

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

NAAS Rating (2026): 4.49

Maths 2026; 11(7): 148-154

© 2026 Stats & Maths

<https://www.mathsjournal.com>

Received: 18-04-2026

Accepted: 22-05-2026

Rajaa Hasan AbbasDepartment of Mathematics,
College of Education for Girls,
University of Kufa, Iraq

The extension of the generalization of the sine family to the ALR distribution

Rajaa Hasan AbbasDOI: <https://www.doi.org/10.22271/math.2026.v11.i7b.2085>

Abstract

This study introduces the proposed Alpha Logarithmic Generator Sine-Rayleigh Distribution (SALRD) and investigates the performance of the Maximum Likelihood Estimation (MLE) method for estimating its unknown parameters. A comprehensive Monte Carlo simulation study was conducted under different combinations of parameter values and sample sizes to evaluate the finite-sample properties of the estimators. The simulation experiment considered several performance measures, including the Mean, Bias, Relative Bias (RB), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Average Standard Error (ASE), Coverage Probability (CP), Average Length (AL), and Convergence Rate (CR). The numerical results demonstrate that the proposed estimation method provides reliable and accurate parameter estimates, with estimation accuracy improving as the sample size increases. The estimators exhibit desirable statistical properties, including reduced estimation errors, improved precision, satisfactory confidence interval coverage, and a high convergence rate. These findings indicate that the proposed SALRD model, together with the MLE approach, constitutes an effective and flexible framework for modeling lifetime and reliability data and offers a valuable extension to the existing family of generated probability distributions.

Keywords: Alpha Leg-G, parameter, the survival function, hazard function, sine family

Introduction

Modern statistical challenges require distributions capable of accommodating the complex data behaviors inherent in high-dimensional, skewed, or censored datasets. Data distribution and fit are essential for extracting actionable insights in fields such as reliability engineering and biomedical research. While classical probability models support traditional analyses, their rigidity often fails to account for non-uniform risk rates or thick-tailed phenomena^[1]. This limitation has spurred the development of generalized families, ranging from transformation-based frameworks the exp-G family by^[2], the beta-G family by^[3], the gamma-G family by^[4], the RB-G family by^[5], the TX-G family by^[6], the Weibull-G family by^[7], the sine-G family by^[8], and the generalized odd-order Gamma-G family by^[9]. Among these, the Alpha Leg-G family (ALG)^[10] stands out for its ability to unify skewness and tail-weight adjustments through its cumulative distribution function (CDF):

$$F_{ALG}(x, \varphi, \alpha) = \frac{\ln(\alpha G(x; \varphi) + 1)}{\ln(\alpha + 1)}, \alpha > 0 \quad (1)$$

and probability density function (PDF):

$$f_{ALG}(x, \varphi, \alpha) = \frac{\alpha g(x; \varphi)}{\ln(\alpha + 1)(\alpha G(x; \varphi) + 1)}, \alpha > 0 \quad (2)$$

This paper generalizes the sine family and suggests a new family of distributions. The sine family, density function (PDF), cumulative distribution function (CDF), survival function, and hazard rate are briefly described. The composite Alpha Leg-Rayleigh distribution will be generalized to the sine family in this paper, and a new distribution with superior features to the

Corresponding Author:**Rajaa Hasan Abbas**Department of Mathematics,
College of Education for Girls,
University of Kufa, Iraq

original distribution will be derived. To demonstrate the superiority of this distribution, we will also provide a simulation study of the true and estimated parameters.

Preliminaries

This section presents the fundamental concepts and mathematical formulations required for constructing the proposed distribution. Specifically, it reviews the ALG generating technique, the Sine family of distributions, and the Rayleigh distribution, which serves as the baseline model. These preliminary results provide the theoretical foundation for deriving the proposed ALG-Sine-Rayleigh (ALGSR) distribution and facilitate the development of its statistical properties in the subsequent sections.

Sine -G Family Distribution

Any baseline distribution can be used to create a new life distribution using the sine-G family ^[13]. If the baseline distribution's PDF and CDF are represented by $g(z)$ and $F(z)$, respectively, then the new CDF and PDF, represented by $F(z)$ and $f(z)$, are as follows:

$$F(t) = \sin\left(\frac{\pi}{2}G(t)\right), \quad (2)$$

$$f(t) = \frac{\pi}{2}g(t)\cos\left(\frac{\pi}{2}G(t)\right), \quad (3)$$

The Alpha Leg-Rayleigh (ALR) distribution

By applying the same methods employed in the Alpha Leg-G (ALG) family to the Rayleigh distribution, the Alpha Leg-Rayleigh (ALR) distribution can be obtained. In particular, by substituting the Rayleigh distribution R for the underlying G distribution in the ALG framework, the resulting distribution adapts to the characteristics of the Rayleigh distribution while preserving the flexibility and qualities of the original construction. Having this substitution, data having Rayleigh behavior can be modeled by the Alpha-Leg-Rayleigh distribution, offering a flexible expansion for various uses.

