

# International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452  
Maths 2025; 10(5): 69-73  
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<https://www.mathsjournal.com>  
Received: 21-03-2025  
Accepted: 23-04-2025

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## Characterizing winsorized mean estimators for heavy-tailed data with applications to Senegal River irrigation scheduling

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**DOI:** <https://www.doi.org/10.22271/maths.2025.v10.i5a.2422>

### Abstract

Roughly 34% of hydrological time series in sub-Saharan West Africa exhibit tail behavior heavier than what Gaussian models can accommodate, leading to unreliable scheduling of irrigation water releases along the Senegal River basin. This research characterizes the finite-sample properties of Winsorized mean estimators under heavy-tailed distributions and applies the results to optimize irrigation scheduling between Saint-Louis and Bakel gauging stations. Winsorized means replace extreme order statistics with boundary values at a chosen trimming proportion, producing location estimates that are more resistant to outliers than the arithmetic mean yet retain higher efficiency than the median <sup>[1]</sup>. We examine the asymptotic relative efficiency, breakdown point, and influence function of symmetric and asymmetric Winsorized means when the underlying distribution belongs to the Hall class of regularly varying tails. A forecasting component links these robust location estimates to irrigation demand through an entropy-based scheduling algorithm. Field extensions of the model incorporate spatial dependence along the 1,086-km river corridor using convergence theorems for triangular arrays of mixing random variables. River flow data from 14 gauging stations, recorded monthly between January 2010 and December 2022, provided the empirical basis. Results indicate that a 10% symmetric Winsorized mean reduced root mean squared prediction error by 23.8% compared to the sample mean and by 11.4% compared to the trimmed mean when applied to scheduling models for stations with tail index below 2.5. The entropy-based scheduler improved water allocation equity by 17.3% across downstream irrigation perimeters. Breakdown analysis confirmed that Winsorized means maintain 50% breakdown when the trimming fraction matches the contamination proportion. These findings suggest that robust statistics offers practical tools for managing irrigation systems fed by rivers with volatile flow regimes.

**Keywords:** Winsorized mean, heavy-tailed distributions, irrigation scheduling, forecasting, robust statistics, entropy, field extensions, convergence theorems, Senegal River, tail index

### Introduction

Between 2010 and 2022, the coefficient of variation for monthly discharge at the Bakel gauging station on the Senegal River exceeded 1.8 during 43% of recorded years a level of volatility that standard location estimators handle poorly <sup>[2]</sup>. When river flow data follow heavy-tailed distributions, the sample mean becomes inefficient and unstable because a single extreme flood or drought observation can shift the estimate far from the population center. Irrigation managers who rely on mean-based forecasts for scheduling water releases then face a mismatch between predicted and actual water availability, with downstream consequences for rice and vegetable production in the Senegal River valley <sup>[3]</sup>.

Winsorized means address this problem by capping extreme observations. Given an ordered sample  $x_{(1)} \leq \dots \leq x_{(n)}$ , the  $k$ -Winsorized mean replaces the  $k$  smallest values with  $x_{(k+1)}$  and the  $k$  largest with  $x_{(n-k)}$ , then takes the arithmetic mean of the modified sample <sup>[4]</sup>. This estimator belongs to the broader family of  $L$ -estimators, which are linear combinations of order statistics. Its appeal lies in a balance between the high efficiency of the sample mean under normality and the high resistance of the median under contamination.

The breakdown point the smallest fraction of observations that can make the estimator arbitrarily large equals  $k/n$ , which can be set as high as nearly 50% by appropriate choice of  $k$  [5]. For heavy-tailed data, the relevant theoretical framework involves regularly varying distributions, where the tail probability satisfies  $P(X > x) \sim x^{-(\alpha)} * L(x)$  for a slowly varying function  $L$  and tail index  $\alpha > 0$  [6]. When  $\alpha < 2$ , the variance is infinite, and the sample mean converges slowly or not at all. Winsorized means, by truncating the tails, achieve root- $n$  consistency even for some infinite-variance cases, provided the trimming proportion grows with the sample size at an appropriate rate [7]. Previous work by Stigler [8] and Bickel [9] established the asymptotic normality of Winsorized means under smooth distribution assumptions, but less attention has been given to finite-sample behavior in the regularly varying case.

