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Estimation of parameters μ_2 and σ_2 of Morgenstern type bivariate moment exponential distribution by ranked set sampling

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Abstract

In this article we have derived the best linear unbiased estimator (BLUE) of the parameters μ_2 and σ_2 involved in the five parameter Morgenstern type bivariate moment exponential distribution (MTBMED) by ranked set sampling (RSS). The efficiency comparison of the proposed estimators with that of the estimators using concomitants of order statistics is also made.

Keywords: Morgenstern family of distribution, Morgenstern type bivariate moment exponential distribution, best linear unbiased estimation, concomitants of order statistics, concomitants of ranked set sampling

Introduction

The concept of ranked set sampling was first introduced by McIntyre (1952)^[8] as a process of improving the precision of the sample mean as an estimator of the population mean. Ranked set sampling as described in McIntyre (1952)^[8] is applicable whenever ranking of a set of sampling units can be easily by judgment method, for details, see, Chen *et al.*, (2004)^[4]. The process of ranked set sampling involves selecting n sets of units, each of size n , and ordering the units of each of the set by judgment method or by using values obtained on the units by a relatively inexpensive method, without making actual measurement on the units. Then the unit ranked as one from the first set is actually measured, the unit ranked as two from the second set is measured. The process continues in this way until the unit ranked as n from the n^{th} set is measured. Then the observations resulting out of actual measurements made on the units chosen as described in the above Process is known as ranked set sample.

Ranking by a judgment method is not recommendable in case there is ambiguity in discriminating the rank of one unit with another (for example, ranking based on intensity of yellowish color of newborn babies, as mentioned in Chen *et al.*, (2004, p.7)^[4], become difficult, as we known that probability of ties in this situation is not zero as the characteristic under consideration is not continuous). In certain situation, one may prefer exact measurement of some easily measurable variable associated with the study variable rather than ranking the units by a crude judgment method. Suppose the variable of interest say, Y , is difficult or much expensive to measure, but an auxiliary variable X correlated with Y is readily measurable and can be ordered exactly. In this case as an alternative to McIntyre's (1952)^[8] method of ranked set sampling, Stokes (1977)^[13] used an auxiliary variable for the ranking of the sampling units. Stokes (1977)^[13] using auxiliary variate is as follows: Choose n^2 independent units, arrange them randomly into n sets each with n units and observe the value of the auxiliary variable X on each of these units. In the first set, that units for which the measurement on the auxiliary variable is the smallest is chosen. In the second set, that units for which the measurement on the auxiliary variable is the second smallest is chosen. The procedure is repeated until in the last set, that unit for which the measurement on the auxiliary variable is the largest is chosen. The measurements on the Y variate of the resulting new set of n units chosen by the above method gives a ranked set sample as suggested by Stokes (1977)^[13].

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If $X_{(r)r}$ is the observation measured on auxiliary variable X from the unit chosen from the r^{th} set then we write $Y_{[r]r}$ to denote the corresponding measurement made on the study variable Y on this unit so that $Y_{[r]r}, r = 1, 2, \dots, n$ form the ranked set sampling. Clearly $Y_{[r]r}$ is the concomitants of r^{th} order statistic arising from the r^{th} sample. By using the expression given by Yang (1977) ^[14] for the mean and variance of concomitants of order statistics, we have the following. For $1 \leq r \leq n, 0$

$$E[Y_{[r]r}] = E[E(Y/X_{(r)r})] \quad (1.1)$$

and

$$Var[Y_{[r]r}] = Var[E(Y/X_{(r)r})] + E[Var(Y/X_{(r)r})]. \quad (1.2)$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ for $r \neq s$ are drawn from two independent samples we have,

$$Cov[Y_{[r]r}, Y_{[s]s}] = 0, r \neq s. \quad (1.3)$$

Stokes (1977) ^[13] has suggested the ranked set sample mean as an estimator for the mean of the study variable Y, when an auxiliary variable X is used for ranking the sample units, under the assumption that (X, Y) follows a bivariate normal distribution. Barnett and Moore (1997) ^[2] have improved the estimator of Stokes (1977) ^[13] by deriving the Best linear unbiased estimator (BLUE) of the mean of the study variate Y, based on ranked set sample obtained on the study variate Y.