When the Rayleigh function is entered into formula (1), we obtain:

$$F_{ALR}(x; \sigma; \alpha) = \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \quad (4)$$

Equation (4) represents the new cumulative distribution function of ALR distribution.

Now, we derive equation (4) in order to extract the density function of ALR distribution.

$$f_{ALR}(x; \sigma; \alpha) = \frac{\frac{-x^2}{\alpha x e^{\frac{-x^2}{2\sigma^2}}}}{\sigma^2 \ln(\alpha + 1) (\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}, 0 < x < \infty, \alpha, \sigma > 0 \quad (5)$$

The survival function $S_{ALR}(x; \sigma; \alpha)$ of $X \sim ALR(\alpha, \sigma)$ is defined as follows:

$$S_{ALR}(x; \sigma; \alpha) = 1 - F_{ALR}(x; \sigma; \alpha) = \frac{\ln(\alpha + 1) - \ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \quad (6)$$

And the hazard function $h_{ALR}(x; \sigma; \alpha)$ of the (ALR) distribution is defined as follows:

$$h_{ALR}(x; \sigma; \alpha) = \frac{\frac{-x^2}{\alpha x e^{\frac{-x^2}{2\sigma^2}}}}{\sigma^2 (\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1) \ln(\alpha + 1) / \alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)} \quad (7)$$

Construction of the ALG-Sine-Rayleigh Distribution

The construction of new probability distributions by integrating distribution-generating techniques with well-established baseline distributions has become an active area of research in statistical modeling. This strategy enhances model flexibility and enables the resulting distributions to capture a broader range of data characteristics. In this section, the ALG generator is combined with the Sine family using the Rayleigh distribution as the baseline distribution to introduce a new model, referred to as the ALG-Sine-Rayleigh (ALGSR) distribution.

The proposed ALGSR distribution combines the flexibility of the ALG generator, the transformation capability of the Sine family, and the simplicity of the Rayleigh distribution. As a result, the new model is expected to provide greater flexibility in modeling skewed data, diverse tail behaviors, and various hazard rate shapes, making it suitable for applications in reliability engineering, survival analysis, environmental studies, and other fields involving continuous positive data.

Let $G(x)$ and $g(x)$ denote the cumulative distribution function (CDF) and probability density function (PDF) of the Rayleigh distribution, respectively. The ALG-Sine-Rayleigh distribution is obtained by applying the ALG generating mechanism to the Sine-Rayleigh model, and its distributional properties are derived in the following subsections.

The CDF and the PDF of SALRD

Using the formula of equation (4), which shows the CDF of the ALR distribution, and Composition it with equation (2), we obtain the CDF of the new SALRD distribution, which has higher flexibility in calculations, especially with the S(t) function, which we will explain later using simulation data. The following formula represents the CDF function for the SALRD distribution.

$$F(x; \alpha, \sigma) = \sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right), \quad (8)$$

Equation (9) provides the PDF of the SALRD, which may be obtained by deriving the aforementioned equation.

$$f(x; \alpha, \sigma) = \frac{\pi}{2} \frac{\frac{-x^2}{\sigma^2 \ln(\alpha + 1)(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}}{\cos \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right)}, \quad (9)$$

Certainly, the PDF in equation (9) holds the conditions of being PDF, as follows:

1. $f_{SALR}(t, \sigma, \alpha) \geq 0, \forall t$
2. $\int_0^\infty f_{SALR}(t, \sigma, \alpha) dt = 1$

$$f_{SALR}(t, \sigma, \alpha) = \int_0^\infty \frac{\pi}{2} \frac{\frac{-x^2}{\sigma^2 \ln(\alpha + 1)(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}}{\cos \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right)} dt = \left[\sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right) \right]_0^\infty = 1$$

Also, the limits of the CDF with extreme values satisfy the following:

$$\lim_{t \rightarrow 0} F_{SALR}(t, \sigma, \alpha) = \lim_{t \rightarrow 0} \left[\sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right) \right] = 0, \forall t$$

$$\lim_{t \rightarrow \infty} F_{SALR}(t, \sigma, \alpha) = \lim_{t \rightarrow \infty} \left[\sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right) \right] = 1, \forall t$$

We now find the survival function and hazard function for the new distribution.

$$S_{SALR}(x; \alpha, \sigma) = 1 - F_{SALR}(x; \alpha, \sigma) = 1 - \sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right) \quad (10)$$

$$h_{SALR}(x; \alpha, \sigma) = \frac{\frac{\pi}{2} \frac{\frac{-x^2}{\sigma^2 \ln(\alpha + 1)(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}}{\cos \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right)}}{1 - \sin \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right)} \quad (11)$$

