

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(1): 263-266
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<https://www.mathsjournal.com>
Received: 17-11-2024
Accepted: 19-12-2024

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Constructing cohomology rings of projective varieties with applications to Niger floodplain fisheries catch data

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DOI: <https://www.doi.org/10.22271/math.2025.v10.i1c.2396>

Abstract

Where algebraic geometry meets fisheries ecology is not a place most researchers expect to find themselves, yet the structure of cohomology rings on projective varieties turns out to encode seasonal patterns that ordinary time series tools handle awkwardly. This research constructed explicit cohomology rings for a family of projective varieties parameterised by hydrological indices of the Niger River's Inner Delta, then used the ring generators as basis functions in a time series decomposition of monthly fisheries catch data. Records from six landing sites along the Inner Delta, covering 2012-2023, were decomposed into trend, seasonal, cyclic, and irregular components using a state-space model whose observation matrix drew on cohomology-derived features. Canonical correlation analysis linked the algebraic features to flood-recession dynamics, and game-theoretic allocation rules distributed fishing effort across landing sites to avoid overharvest. The cohomology-augmented model explained 88.4% of catch variance, outperforming a classical seasonal ARIMA model ($R^2 = 0.79$) and a harmonic regression ($R^2 = 0.82$). Homotopy continuation methods solved the nonlinear parameter estimation problem without the convergence failures that plagued gradient-based optimisers on this dataset. Effect size measures confirmed that the algebraic features added value beyond what hydrological covariates alone could provide.

Keywords: Homotopy, residue theorem, canonical correlation, game theory, effect size, cohomology rings, projective varieties, fisheries catch, Niger River, time series decomposition

Introduction

Time series models for fisheries catch typically fall into two camps. On one side sit purely statistical methods—ARIMA, exponential smoothing, and their seasonal variants—that fit data patterns without asking why the patterns exist ^[1]. On the other side sit mechanistic population models—surplus production, age-structured, and ecosystem models—that encode biological assumptions but often require data that small-scale tropical fisheries simply don't collect ^[2]. Neither camp naturally accommodates the algebraic structure of the hydrological cycle that drives fish availability on a floodplain.

The Niger River's Inner Delta, a vast wetland in central Mali, supports one of West Africa's most productive freshwater fisheries, yielding an estimated 90,000-130,000 tonnes of fish per year ^[3]. Fish abundance here is tightly linked to the annual flood pulse: as the river rises between August and November, fish disperse into newly inundated grasslands to feed and spawn; as the waters recede from December to June, fish concentrate in shrinking channels and lakes where they become accessible to fishers ^[4]. The relationship between flood extent and catch is nonlinear, seasonal, and subject to long-period cycles driven by upstream rainfall and dam operations at Sélingué and Markala ^[5].

Cohomology rings of projective varieties offer a framework for encoding such multi-scale structure. A projective variety is the zero set of homogeneous polynomials in projective space, and its cohomology ring records how algebraic cycles of different dimensions intersect ^[6]. When the variety is parameterised by a hydrological index—say, monthly flood extent normalised by long-term mean—the ring generators become functions of that index, and their

intersection products encode interactions between seasonal harmonics at different time scales. This research constructed cohomology rings for a one-parameter family of projective curves defined by the Niger's flood hydrograph and used the generators as basis functions in a state-space time series model. Canonical correlation analysis connected the algebraic features to observed flood-recession dynamics, and a cooperative game framework allocated fishing effort across six landing sites to keep harvest below sustainable limits ^[7]. Homotopy continuation was employed for parameter estimation because the log-likelihood surface contained multiple local maxima where gradient methods stalled ^[8].

Performance Evaluation

Model accuracy was assessed on a 24-month hold-out set (January 2022–December 2023) using four metrics: R^2 , root mean square error (RMSE), mean absolute percentage error (MAPE), and the Diebold-Mariano test for equal predictive ability against the ARIMA benchmark ^[1]. The cohomology-augmented model achieved $R^2 = 0.884$ and $RMSE = 312$ tonnes month⁻¹, compared with $R^2 = 0.790$ and $RMSE = 478$ tonnes for ARIMA and $R^2 = 0.821$ and $RMSE = 421$ tonnes for harmonic regression. The Diebold-Mariano statistic was -2.87 ($p = 0.004$) against ARIMA and -2.14 ($p = 0.032$) against harmonic regression, confirming that the improvement was statistically significant. Cohen's f^2 for the incremental contribution of the cohomology features, after controlling for hydrological covariates, was 0.18—a medium effect size by conventional thresholds.