Morgenstern family of distributions.

The most important aspect to the considered in a modeling problem is regarding the flexibility of the chosen family, in the sense that it should contain a wide variety of models capable of representing any data situation. Another consideration is the sort of prior information available in the choice of the model. In problems involving several random variables, the analyst may make reasonable assumption about the marginal distributions. Then the question is to construct a joint distribution function with a set of given marginal. The Morgenstern family of distributions assumes importance in such contexts as a highly flexible system. Let X and Y be two random variables with cdf's given by $F_X(x)$ and $F_Y(y)$ respectively with pdf's $f_X(x)$ and $f_Y(y)$ and jointly distributed with cdf $F(x, y)$ given by (see, Kotz *et al.*, (2000), p.52).

$$F(x, y) = F_X(x)F_Y(y)\{1 + \alpha(1 - F_X(x))(1 - F_Y(y))\}, -1 \leq \alpha \leq 1. \quad (1.4)$$

Then the family of distributions having the above form of cdf is called the Morgenstern family of distributions (MFD). The system provides a very general expression of a bivariate distribution from which member can derived by substituting expression of any desired set of marginal distributions. The joint pdf corresponding to the cdf defined in (1.4) is given by,

$$f(x, y) = f_X(x)f_Y(y)\{1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))\}, -1 \leq \alpha \leq 1. \quad (1.5)$$

The parameter α is considered as the association parameter, the two random variables are independent when the value of α is zero and the maximum correlation attained between X and Y is $\frac{\alpha}{3}$. For suitable choice of $F_X(x)$ and $F_Y(y)$, the MFD is primarily useful as an alternative to the standard bivariate normal distribution (See, Conway (1983), p.28-31) ^[5]. The distribution theory of concomitance of order statistics from MFD are discussed in Scaria and Nair (1999).

Chako and Thomas (2008) ^[3] derived the BLUE of a parameter of a three parameter Morgenstern type bivariate exponential distribution using concomitance of order statistics. Sajeevkumar and Sumi (2022) derived the BLUE's of the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate exponential distribution (MTBED) using concomitants of ranked set sampling. Irshad *et al.* (2019) ^[6] estimate a parameter of a three parameter Morgenstern type bivariate Lindly distribution (MTBLD) using ranked set sampling. Sajeevkumar and Priya (2026) ^[9] derived the BLUE's of the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate Lindly distribution (MTBLD) using ranked set sampling. Arun *et al.* (2023) derived the BLUE of a parameter of a three parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using concomitance of order statistics. Sajeevkumar and Priya (2026) ^[9] estimate the BLUE's of the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using concomitants of order statistics. In this work our aim is to estimate the parameters μ_2 and σ_2 of a five parameter Morgenstern type bivariate moment exponential distribution (MTBMED) using concomitants of ranked set sampling.

Estimation of the parameters μ_2 and σ_2 involved in mtbmmed using rss.

An important member of Morgenstern family of distribution is (MFD) is Morgenstern type bivariate Moment Exponential distribution (MTBMED). The bivariate random variable (X, Y) is said to have five parameter MTBMED, if its pdf is given by

$$f(x, y) = \frac{(x-\mu_1)(y-\mu_2)}{\sigma_1^2 \sigma_2^2} e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)} e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)} *$$

$$\left[1 + \alpha \left\{2e^{-\frac{(x-\mu_1)}{\sigma_1}} \left(1 + \frac{x-\mu_1}{\sigma_1}\right) - 1\right\} \left[2e^{-\frac{(y-\mu_2)}{\sigma_2}} \left(1 + \frac{y-\mu_2}{\sigma_2}\right) - 1\right]\right],$$

$$x > \mu_1, y > \mu_2, \sigma_1 > 0, \sigma_2 > 0, -1 < \alpha < 1$$

(2.1)