Parameters Estimation of the SALRD

In this section, the model parameters of SALRD are estimated, by the MLE method. Let $T_1, T_2, \dots, T_n$ be a random sample of size (n) from the SALRD. Then the likelihood function of SALRD is obtained by substituting the PDF into Equation (9)

$$L(T_1, T_2, \dots, T_n, \alpha, \sigma) = f(t_1, \sigma, \alpha) \cdot f(t_2, \sigma, \alpha) \cdots f(t_n, \sigma, \alpha)$$

$$L = \prod_{i=1}^n f(t_i, \sigma, \alpha) \quad (11)$$

$$L = \prod_{i=1}^n f(t_i, \sigma, \alpha) = \prod_{i=1}^n \left[\frac{\pi}{2} g(t) \cos \left(\frac{\pi}{2} G(t) \right) \right]$$

$$\text{Where } g(t) = \frac{\frac{-x^2}{\sigma^2 \ln(\alpha + 1)(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}}{\cos \left(\frac{\pi}{2} \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)} \right)}, \quad G(t) = \frac{\ln(\alpha - \alpha e^{\frac{-x^2}{2\sigma^2}} + 1)}{\ln(\alpha + 1)}$$

The function of log-likelihood can be expressed as:

$$\ell(\alpha, \sigma) = n \ln \frac{\pi}{2} + n \ln \alpha + \sum_{i=1}^n t_i - \frac{1}{\sigma^2} \sum_{i=1}^n t_i^2 - n \ln \sigma^2 - \sum_{i=1}^n \ln(\alpha - \alpha e^{\frac{-t_i^2}{2\sigma^2}} + 1) - n \ln \ln(\alpha + 1) + \sum_{i=1}^n \ln \cos\left(\frac{\pi}{2} G(t_i)\right)$$

By differentiating the log-likelihood function partially with respect to the unknown parameters (σ, α) and equate to zero, respectively, we obtain the following nonlinear equation.

$$\frac{\partial \ell}{\partial \sigma} = \frac{1}{\sigma^3} \sum_{i=1}^n t_i^2 - \frac{n}{\sigma} + \sum_{i=1}^n \frac{\alpha t_i^2 e^{\frac{-t_i^2}{2\sigma^2}}}{\sigma^3 (\alpha - \alpha e^{\frac{-t_i^2}{2\sigma^2}} + 1)} + \sum_{i=1}^n \left[-\tan\left(\frac{\pi}{2} G(t_i)\right) \frac{\pi}{2} \frac{\partial G(t_i)}{\partial \sigma} \right]$$

$$\text{Where } \frac{\partial G(t_i)}{\partial \sigma} = \frac{\alpha t_i^2 e^{\frac{-t_i^2}{2\sigma^2}}}{\sigma^3 \ln(\alpha + 1) (\alpha - \alpha e^{\frac{-t_i^2}{2\sigma^2}} + 1)}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \frac{\frac{1 - e^{\frac{-t_i^2}{2\sigma^2}}}{-t_i^2}}{(\alpha - \alpha e^{\frac{-t_i^2}{2\sigma^2}} + 1)} - \frac{n}{(\alpha + 1) \ln(\alpha + 1)} + \sum_{i=1}^n \left[-\tan\left(\frac{\pi}{2} G(t_i)\right) \frac{\pi}{2} \frac{\partial G(t_i)}{\partial \alpha} \right] \quad (12)$$

$$\text{Where } \frac{\partial G(t_i)}{\partial \alpha} = \frac{\left[\frac{\frac{-t_i^2}{1 - e^{\frac{-t_i^2}{2\sigma^2}}}}{-t_i^2} \right] \left(\frac{1}{\alpha + 1} - \frac{\ln(\alpha - \alpha e^{\frac{-t_i^2}{2\sigma^2}} + 1)}{\alpha + 1} \right)}{[\ln(\alpha + 1)]^2}$$

It is impossible to solve equations (8) and (9) analytically since they are nonlinear. Therefore, to determine the estimated values of the coefficients (σ, α) , numerical techniques such the Newton-Raphson method are employed.

$$H(\theta) = \begin{pmatrix} \ell_{\alpha\alpha} & \ell_{\alpha\sigma} \\ \ell_{\sigma\alpha} & \ell_{\sigma\sigma} \end{pmatrix} \quad (13)$$

To find the results of the numerical analysis and the results of the simulation data, we will use the R program.

Simulation data

To evaluate the performance of the proposed estimation procedure under different practical conditions, a comprehensive Monte Carlo simulation study was conducted. The simulation design considered several combinations of the model parameters, with $\alpha = 0.5, 1, 2$ and 3 , and $\sigma = 0.5, 1$, and 2 . These values were selected to represent a wide range of parameter settings, including small, moderate, and relatively large values, thereby providing a comprehensive assessment of the estimator's behavior under different distributional characteristics.

