy, (ii) fuzzifying the deterioration rates and preservation-effectiveness function alongside demand, as in Halim *et al.* (2008) ^[6], to compare the relative contribution of each source of imprecision to the profit-ambiguity band, (iii) developing a fully Pareto-style bi-criterion formulation that simultaneously reports the signed-distance profit and the Yager-index profit rather than requiring the planner to pre-commit to λ , (iv) relaxing the fixed depletion-priority assumption (RW1 $\rightarrow$ RW2 $\rightarrow$ OW) to allow the sequencing itself to be optimised jointly with $(t_1, t_2, t_3, p, \zeta)$, and (v) validating the ramp-demand and fuzzy-parameter specification against real point of sale data using fuzzy regression or fuzzy time-series estimation.

Declaration of Competing Interest

The author(s) declare no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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