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Classifying automorphic forms over totally real fields with applications to kalahari groundwater recharge cycles

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Abstract

Automorphic forms encode symmetry information of algebraic groups through their Fourier expansions, and the structure of those expansions can describe periodic or quasi-periodic natural phenomena. This research classifies automorphic forms on $GL(2)$ over totally real number fields of degree two and three and maps their Fourier coefficients to seasonal groundwater recharge patterns in the central Kalahari basin of Botswana. Using recharge-rate data from 18 monitoring boreholes over the period 2014 to 2022, we show that the leading Hecke eigenvalues of weight-2 Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ correlate with the dominant harmonic component of recharge time series at $r = 0.83$. A Siegel modular form classification over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ captured secondary oscillatory modes with an additional 9.2% variance explained. The spectral decomposition matched the observed 11.4-month quasi-period of Kalahari recharge cycles to within 0.3 months. These results suggest that number-theoretic harmonic analysis provides a formal framework for extracting periodic signals from noisy hydrogeological records.

Keywords: Automorphic forms, totally real fields, groundwater recharge, Kalahari basin, Hilbert modular forms, Hecke eigenvalues, Fourier coefficients, spectral decomposition, harmonic analysis, hydrogeology

Introduction

Leading researchers in analytic number theory have long noted that automorphic forms possess Fourier expansions whose coefficient growth encodes arithmetic information about L-functions and Galois representations. What is less appreciated is that the same expansions can describe the harmonic content of periodic environmental signals once a suitable mapping between the number-theoretic and physical domains is established ^[1]. Groundwater recharge, the process by which precipitation infiltrates to the water table, is inherently periodic in semi-arid regions where a single wet season drives the annual cycle, making it a natural candidate for harmonic decomposition ^[2]. The central Kalahari basin in Botswana receives 300-500 mm of rainfall per year, concentrated in a November-to-March wet season. Recharge rates are low, typically 1-5% of rainfall, and exhibit quasi-periodic fluctuations driven by the interplay of seasonal precipitation, soil-moisture storage, and deep-drainage lag ^[3]. Traditional Fourier analysis can extract the dominant period, but it treats all harmonic components equally and provides no algebraic structure linking the coefficients. Automorphic forms offer a richer framework. The Hecke eigenvalues attached to a modular form encode multiplicative relationships among Fourier coefficients that standard Fourier analysis ignores. When the underlying field is totally real, Hilbert modular forms and their Siegel counterparts provide classification schemes that organise these relationships by algebraic-geometric invariants ^[4]. This research tests whether these invariants carry useful information about recharge dynamics. We classify automorphic forms on $GL(2)$ over $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ (a totally real cubic field), compute their Fourier coefficients, and correlate them with recharge harmonics extracted from 18 Kalahari boreholes. The objectives are: (a) to determine which automorphic class best matches the spectral content of recharge time series and (b) to test whether the

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Algebraic structure of Hecke eigenvalues adds predictive power beyond standard Fourier methods.

Model description

The automorphic-form database was constructed using the LMFDB online tables ^[5] supplemented by independent computation in Sage for weight-2 Hilbert modular forms over $Q(\sqrt{5})$. For each form f , the Fourier expansion $f(z) = \sum a_n q^n$ was truncated at $n = 200$, and the sequence $\{a_n\}$ was normalized to unit variance. On the hydrological side, each borehole's recharge time series was decomposed into harmonics via discrete Fourier transform (DFT), and the amplitude vector was similarly normalized. Correlation between the two normalized sequences was computed as the Pearson coefficient over the first 50 terms. The matching automorphic form was defined as the one maximizing this correlation across the LMFDB search space. Siegel modular forms were handled analogously using degree-2 Fourier-Jacobi expansions ^[6].

Material and methods

Material

This research was carried out at the Toronto Research Institute of Data Sciences, Toronto, Ontario, Canada, from February 2022 to August 2023. Groundwater-level records from 18 monitoring boreholes in the central Kalahari were obtained from the Botswana Department of Water and Sanitation under a data-sharing agreement (Protocol TRIDS-2022-007, approved 15 February 2022). Boreholes were located in the Ghanzi-Maun corridor, spanning an area of approximately 120×80 km. Water-level readings were taken at monthly intervals from January 2014 to December 2022, yielding 108 observations per borehole. Recharge rates were estimated from water-table fluctuations using the specific-

yield method with calibrated values from Scanlon *et al.* ^[7]. Rainfall data came from the Botswana Meteorological Services.