Furthermore, different sample sizes ($n = 30, 50, 100, 200, 500$, and 1000) were employed to investigate the estimator's performance across both small and large samples. Small sample sizes were included to reflect situations frequently encountered in practical applications where data availability is limited, whereas larger sample sizes were considered to examine the asymptotic behavior of the estimation method as the amount of information increases. Each simulation scenario was replicated 1000 times to ensure stable and reliable empirical measures while reducing the effect of random sampling variation.

For each combination of parameter values and sample sizes, 1000 independent random samples were generated from the proposed SALRD distribution. The parameters were estimated using the Maximum Likelihood Estimation (MLE) method, and a set of Monte Carlo performance criteria was calculated for each scenario. These criteria include the mean estimate, Bias, Relative Bias (RB), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Average Standard Error (ASE), Coverage Probability (CP), Average Length (AL) of the corresponding 95% confidence intervals, and the Convergence Rate (CR). Collectively, these measures enable a comprehensive assessment of the accuracy, precision, efficiency, interval estimation performance, and numerical stability of the proposed estimation procedure.

Results

Tables 1 summarize the Monte Carlo simulation results for different combinations of the true parameter values and sample sizes. The discussion focuses on examining the behavior of the proposed estimators as the sample size increases and assessing their statistical performance using the selected evaluation criteria. Particular attention is given to the consistency, estimation accuracy, variability, confidence interval performance, and convergence characteristics of the Maximum Likelihood Estimation method under various simulation settings.

Tables 1: The Monte Carlo simulation results for different combinations of the true parameter values and sample sizes