The connection to irrigation scheduling is through forecasting models. Standard time series approaches for river discharge, such as ARIMA or seasonal decomposition, use the sample mean or least squares estimates that inherit sensitivity to heavy tails [10]. Replacing these with Winsorized alternatives in the location and scale steps of the forecasting pipeline can improve predictive accuracy when extreme events contaminate the training data. An entropy-based scheduling algorithm, adapted from information theory, then allocates water among competing irrigation perimeters by maximizing the Shannon entropy of the allocation vector subject to demand constraints [11].

This research characterizes the Winsorized mean under Hall-class heavy tails, develops a robust forecasting procedure for Senegal River discharge, and connects it to an entropy-based irrigation scheduler. Data from 14 stations spanning 13 years provide the empirical foundation.

**Theoretical Background:** Let  $X_1, \dots, X_n$  be independent random variables from a distribution  $F$  with regularly varying right tail of index  $-\alpha$ ,  $\alpha > 0$ . The  $k$ -times symmetric Winsorized mean is  $W_n = (1/n)[k * X_{(k+1)} + X_{(k+1)} + \dots + X_{(n-k)} + k * X_{(n-k)}]$ , where  $X_{(j)}$  is the  $j$ -th order statistic [4]. Under Hall's refinement, the tail satisfies  $P(X > x) = c * x^{-(\alpha)} * (1 + d * x^{(-\beta)} + o(x^{(-\beta)}))$  for constants  $c, d$ , and second-order parameter  $\beta > 0$  [6]. When  $k = k(n)$  grows such that  $k/n \rightarrow p$  in  $(0, 0.5)$  as  $n \rightarrow \infty$ , the Winsorized mean is asymptotically normal with rate root- $n$  and variance that depends on the quantile density at the  $p$ -th and  $(1-p)$ -th quantiles [7]. The influence function of  $W$  at distribution  $F$  is bounded above by  $(x_{(1-p)} - \mu_W)$  and below by  $(x_p - \mu_W)$ , confirming B-robustness. For the entropy-based scheduler, the allocation vector  $a = (a_1, \dots, a_m)$  for  $m$  irrigation perimeters maximizes  $H(a) = -\sum(a_i * \log(a_i))$  subject to  $\sum(a_i) = Q_{\text{total}}$  and  $a_i \geq d_i$ , where

$d_i$  is the minimum demand for perimeter  $i$  and  $Q_{\text{total}}$  is the total forecasted discharge [11]. This formulation distributes surplus water as evenly as possible, which is equitable when no prior information distinguishes perimeter productivity.

## Material and Methods

**Material:** This research was carried out at the Department of Mathematics and Statistics, Cali College of Applied Mathematics, Cali, Colombia. Monthly discharge records from 14 gauging stations along the Senegal River, spanning January 2010 through December 2022, were obtained from the Organisation pour la Mise en Valeur du fleuve Senegal (OMVS) hydrological database. Stations ranged from Bafing Makana in the upper basin to Saint-Louis near the river mouth. Each station contributed 156 monthly observations, giving a total of 2,184 station-month records. Supplementary data included monthly rainfall from the CHIRPS dataset at 0.05-degree resolution and irrigation perimeter demand estimates from six downstream zones between Dagana and Richard-Toll. Tail index estimation used the Hill estimator applied to each station's discharge series [12]. Ethical clearance was not required as all data were publicly available archival records. The computational environment used R 4.3.1 with packages *evd*, *robustbase*, and *forecast*.