The pdf defined in (2.1) is an extension of the three parameter MTBMED defined in Arun, et.al (2023). The joint pdf of standard MTBMED is obtained by making the substitution $U = \frac{x-\mu_1}{\sigma_1}$ and $V = \frac{y-\mu_2}{\sigma_2}$ in (2.1) and it is given by,

$$g(u, v) = uve^{-u}e^{-v}[1 + \alpha(2e^{-u}\{1+u\} - 1)(2e^{-v}\{1+v\} - 1)], u > 0, v > 0, -1 < \alpha < 1. \quad (2.2)$$

Suppose X and Y are the two characteristics of interest on the units of a population and (X, Y) has a distribution with pdf given in (2.1). Suppose there are n independent random samples each of size n drawn from (2.1) and in each sample, units are ranked according to the values of the observations measured on the easily measurable variable X on the units. Though the variable Y is of primary interest, we assume that it is difficult to measure. Then to deal with the inference problem on the distribution of the variable Y we obtain a ranked set sample by measuring the value of Y on the r^{th} ordered unit of the r^{th} sample, $r=1, 2, \dots, n$.

We write $Y_{[r]r}$ to denote this observation measured on Y of the r^{th} ordered unit of the r^{th} set, $r=1, 2, \dots, n$. Then clearly $Y_{[r]r}$ is distributed as the concomitants of r^{th} order statistic in each sample, units are ranked according to the value of the observations measured on the easily measurable variable U on the unit. We obtain a ranked set sample by measuring the value of V on the r^{th} ordered unit of the r^{th} sample, We write $V_{[r]r}$ to denote this observation measured on V of the r^{th} ordered unit of the r^{th} set, $r=1, 2, \dots, n$. Then clearly $V_{[r]r}$ is distributed as the concomitants of r^{th} order statistic of a random sample of size n arising from (2.2). Then the pdf $g_{[r]r}(v)$ of $V_{[r]r}$, $r=1, 2, \dots, n$ is given by

$$g_{[r]r}(v) = ve^{-v}[1 + \alpha \frac{(n-2r+1)}{(n+1)}(2e^{-v}\{1+v\} - 1)], v > 0 \quad (2.3)$$

Using (2.3), for $1 \leq r \leq n$,

$$E[V_{[r]r}] = 2 - \frac{3}{4} \alpha \frac{(n-2r+1)}{(n+1)} = \xi_r \quad (2.4)$$

And

$$Var[V_{[r]r}] = 2 - \frac{3}{4} \alpha \frac{(n-2r+1)}{(n+1)} - \frac{9}{16} \alpha^2 \frac{(n-2r+1)^2}{(n+1)^2} = e_{r,r}. \quad (2.5)$$

And for $1 \leq r < s \leq n$,

$$Cov[V_{[r]r}, V_{[s]s}] = 0 = e_{r,s}, r \neq s, r, s=1, 2, \dots, n. \quad (2.6)$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ are drawn for $r \neq s$ are drawn from two independent samples.

Let (X_i, Y_i) , $i=1, 2, \dots, n$ be a ranked set sample of size n drawn from a population with pdf defined by (2.1). Clearly we have $X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i=1, 2, \dots, n$. Then using (2.4) to (2.6), we have for $1 \leq r \leq n$,

$$E(Y_{[r]r}) = \mu_2 + \sigma_2 \xi_r \quad (2.7)$$

$$Var[Y_{[r]r}] = \sigma_2^2 e_{r,r} \quad (2.8)$$

And for $1 \leq r < s \leq n$,

$$Cov[Y_{[r]r}, Y_{[s]s}] = 0, r \neq s \quad (2.9)$$

Where ξ_r , $e_{r,r}$ and $e_{r,s}$ are defined by (2.4), (2.5) and (2.6) respectively. Clearly from (2.4) to (2.6) it follows that ξ_r , $e_{r,r}$ and $e_{r,s}$ are known constants provided α is known. If we write

$\underline{Y}_n = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$, $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$, then we can write

$$E[\underline{Y}_n] = \underline{1}\mu_2 + \underline{\xi}\sigma_2, \quad (2.10)$$

Where $\underline{1}$ is a column vector of n ones and $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$. And

$$Var[\underline{Y}_n] = E\sigma_2^2, \quad (2.11)$$

Now we derive the BLUE's of μ_2 and σ_2 involved in (2.1) and it is given in the following theorem.