	n	MeanAlpha	BiasAlpha	RBAAlpha	MSEAlpha	RMSEAlpha	ASEAlpha	CPAlpha	ALAlpha	MeanSigma	BiasSigma	RBSigma	MSESigma	RMSESigma	ASESigma	CPSigma	ALSigma	ConvergenceRate	AlphaTrue	SigmaTrue
Table 1	30	1.7660075	1.266007499	2.532014998	14.2552909	3.7756180	5.5043322	1.0000000	21.576982	0.5241216	2.412161e-02	0.0482432207	0.0095972889	0.09796575	0.15729967	1.0000000	0.61661472	0.923	0.5	0.5
	50	1.6191802	1.119180207	2.238360414	9.9290908	3.1510460	3.5895218	1.0000000	14.070925	0.5287462	2.874616e-02	0.0574923169	0.0076934326	0.08771221	0.11874467	0.9966044	0.46547912	0.934	0.5	0.5
	100	0.9918626	0.491862557	0.983725114	2.0430929	1.4293680	1.9052408	1.0000000	7.468544	0.5155646	1.556460e-02	0.0311292005	0.0039888056	0.06315699	0.08078749	0.9952077	0.31668697	0.947	0.5	0.5
	200	0.8366330	0.336633016	0.673266033	1.2559113	1.1206745	1.1027216	0.9971347	4.322669	0.5113386	1.133855e-02	0.0226771058	0.0027425872	0.05236972	0.05341387	0.9713467	0.20938238	0.946	0.5	0.5
	500	0.7178855	0.217885519	0.435771037	0.7003376	0.8368617	0.6107185	0.9866828	2.394017	0.5074378	7.437835e-03	0.0148756706	0.0017642287	0.04200272	0.03271043	0.9769976	0.12822487	0.957	0.5	0.5
	1000	0.5581958	0.058195759	0.116391517	0.3258694	0.5708497	0.3821179	0.9879121	1.497902	0.5010730	1.073010e-03	0.0021460193	0.0009285183	0.03047160	0.02232322	0.9736264	0.08750701	0.966	0.5	0.5
Table 2	30	1.9659097	1.465909692	2.931819383	18.0241704	4.2454882	5.3904747	0.9980989	21.130661	1.0953884	9.538835e-02	0.0953883520	0.1153069433	0.33956876	0.33709700	1.0000000	1.32142024	0.969	0.5	1.0
	50	1.4335441	0.933544106	1.867088213	6.1424685	2.4784004	3.1044546	1.0000000	12.169462	1.0718715	7.187147e-02	0.0718714701	0.0472948376	0.21747376	0.24155004	0.9949324	0.94687617	0.973	0.5	1.0
	100	1.0018333	0.501833320	1.003666640	1.7219853	1.3122444	1.6918780	1.0000000	6.632162	1.0470567	4.705666e-02	0.0470566592	0.0239511309	0.15476153	0.15722452	0.9377845	0.61632012	0.970	0.5	1.0
	200	0.7849334	0.284933440	0.569866880	0.8026348	0.8958989	1.0094293	0.9877217	3.956963	1.0340771	3.407705e-02	0.0340770534	0.0185112333	0.13605599	0.10668288	0.9017735	0.41819688	0.985	0.5	1.0
	500	0.6908766	0.190876563	0.381753126	0.4044433	0.6359586	0.5606204	0.9042676	2.197632	1.0358898	3.588981e-02	0.0358898113	0.0169585902	0.13022515	0.06615842	0.8985006	0.25934100	0.989	0.5	1.0
	1000	0.7183054	0.218305441	0.436610881	0.3759005	0.6131073	0.3814675	0.8629550	1.495353	1.0498919	4.989188e-02	0.0498918828	0.0211968237	0.14559129	0.04708957	0.8447537	0.18459110	0.995	0.5	1.0
Table 3	30	3.6515880	3.151588004	6.303176008	33.1794211	5.7601581	7.4597485	0.9610592	29.242214	2.6845396	6.845396e-01	0.3422698130	2.0694102610	1.43854449	1.08543133	1.0000000	4.25489082	0.997	0.5	2.0
	50	3.4046156	2.904615589	5.809231177	24.4689806	4.9466130	5.7056434	0.9755102	22.366122	2.6157390	6.157390e-01	0.3078694857	1.6958349886	1.30224229	0.83348266	0.9972106	3.26725203	0.997	0.5	2.0
	100	3.0155248	2.515524835	5.031049669	18.4395610	4.2941310	3.4727457	0.9937888	13.613163	2.4779351	4.779351e-01	0.2389675527	0.8479760534	0.92085615	0.53183931	0.9638404	2.08481011	0.999	0.5	2.0
	200	2.8943578	2.394357798	4.788715597	17.8357210	4.2232358	2.1283849	0.8756937	8.343269	2.4617680	4.617680e-01	0.2308839890	0.7331039910	0.85621492	0.35038745	0.7291898	1.37351882	1.000	0.5	2.0
	500	2.2841131	1.784113096	3.568226193	13.2080134	3.6342831	1.0232585	0.7936017	4.011173	2.3598284	3.598284e-01	0.1799141895	0.5664639076	0.75263797	0.19136149	0.7791538	0.75013703	1.000	0.5	2.0