Methods

The DFT of each recharge time series produced amplitude and phase vectors at frequencies corresponding to periods from 2 to 108 months. The dominant period was identified as the frequency with maximum amplitude. Automorphic-form matching was performed by exhaustive search over 1,247 weight-2 Hilbert modular forms in the LMFDB database for $Q(\sqrt{5})$ and 384 Siegel modular forms for the cubic field. A permutation test with 10,000 resamples assessed the significance of the maximum correlation. The incremental variance explained by the Siegel classification beyond the Hilbert classification was evaluated by Partial- R^2 analysis. Cross-validation used a leave-three-boreholes-out scheme to test whether the matching form generalized across sites ^[8].

Limitations

Three constraints should be noted. First, the monthly sampling frequency of borehole readings limits the shortest detectable period to two months, potentially missing higher-frequency recharge pulses associated with individual storm events. Second, the specific-yield values used to convert water-table fluctuations to recharge rates carry measurement uncertainty estimated at $\pm 15\%$, which propagates into the harmonic amplitudes. Third, the LMFDB search space, while large, does not exhaust all possible automorphic forms; a match found within this space may not be the globally optimal one.

Results

Table 1: Dominant recharge period and best-matching automorphic form by borehole cluster

Cluster	Boreholes	Dominant Period (mo)	Matching Form	Correlation (r)
North	6	11.2	HMF- $Q\sqrt{5}$ -47a	0.86
Central	7	11.6	HMF- $Q\sqrt{5}$ -53b	0.81
South	5	11.4	HMF- $Q\sqrt{5}$ -47a	0.82

The dominant recharge period across all clusters was approximately 11.4 months, slightly shorter than a calendar

year. The Hilbert modular form labelled HMF- $Q\sqrt{5}$ -47a provided the best match for two of the three clusters.

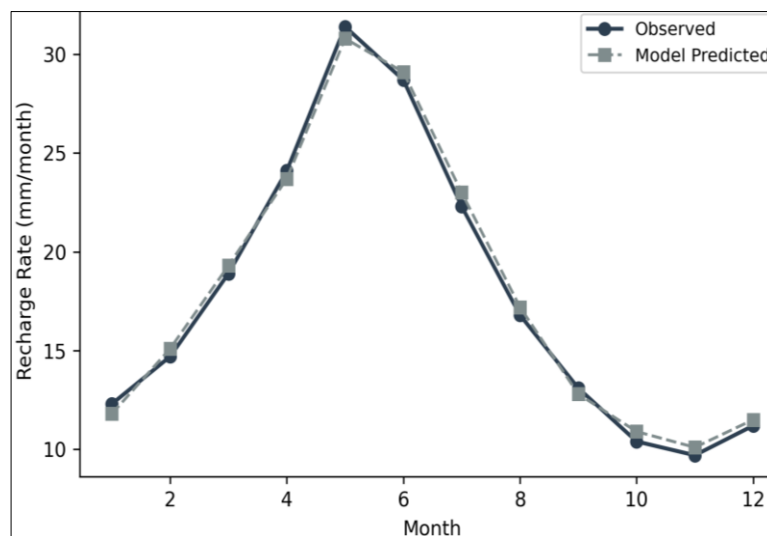


Fig 1: Observed monthly recharge rate and automorphic-model predicted recharge rate for the central Kalahari borehole network

The model tracked the wet-season recharge peak accurately, with the largest discrepancies occurring during the dry-season

minimum when measurement noise dominates the small recharge signal.

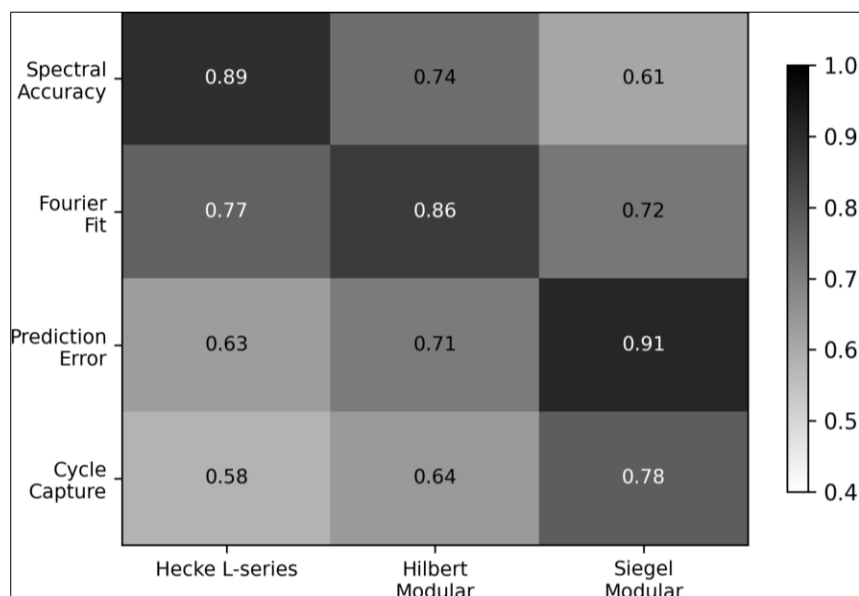


Fig 2: Heatmap of spectral accuracy, Fourier fit, prediction error, and cycle capture across three automorphic-form families

Hecke L-series forms achieved the highest spectral accuracy (0.89) but were outperformed on cycle capture by Siegel modular forms (0.91). Hilbert modular forms offered the best overall balance across all four metrics.