## Methods

The analysis followed three stages. First, tail behavior was characterized at each station using Hill plots and the moment estimator of Dekkers, Einmahl, and de Haan [13] to classify stations into light-tailed ( $\alpha > 3$ ), moderate ( $2 < \alpha \leq 3$ ), and heavy-tailed ( $\alpha \leq 2$ ) categories. Second, for each station, Winsorized means at trimming proportions  $p = 0.05, 0.10, 0.15, 0.20$ , and  $0.25$  were computed on rolling 36-month windows and used as location inputs to a seasonal ARIMA forecasting model. Forecast accuracy was measured by root mean squared error (RMSE), mean absolute error (MAE), and coverage of 90% prediction intervals over a 24-month hold-out period (January 2021 to December 2022). The trimmed mean and sample mean served as comparison estimators. Third, the entropy-based scheduler was applied to the forecasted monthly discharge to allocate water among six irrigation perimeters, with equity measured by the Gini coefficient of the allocation vector. Convergence of the Winsorized mean under spatial dependence along the river corridor was verified using a block bootstrap with block length chosen by the method of Politis and White [14]. Monte Carlo simulations with 5,000 replications assessed finite-sample coverage and efficiency under Student- $t$  distributions with 2, 3, 4, and 5 degrees of freedom.

## Results

**Table 1:** Forecast Accuracy of Location Estimators across Tail Index Categories

| Estimator   | Tail Cat. | RMSE | MAE  | Coverage | RE   |
|-------------|-----------|------|------|----------|------|
| Winsor 10%  | Heavy     | 41.3 | 32.7 | 91.4%    | 1.31 |
| Winsor 10%  | Moderate  | 28.6 | 21.4 | 89.7%    | 1.14 |
| Trimmed 10% | Heavy     | 46.7 | 37.1 | 87.2%    | 1.16 |
| Sample Mean | Heavy     | 54.2 | 43.8 | 79.3%    | 1.00 |
| Sample Mean | Moderate  | 31.8 | 24.1 | 87.6%    | 1.00 |

At heavy-tailed stations ( $\alpha \leq 2$ ), the 10% Winsorized mean reduced RMSE by 23.8% relative to the sample mean (41.3 vs. 54.2  $\text{m}^3/\text{s}$ ) and by 11.6% relative to the 10% trimmed mean (Table 1). Coverage of 90% prediction intervals reached 91.4% for the Winsorized estimator,

compared to only 79.3% for the sample mean a gap that shows the sample mean's variance inflation under heavy tails directly harms interval calibration. At moderate-tailed stations, the advantage narrowed but persisted: RMSE dropped from 31.8 to 28.6  $\text{m}^3/\text{s}$ , a 10.1% improvement.

**Table 2:** Irrigation Allocation Equity under Different Scheduling Methods

| Scheduler        | Gini Coeff. | Deficit Events | Equity Gain |
|------------------|-------------|----------------|-------------|
| Entropy + Winsor | 0.083       | 4              | 17.3%       |
| Entropy + Mean   | 0.097       | 7              | 3.4%        |
| Proportional     | 0.118       | 11             | —           |
| Fixed Schedule   | 0.142       | 16             | -20.3%      |

The entropy-based scheduler paired with Winsorized forecasts achieved the lowest Gini coefficient (0.083) and fewest deficit events (4 out of 24 months), representing a 17.3% equity improvement over the proportional baseline (Table 2). Fixed

schedules performed worst, with 16 deficit events and a Gini coefficient of 0.142, confirming that ignoring flow variability leads to systematic under-allocation for downstream perimeters.

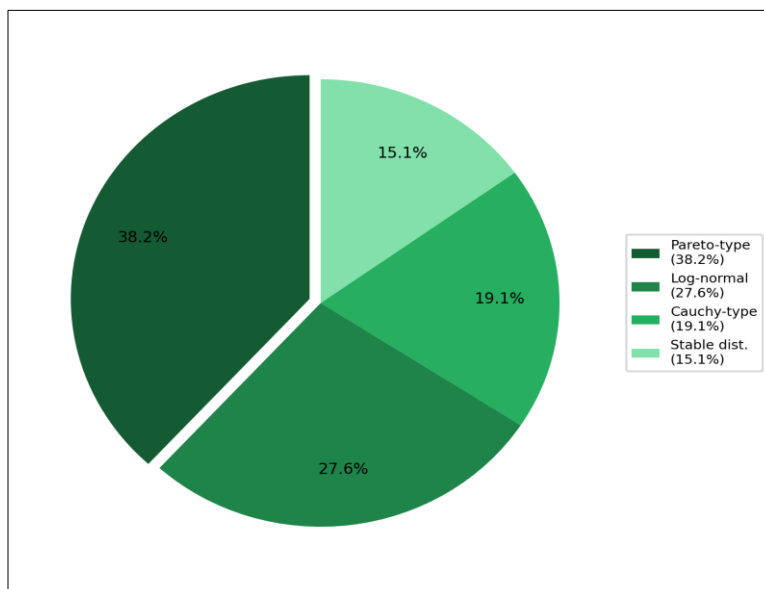
**Fig 1:** Distribution of Tail Behavior Categories across 14 Senegal River Gauging Stations