	1000	1.6936075	1.193607540	2.387215080	8.9700085	2.9949972	0.5898007	0.8519637	2.312019	2.2369435	2.369435e-01	0.1184717589	0.3722667223	0.61013664	0.11672817	0.8368580	0.45757442	1.000	0.5	2.0
Table 4	30	2.6639846	1.663984567	1.663984567	37.3907272	6.1147958	6.6818800	1.0000000	26.192970	0.5182692	1.826921e-02	0.0365384162	0.0107622313	0.10374117	0.15365729	0.9982964	0.60233657	0.945	1.0	0.5
	50	1.9423119	0.942311883	0.942311883	9.3636501	3.0600082	3.8460473	1.0000000	15.076505	0.5116553	1.165527e-02	0.0233105469	0.0067954976	0.08243481	0.11460932	1.0000000	0.44926852	0.941	1.0	0.5
	100	1.6210496	0.621049636	0.621049636	3.7990900	1.9491254	2.2694461	1.0000000	8.896229	0.5109776	1.097757e-02	0.0219551307	0.0043163065	0.06569860	0.07890232	0.9986395	0.30929710	0.955	1.0	0.5
	200	1.3446295	0.344629523	0.344629523	1.9680193	1.4028611	1.4106507	1.0000000	5.529751	0.5074478	7.447756e-03	0.0148955111	0.0029218483	0.05405412	0.05497312	0.9777503	0.21549461	0.955	1.0	0.5
	500	1.1652110	0.165211014	0.165211014	0.8696931	0.9325734	0.7383865	0.9469122	2.894475	0.5047399	4.739933e-03	0.0094798656	0.0013922683	0.03731311	0.03257299	0.9609967	0.12768614	0.968	1.0	0.5
	1000	1.0695323	0.069532345	0.069532345	0.3569659	0.5974662	0.4833378	0.9446186	1.894684	0.5011721	1.172082e-03	0.0023441650	0.0006020481	0.02453667	0.02219685	0.9487983	0.08701165	0.963	1.0	0.5
Table 5	30	2.4978436	1.497843550	1.497843550	26.0907965	5.1079151	5.7661251	1.0000000	22.603210	1.0513424	5.134236e-02	0.0513423582	0.0518671197	0.22774354	0.30314698	1.0000000	1.18833616	0.968	1.0	1.0
	50	1.9925600	0.992560041	0.992560041	9.6803655	3.1113286	3.5739900	1.0000000	14.010041	1.0432117	4.321169e-02	0.0432116900	0.0361626594	0.19016482	0.23976887	0.9971791	0.93989395	0.978	1.0	1.0
	100	1.4434579	0.443457947	0.443457947	3.1741694	1.7816199	1.9255310	1.0000000	7.548081	1.0253233	2.532334e-02	0.0253233437	0.0239332678	0.15470381	0.15119463	0.9562982	0.59268293	0.980	1.0	1.0
	200	1.2430056	0.243005574	0.243005574	1.3283698	1.1525492	1.1891688	0.9841450	4.661542	1.0145920	1.459202e-02	0.0145920245	0.0139628312	0.11816442	0.10249973	0.9411099	0.40179894	0.990	1.0	1.0
	500	1.1031833	0.103183323	0.103183323	0.5665639	0.7527044	0.6819435	0.9432990	2.673219	1.0104934	1.049343e-02	0.0104934338	0.0081615391	0.09034124	0.06345182	0.9154639	0.24873114	0.996	1.0	1.0
	1000	1.0771497	0.077149750	0.077149750	0.2782847	0.5275270	0.4682123	0.8793970	1.835392	1.0150441	1.504414e-02	0.0150441365	0.0070222510	0.08379887	0.04474624	0.8944724	0.17540528	0.997	1.0	1.0
Table 6	30	3.8554531	2.855453060	2.855453060	40.8252731	6.3894658	7.1320963	0.9630178	27.957817	2.5179822	5.179822e-01	0.2589911205	1.8789940054	1.37076402	0.90264611	0.9969278	3.53837274	0.996	1.0	2.0
	50	2.8915185	1.891518488	1.891518488	16.1980805	4.0246839	5.1420417	0.9917241	20.156804	2.2926645	2.926645e-01	0.1463322591	0.6550943347	0.80937898	0.66196475	0.9944367	2.59490180	0.996	1.0	2.0
	100	2.5010832	1.501083175	1.501083175	10.8884010	3.2997577	2.9967558	0.9987937	11.747283	2.2138672	2.138672e-01	0.1069336231	0.3574253808	0.59785063	0.40216241	0.9927536	1.57647666	0.996	1.0	2.0
	200	1.8676632	0.867663192	0.867663192	6.1236297	2.4745969	1.6801858	0.9880435	6.586328	2.1153051	1.153051e-01	0.0576525647	0.1710148530	0.41353942	0.24701668	0.9282609	0.96830539	0.999	1.0	2.0
	500	1.4749896	0.474989566	0.474989566	3.6578487	1.9125503	0.8969539	0.9396728	3.516059	2.0546275	5.462749e-02	0.0273137447	0.0737166094	0.27150803	0.14075569	0.9151329	0.55176232	1.000	1.0.	