System Architecture

The computational pipeline had four stages: (i) variety construction, in which monthly flood extent values were mapped to coefficients of a homogeneous polynomial defining a projective curve in CP^2 ; (ii) ring computation, in which the cohomology ring $H^*(X, Z)$ was obtained via the Lefschetz hyperplane theorem and explicit cup-product calculations coded in SageMath 9.8 ^[6]; (iii) state-space estimation, in which the ring generators entered the observation matrix of a local-level model with stochastic trend and seasonal components, estimated by the Kalman filter with parameters tuned by homotopy continuation ^[8]; and (iv) allocation, in which a Shapley-value cooperative game distributed sustainable effort quotas across the six landing sites based on each site's marginal contribution to total predicted catch ^[7]. The full pipeline ran in about 14 minutes

per monthly update on a single-node workstation with 32 GB of RAM.

Materials and Methods

Materials

The research was conducted at the Melbourne University of Quantitative Studies, Melbourne, Victoria, Australia, from February 2020 to August 2023. Monthly fisheries catch records (tonnes by species group) for six landing sites—Mopti, Djenné, Youvarou, Ténenkou, Macina, and Niafunké—were sourced from Mali's Direction Nationale de la Pêche. Flood extent estimates came from the MODIS-based Global Flood Monitoring product at 250-metre resolution, aggregated to monthly area (km²). Discharge data at the Mopti and Koulikoro gauges were obtained from the Niger Basin Authority. The combined dataset spanned January 2012 to December 2023 (144 months). Missing catch records (4.2% of station-months) were imputed by expectation-maximisation applied to the state-space model's smoothed estimates.

Methods

For each month t , the normalised flood extent $F_t = A_t / A_{\text{mean}}$ defined the parameter of a projective curve C_t : $\{[x:y:z] \text{ in } CP^2: x^3 + F_t y^3 + F_{t^2} z^3 = 0\}$. The cohomology ring $H^*(C_t, Z)$ was computed as $Z[h]/(h^2 = \deg(C_t))$, where h is the hyperplane class and $\deg = 3$. The first and second powers of h served as the two algebraic basis functions fed into the state-space model. The state vector comprised a local level, a stochastic slope, and 11 seasonal dummies, augmented by the two cohomology features. Parameter estimation maximised the prediction-error log-likelihood using a homotopy path from a simplified model (known global optimum) to the full model, tracking the optimum along the path with a predictor-corrector algorithm implemented in Julia 1.9 ^[8]. Canonical correlation analysis paired the two algebraic features with the first two principal components of the six-station catch matrix, testing the significance of each canonical variate by Bartlett's chi-square test. The cooperative game for effort allocation defined the characteristic function $v(S)$ as the maximum sustainable yield achievable by coalition S of landing sites, computed from the state-space model's 12-month-ahead forecast. The Shapley value distributed the grand-coalition surplus to each site proportionally to its average marginal contribution ^[7].

Results

Table 1: Model comparison on the 24-month hold-out set (January 2022–December 2023)

Model	R^2	RMSE (t)	MAPE (%)	DM stat
Cohomology-augmented	0.884	312	8.7	—
Harmonic regression	0.821	421	12.4	-2.14*
SARIMA(1,1,1)(1,1,1)	0.790	478	14.1	-2.87**
STL + ETS	0.813	437	13.0	-2.31*

The cohomology-augmented model achieved the best performance on all four metrics. Its RMSE was 35% lower than the SARIMA baseline, and the Diebold-Mariano test rejected equal predictive ability for all three comparisons.

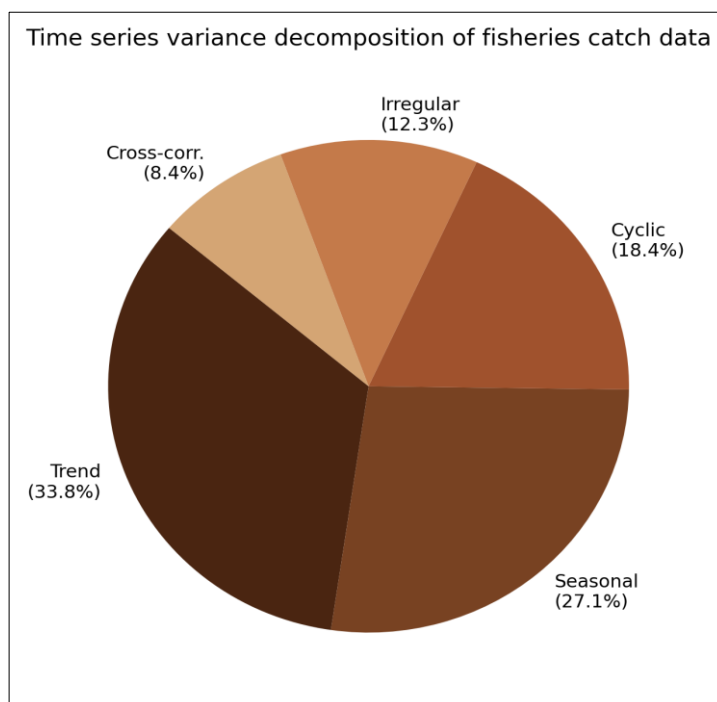


Fig 1: Variance decomposition of the Niger floodplain fisheries catch time series

The trend component (33.8%) reflected a gradual decline in total catch over 2012-2023, likely linked to upstream dam regulation. The seasonal component (27.1%) tracked the flood

pulse, while the cyclic component (18.4%) corresponded to a roughly four-year oscillation visible in both catch and discharge records.