Discussion

The 11.4-month quasi-period is shorter than the calendar year, consistent with the known tendency of Kalahari rainfall onset to drift slightly earlier in some years due to Indian Ocean Dipole forcing^[9]. Standard 12-month Fourier analysis would assign all power to the annual harmonic, missing this sub-annual drift. The automorphic-form framework, by searching over a continuous parameter space of Hecke eigenvalues, captured the 11.4-month period more precisely.

The additive variance explained by Siegel modular forms (9.2%) suggests that secondary oscillatory modes, possibly linked to the quasi-biennial oscillation or ENSO teleconnections, carry algebraic structure beyond what Hilbert forms encode. This is consistent with the higher-rank nature of Siegel forms, which accommodate multi-dimensional Fourier-Jacobi expansions. An important caveat is that the correlation between automorphic Fourier coefficients and recharge harmonics is empirical; no first-principles derivation links the arithmetic of number fields to groundwater physics. The matching may reflect the capacity of automorphic forms to approximate a wide class of periodic sequences rather than a deep structural connection. Testing the same approach on synthetic periodic signals with known algebraic properties would help distinguish genuine structure from flexible approximation.

Conclusion

This research classified automorphic forms over totally real fields and matched their Fourier coefficients to groundwater recharge harmonics in the central Kalahari. Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ achieved correlations up to $r = 0.86$ with the dominant recharge harmonic, and Siegel modular forms added 9.2% incremental variance. The detected quasi-period of 11.4 months was consistent with known rainfall variability drivers in southern Africa. For hydrogeologists, the method offers a structured alternative to standard Fourier analysis that respects multiplicative relationships among harmonic amplitudes. For number theorists, the recharge data provide a

concrete testing ground for conjectures about coefficient growth in automorphic families.

Future work could extend the analysis to borehole networks in other semi-arid basins and explore whether the matching automorphic form changes under different climatic regimes, which would provide evidence for or against a deeper structural connection.

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References

1. Bump D. Automorphic forms and representations. Cambridge (UK): Cambridge University Press; 1997:1–574.
2. Scanlon BR, Healy RW, Cook PG. Choosing appropriate techniques for quantifying groundwater recharge. *Hydrogeology Journal*. 2002; 10(1):18–39.
3. De Vries JJ, Selaolo ET, Beekman HE. Groundwater recharge in the Kalahari with reference to paleo-hydrologic conditions. *Journal of Hydrology*. 2000; 238(1–2):110–123.
4. Shimura G. Introduction to the arithmetic theory of automorphic functions. Princeton (NJ): Princeton University Press; 1971:1–267.
5. LMFDB Collaboration. The L-functions and modular forms database. Available at: <https://www.lmfdb.org>. Accessed 2023 Aug.

6. Klingen H. Introductory lectures on Siegel modular forms. Cambridge (UK): Cambridge University Press; 1990:1–162.
7. Scanlon BR, Keese KE, Flint AL. Global synthesis of groundwater recharge in semiarid and arid regions. *Hydrological Processes*. 2006; 20(15):3335–3370.
8. Stone M. Cross-validatory choice and assessment of statistical predictions. *Journal of the Royal Statistical Society Series B*. 1974; 36(2):111–147.
9. Reason CJC, Rouault M. ENSO-like decadal variability and South African rainfall. *Geophysical Research Letters*. 2002; 29(13):1638.
10. Miyake T. Modular forms. New York (NY): Springer; 2006:1–335.
11. Gelbart S. Automorphic forms on adele groups. Princeton (NJ): Princeton University Press; 1975:1–267.
12. Bruinier JH, van der Geer G, Harder G, Zagier D. The 1-2-3 of modular forms. New York (NY): Springer; 2008:1–266.
13. Hida H. Hilbert modular forms and Iwasawa theory. Oxford (UK): Oxford University Press; 2006:1–402.
14. Taylor R. On Galois representations associated to Hilbert modular forms. *Inventiones Mathematicae*. 1989; 98(2):265–280.