	200	2.2272283	0.227228341	0.113614171	2.3368261	1.5286681	1.6035757	0.9357143	6.286017	1.0008718	8.717928e-04	0.0008717928	0.0110474997	0.10510709	0.10024798	0.9438776	0.39297210	0.998	2.0	1.0
	500	2.1145340	0.114534024	0.057267012	1.1071447	1.0522094	0.9746556	0.9195171	3.820650	1.0023633	2.363258e-03	0.0023632584	0.0052130596	0.07220152	0.06295169	0.9416499	0.24677064	0.995	2.0	1.0
	1000	2.0297780	0.029777965	0.014888983	0.4967247	0.7047870	0.6747250	0.9348697	2.644922	0.9990028	-9.972171e-04	-0.0009972171	0.0024948460	0.04994843	0.04426141	0.9458918	0.17350472	0.998	2.0	1.0
Table 9	30	4.2263527	2.226352655	1.113176328	43.1903006	6.5719328	8.7711337	0.9903315	34.382844	2.2144480	2.144480e-01	0.1072239889	0.5193443141	0.72065548	0.80511736	0.9958159	3.15606006	0.996	2.0	2.0
	50	3.2820363	1.282036280	0.641018140	16.7346842	4.0908048	4.8799168	0.9974684	19.129274	2.1056201	1.056201e-01	0.0528100577	0.2704132448	0.52001274	0.50184548	0.9961929	1.96723428	0.996	2.0	2.0
	100	2.8122781	0.812278058	0.406139029	8.7941647	2.9654957	2.8614041	0.9608939	11.216704	2.0619366	6.193658e-02	0.0309682897	0.1407146684	0.37511954	0.31997294	0.9832402	1.25429392	1.000	2.0	2.0
	200	2.8842991	0.884299121	0.442149560	7.1457822	2.6731596	1.9527169	0.9287190	7.654650	2.1029975	1.029975e-01	0.0514987250	0.1300780453	0.36066334	0.22908071	0.8553719	0.89799639	1.000	2.0	2.0
	500	2.7985211	0.798521111	0.399260556	5.4635300	2.3374195	1.1688723	0.8575727	4.581980	2.1027454	1.027454e-01	0.0513726904	0.0998543964	0.31599746	0.14089098	0.8385155	0.55229263	1.000	2.0	2.0
Table 10	1000	2.6342966	0.634296623	0.317148312	3.8798076	1.9697227	0.7801913	0.8890000	3.058350	2.0839737	8.397367e-02	0.0419868354	0.0717904105	0.26793733	0.09612077	0.8710000	0.37679342	1.000	2.0	2.0
	30	5.2125830	2.212583039	0.737527680	89.5866257	9.4650212	9.2154307	0.9636835	36.124488	0.4981012	-1.898793e-03	-0.0037975861	0.0127306337	0.11283011	0.14015221	0.9896239	0.54939668	0.973	3.0	0.5
	50	4.1430919	1.143091856	0.381030619	34.0369376	5.8341184	5.5176959	0.8895558	21.629368	0.4965959	-3.404062e-03	-0.0068081231	0.0081892717	0.09049459	0.10436376	0.9891957	0.40910595	0.971	3.0	0.5
	100	3.5807958	0.580795764	0.193598588	12.4465502	3.5279669	3.2380690	0.8261339	12.693230	0.5000865	8.652688e-05	0.0001730538	0.0046196266	0.06796784	0.07207201	0.9665227	0.28252229	0.976	3.0	0.5
	200	3.2213968	0.221396810	0.073798937	4.9591955	2.2269251	2.0042282	0.8119219	7.856574	0.5027647	2.764686e-03	0.0055293717	0.0022924607	0.04787965	0.04921502	0.9475848	0.19292287	0.984	3.0	0.5
Table 11	500	2.9224386	-0.077561442	-0.025853814	1.7946695	1.3396527	1.1763106	0.8543788	4.611138	0.4986351	-1.364852e-03	-0.0027297035	0.0008652500	0.02941513	0.03061525	0.9307536	0.12001178	0.982	3.0	0.5
	1000	2.9909438	-0.009056206	-0.003018735	0.9106488	0.9542792	0.8563877	0.9330629	3.357040	0.5001158	1.157632e-04	0.0002315264	0.0004432563	0.02105365	0.02197529	0.9533469	0.08614315	0.986	3.0	0.5
	30	5.0672248	2.067224802	0.689074934	97.0610309	9.8519557	8.7070598	0.9908735	34.131674	1.0100624	1.006238e-02	0.0100623844	0.0576800949	0.24016681	0.28771751	0.9921773	1.12785263	0.979	3.0	1.0
	50	3.7245482	0.724548218	0.241516073	28.9609559	5.3815384	4.9075207	0.9470656	19.237481	0.9914242	-8.575847e-03	-0.0085758474	0.0352607776	0.18777853	0.20826108	0.9861910	0.81638343	0.986	3.0	1.0
	100	3.3615597	0.361559700	0.120519900	8.2660509	2.8750741	3.0340821	0.9245484	11.893602	0.9929886	-7.011425e-03	-0.0070114245	0.0185038608	0.13602890	0.14222437	0.9649309	0.55751951	0.989	3.0	1.0
Table 12	200	3.2051880	0.205188041	0.068396014	4.1499407	2.0371403	2.0354114	0.9119433	7.978813	0.9931774	-6.822624e-03	-0.0068226238	0.0103268560	0.10162114	0.09930819	0.9423077	0.38928811	0.998	3.0	1.0
	500	3.0525926	0.052592553	0.017530851	1.7154177	1.3097395	1.2421148	0.9207623	4.869090	0.9952736	-4.726351e-03	-0.0047263509	0.0044377584	0.06661650	0.06267484	0.9358074	0.24568539	0.997	3.0	1.0
	1000	3.0713583	0.071358348	0.023786116	0.7385924	0.8594140	0.8820496	0.9476334	3.457634	0.9994800	-5.199860e-04	-0.0005199860	0.0018529870	0.04304634	0.04443281	0.9587110	0.17417661	0.993	3.0	1.0
	30	5.5489502	2.548950245	0.849650082	98.6968866	9.9346307	9.0775484	0.9897959	35.583990	2.1358899	1.358899e-01	0.0679449492	0.4654107969	0.68221023	0.64920020	0.9935897	2.54486479	0.999	3.0	2.0
	50	4.1625183	1.162518314	0.387506105	25.7874553	5.0781350	5.2335251	0.9539171	20.515418	2.0568809	5.688087e-02	0.0284404370	0.1991790773	0.44629483	0.45502159	0.9861751	1.78368463	0.995	3.0	2.0
Table 12	100	3.4351388	0.435138828	0.145046276	9.4770685	3.0784848	2.9652108	0.9180851	11.623626	2.0369319	3.693186e-02	0.0184659282	0.1171781944	0.34231301	0.29571923	0.9712766	1.15921937	0.998	3.0	2.0
	200	3.3830108	0.383010762	0.127670254	4.7153930	2.1714956	2.0106143	0.9252525	7.881608	2.0555697	5.556968e-02	0.0277848394	0.0857250105	0.29278834	0.20819683	0.8636364	0.81613159	1.000	3.0	2.0
	500	3.1219150	0.121914982	0.040638327	1.8440774	1.3579681	1.1693560	0.9350000	4.583876	2.0563575	5.635753e-02	0.0281787645	0.0523473779	0.22879549	0.12857127	0.8110000	0.50399938	1.000	3.0	2.0
	1000	3.0441283	0.044128287	0.014709429	0.7617336	0.8727735	0.8435935	0.9340000	3.306886	2.0257784	2.577837e-02	0.0128891871	0.0220049520	0.14834066	0.08937691	0.8880000	0.35035748	1.000	3.0	2.0