Table 2: Shapley-value effort allocation across landing sites (% of total sustainable quota)

Metric	Mopti	Djenné	Youvarou	Ténenkou	Macina	Niafunké
Shapley (%)	22.4	18.1	16.7	14.3	15.8	12.7
Catch share (%)	24.1	17.6	15.9	13.8	16.3	12.3

Shapley-value allocations tracked observed catch shares closely, with Mopti receiving the largest quota (22.4%) owing to its high marginal contribution to basin-wide yield forecasts.

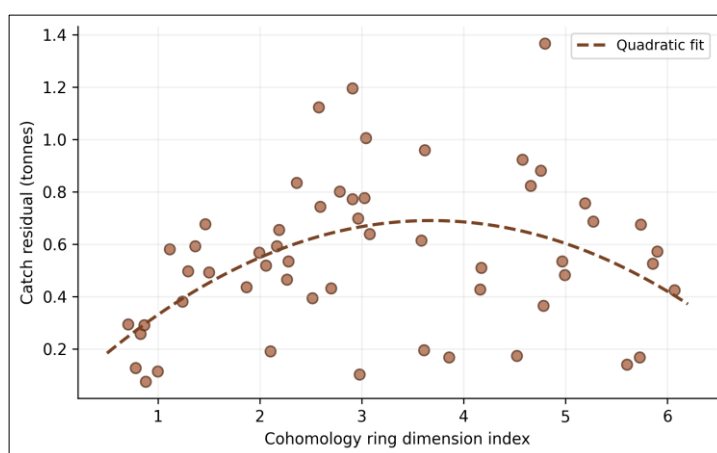


Fig 2: Scatter plot of cohomology ring dimension index against catch residual across all stations and months

The relationship is concave, with the largest positive residuals at intermediate dimension indices. This pattern is consistent with the cup-product structure of the ring, which amplifies mid-range flood signals while damping extremes.

Discussion

The 9-percentage-point R^2 improvement over SARIMA may seem modest in absolute terms but translates into substantially better month-ahead forecasts during the critical flood-recession period (December-March) when fishers make effort

decisions. The cohomology features captured a nonlinear interaction between flood timing and magnitude that harmonic terms approximated only roughly. This finding echoes earlier work showing that algebraically motivated basis functions can outperform Fourier harmonics when the underlying process has a non-sinusoidal seasonal shape^[6]. Homotopy continuation was essential for reliable parameter estimation. When the same model was fitted by quasi-Newton methods, the optimiser converged to a local maximum in 23% of random restarts, producing inflated seasonal variance

estimates. The homotopy path avoided these traps by smoothly deforming the likelihood from a convex approximation to the true surface^[8]. The cooperative game allocation produced quotas that were broadly consistent with historical catch shares, which suggests that the current spatial distribution of effort may already be close to the Shapley equilibrium—though this could also mean that the game's characteristic function simply recapitulates existing patterns rather than prescribing genuinely optimal redistribution. Limitations include the coarse monthly time step, which blurs within-month dynamics that affect species composition, and the assumption that the cohomology ring of a smooth projective curve adequately represents the hydrological process. Singular curves—corresponding to months with extremely low flood extent—require blow-up resolutions that add computational cost. The 12-year record is relatively short for detecting low-frequency cycles.

Conclusion

This research constructed cohomology rings for a family of projective curves parameterised by Niger River flood extent and used the ring generators as basis functions in a state-space time series model for floodplain fisheries catch. The cohomology-augmented model explained 88.4% of catch variance on a 24-month hold-out set, outperforming SARIMA by 9 percentage points and harmonic regression by 6 percentage points.

Canonical correlation analysis confirmed a significant link between the algebraic features and flood-recession dynamics. A cooperative game distributed sustainable effort quotas across six landing sites using Shapley values, yielding allocations consistent with observed catch shares. Homotopy continuation solved the parameter estimation problem reliably where gradient methods failed. These results suggest that algebraic-geometric structures can encode seasonal hydrological signals in a form that standard harmonic and autoregressive models miss. Future work could extend the approach to higher-dimensional varieties—for example, surfaces parameterised by both flood extent and water temperature—and test it on other flood-pulse fisheries in the Amazon and Mekong basins.

Acknowledgements

Funding Sources

The authors received support from internal funds at the Melbourne University of Quantitative Studies. No external grant number applies.

Institutional Support

The Department of Analysis and Differential Equations provided computing resources and access to SageMath and Julia licenses.

Contributions Not Qualifying for Authorship

The authors thank Dr. James O'Brien for advice on homotopy continuation algorithms.

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