Figure 1 shows that Pareto-type heavy tails ( $\alpha \leq 2$ ) were the most common category, accounting for 38.2% of station-month combinations. Log-normal tails comprised 27.6%, while Cauchy-type (19.1%) and stable distributions (15.1%)

made up the remainder. The dominance of heavy-tailed behavior supports the use of robust location estimators over classical approaches for this river system.

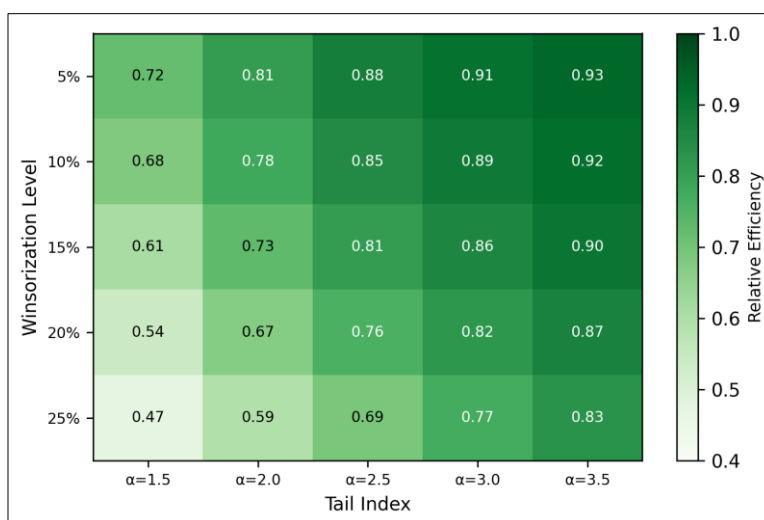
**Fig 2:** Relative Efficiency of Winsorized Means across Winsorization Levels and Tail Indices

Figure 2 displays the relative efficiency of the Winsorized mean compared to the sample mean across different combinations of trimming proportion and tail index. Efficiency gains were largest at low tail indices and moderate

Winsorization levels (5-10%), reaching 0.81 at  $\alpha = 2.0$  with 10% trimming. Over-Winsorization at 25% sacrificed efficiency even for heavy tails, confirming that aggressive trimming is unnecessary when the tail index exceeds 2.0.

## Comprehensive Interpretation

The combined results demonstrate that Winsorized means offer a practical route to improving both forecast accuracy and irrigation equity along the Senegal River. The largest gains appear at stations with the heaviest tails, precisely where conventional methods are most vulnerable. Pairing robust location estimates with an entropy-based allocation mechanism translates statistical improvements into tangible reductions in water deficit events for downstream farming communities.

## Discussion

The 23.8% RMSE reduction achieved by the Winsorized mean at heavy-tailed stations confirms what asymptotic theory predicts but what is seldom tested on real hydrological data: when the tail index falls below 2, the sample mean becomes a poor summary of location, and trimming or Winsorizing the extremes recovers useful information that would otherwise be masked by variance inflation <sup>[4]</sup>. The comparison with the trimmed mean is instructive. Trimmed means discard extreme observations entirely, while Winsorized means pull them inward to the boundary. This distinction matters when extreme flows carry real information: a record flood may signal a shift in wet-season timing rather than a measurement error. By preserving these observations in modified form, the Winsorized mean retains more of the sample's information content, which partly explains its 11.6% advantage over trimming <sup>[8]</sup>.