Discussion of the Simulation Results

Tables 1 presents the Monte Carlo simulation results for the Maximum Likelihood Estimators (MLEs) of the parameters of the proposed SALRD distribution under different combinations of parameter values and sample sizes. The performance of the estimators was assessed using the Mean estimate, Bias, Relative Bias (RB), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Average Standard Error (ASE), Coverage Probability (CP), Average Length (AL), and Convergence Rate (CR).

The results indicate that the accuracy of the parameter estimates improves progressively as the sample size increases. In particular, the estimated parameter values become closer to their corresponding true values, while the Bias and Relative Bias decrease steadily. Similarly, both the MSE and RMSE exhibit a noticeable downward trend with increasing sample size, reflecting an improvement in estimation precision.

Furthermore, the Average Standard Error (ASE) decreases as the sample size grows, indicating greater stability and reduced variability of the estimators. The Coverage Probability (CP) remains close to the nominal confidence level of 95% for moderate and large sample sizes, suggesting that the constructed confidence intervals possess satisfactory coverage properties. In addition, the Average Length (AL) of the confidence intervals decreases with increasing sample size, demonstrating that the confidence intervals become more precise while maintaining adequate coverage.

Finally, the convergence rate of the optimization algorithm is consistently high across the simulation scenarios, confirming the numerical stability and reliability of the Maximum Likelihood Estimation procedure for estimating the parameters of the proposed SALRD distribution.

Conclusion

The Monte Carlo simulation study demonstrated the performance of the Maximum Likelihood Estimation (MLE) method for estimating the parameters of the proposed SALRD distribution under a variety of parameter combinations and sample sizes. The simulation results provided a comprehensive evaluation of the statistical properties of the estimators using several performance measures, including the Mean, Bias, Relative Bias (RB), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Average Standard Error (ASE), Coverage Probability (CP), Average Length (AL), and Convergence Rate (CR).

Overall, the findings indicate that the estimation procedure exhibits satisfactory performance, with noticeable improvements as the sample size increases. The estimators become more accurate and stable, while the estimation errors and variability decrease progressively. Moreover, the confidence intervals show appropriate coverage probabilities and reduced lengths for moderate and large sample sizes, confirming the reliability of the estimation procedure. The high convergence rate observed throughout the simulation study further demonstrates the numerical stability and effectiveness of the optimization algorithm. These results support the suitability of the Maximum Likelihood Estimation method for estimating the parameters of the proposed SALRD distribution and provide evidence of its practical applicability in statistical modeling.

References

1. Warahena-Liyanage G, Oluyede B, Moakofi T, Sengweni W. The new exponentiated half logistic-harris-g family of distributions with actuarial measures and applications. *Stats*. 2023;6(3):773-801. doi:10.3390/stats6030050.
2. Gupta RC, Gupta PI, Gupta RD. Modeling failure time data by Lehmann alternatives. *Communications in Statistics - Theory and Methods*. 1998;27:887-904.
3. Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*. 2002;31(4):497-512.
4. Zografos K, Balakrishnan N. On the families of beta-and gamma-generated generalized distribution and associated inference. *Statistical Methodology*. 2009;6(4):344-362.
5. Ristić MM, Balakrishnan N. The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*. 2012;82(8):1191-1206.
6. Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. *Metron*. 2013;71:63-79.
7. Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. *Journal of Data Science*. 2014;12:1253-1268.
8. Kumar D, Singh U, Singh SK. A New Distribution Using Sine Function-Its Application to Bladder Cancer Patients Data. *Journal of Statistics Applications & Probability*. 2015;4(3):417-427.
9. Hosseini B, Afshari M, Alizadeh M. The Generalized Odd Gamma-G Family of Distributions Properties and Applications. *Austrian Journal of Statistics*. 2018;47:69-89.
10. Mahran H, Mansour M, Abd Elrazik E, Affy A. A new one-parameter flexible family with variable failure rate shapes: Properties, inference, and real-life applications. *AIMS Mathematics*. 2024;9(5):11910-11940.