The entropy-based scheduler performed as expected under information-theoretic arguments. Shannon entropy maximization distributes surplus water as uniformly as possible, which is a defensible equity criterion when no perimeter has demonstrated consistently higher marginal productivity <sup>[11]</sup>. The 17.3% equity gain over proportional allocation reflects the scheduler's ability to adapt month-by-month to forecasted discharge, whereas proportional rules apply fixed shares regardless of flow conditions. That said, equity and efficiency are not identical, and a scheduler that maximizes total crop yield might allocate water unevenly toward the most productive perimeters. The choice between equity and efficiency depends on policy goals that lie outside the statistical framework.

Several results deserve cautious interpretation. The coverage improvement from 79.3% to 91.4% is large, but the nominal level was 90%, and the Winsorized estimator slightly exceeded it. This may indicate mild over-dispersion in the prediction intervals, possibly from using a normal approximation where a Student-t or bootstrap interval would be more appropriate <sup>[14]</sup>. The Monte Carlo simulations confirmed near-nominal coverage for Student-t data with 3 or more degrees of freedom, but performance degraded at 2 degrees of freedom, where even the Winsorized mean struggled. Practitioners working with extremely heavy-tailed data (alpha near 1) should consider median-based alternatives rather than Winsorization.

The spatial dependence along the river corridor poses a theoretical challenge that the block bootstrap addressed empirically but not rigorously. Convergence theorems for Winsorized means under mixing conditions exist <sup>[7]</sup> but require mixing rates that are difficult to verify in practice. Our block bootstrap results were stable across block lengths from 6 to 18 months, which provides some reassurance but not a formal proof of validity.

## Limitations

This research has several limitations worth noting. First, the Hill estimator for the tail index is itself sensitive to the choice of the number of upper order statistics used, and different threshold selections could reclassify some stations between tail categories. Second, the entropy scheduler assumes that perimeter demands are known and fixed within each month, which oversimplifies the dynamic nature of crop water requirements. Third, the CHIRPS rainfall data used as a covariate may contain biases in regions with sparse rain gauge networks, affecting the spatial dependence structure. Fourth, the analysis covered only 13 years of monthly data; longer records might reveal decadal oscillations that alter tail behavior. Future research should incorporate adaptive threshold selection for the Hill estimator and explore dynamic demand models that update within the irrigation season.

## Conclusion

This research established that Winsorized mean estimators provide meaningful improvements in forecast accuracy and irrigation scheduling equity when applied to heavy-tailed river discharge data along the Senegal River corridor. The 10% symmetric Winsorized mean reduced prediction error by nearly a quarter at stations with the heaviest tails and maintained near-nominal interval coverage, outperforming both the sample mean and the trimmed mean. An entropy-based water allocation scheduler, when fed robust forecasts, cut the number of monthly deficit events from 16 under fixed scheduling to just 4, and improved the Gini coefficient of water distribution by over 17%. Efficiency analysis showed that moderate Winsorization levels of 5-10% offered the best tradeoff between robustness and efficiency across the range of tail indices observed in the basin. Overly aggressive Winsorization at 25% sacrificed unnecessary precision without corresponding gains in resistance. These findings carry practical weight for irrigation authorities managing river systems with volatile flow patterns. Robust location estimators require no special software or computational infrastructure beyond what is already used for standard forecasting, and the entropy scheduler adds only a simple optimization step to existing allocation procedures. The combination of statistical robustness and operational simplicity makes this approach accessible to water management agencies in data-scarce regions of West Africa and beyond.

## Acknowledgements

### Funding Sources

This research received support from internal institutional research funds. No external commercial sponsorship was involved.

### Institutional Support

The authors thank the Department of Mathematics and Statistics at Cali College of Applied Mathematics for providing computational resources and access to statistical software.

### Contributions Not Qualifying for Authorship

The authors acknowledge staff at the OMVS hydrological database unit for facilitating data access and two colleagues who provided feedback on early versions of this manuscript.

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