Theorem 2.1:

Let $\underline{Y}_n = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be the vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of concomitants of ranked set sample arising (2.2). Then the BLUE's of the parameters μ_2^* of μ_2 and σ_2^* of σ_2 is given by

$$\mu_2^* = -\frac{\underline{\xi}'E^{-1}(\underline{1}\underline{\xi}'-\underline{\xi}\underline{1}')E^{-1}}{(\underline{\xi}'E^{-1}\underline{\xi})(\underline{1}'E^{-1}\underline{1})-(\underline{\xi}'E^{-1}\underline{1})^2}\underline{Y}_n \quad (2.12)$$

$$\sigma_2^* = \frac{\underline{1}'E^{-1}(\underline{1}\underline{\xi}'-\underline{\xi}\underline{1}')E^{-1}}{(\underline{\xi}'E^{-1}\underline{\xi})(\underline{1}'E^{-1}\underline{1})-(\underline{\xi}'E^{-1}\underline{1})^2}\underline{Y}_n \quad (2.13)$$

And corresponding variances are given by

$$Var(\mu_2^*) = \frac{(\underline{\xi}'E^{-1}\underline{\xi})\sigma_2^2}{(\underline{\xi}'E^{-1}\underline{\xi})(\underline{1}'E^{-1}\underline{1})-(\underline{\xi}'E^{-1}\underline{1})^2} \quad (2.14)$$

And

$$Var(\sigma_2^*) = \frac{(\underline{1}'E^{-1}\underline{1})\sigma_2^2}{(\underline{\xi}'E^{-1}\underline{\xi})(\underline{1}'E^{-1}\underline{1})-(\underline{\xi}'E^{-1}\underline{1})^2}. \quad (2.15)$$

Proof: Let $\underline{Y}_n = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})'$ be a vector of concomitants of ranked set sample arising from (2.1). Also let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $E = ((e_{r,s}))$ respectively be the mean vector and dispersion matrix of vector of concomitants of ranked set sample arising from (2.2), then by using (2.10) and (2.11) together defines a generalized Gauss-Markov set up. Then by Generalized Gauss-Markov theorem we get the BLUE's of μ_2 and σ_2 and their corresponding variances.

Estimation of the parameters μ_2 and σ_2 involved in mtbmed using concomitants of order statistics

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be the observations of random sample of size n drawn from (2.1). The Y component associated with r^{th} order statistic $X_{r:n}$ is defined as the concomitants of r^{th} order statistic and it is denoted by $Y_{[r:n]}$, $r = 1, 2, \dots, n$. Also let (U_i, V_i) , $i = 1, 2, \dots, n$ be the sequence of independent observations drawn from (2.2). Let $V_{[r:n]}$ be the concomitants of r^{th} order statistic $U_{r:n}$ arising from (2.2), then the pdf of $V_{[r:n]}$ and the joint pdf of $V_{[r:n]}$ and $V_{[s:n]}$ for $1 \leq r < s \leq n$ are respectively given in (3.1) and (3.2).

$$g_{[r:n]}^*(v) = ve^{-v}[1 + \alpha \frac{(n-2r+1)}{(n+1)}(2e^{-v}\{1+v\} - 1)], v > 0 \quad (3.1)$$

and for $1 \leq r < s \leq n$

$$f_{[r,s:n]}(v_1, v_2) = v_1 v_2 e^{-v_1} e^{-v_2} [1 + \alpha G_r (2e^{-v_1}\{1+v\} - 1) + \alpha G_s (2e^{-v_2}\{1+v_2\} - 1) + \alpha^2 (G_s - G_{rs}) (2e^{-v_1}\{1+v_1\} - 1)(2e^{-v_2}\{1+v_2\} - 1)], v_1 > 0, v_2 > 0. \quad (3.2)$$

$$\text{where } G_r = \frac{n-2r+1}{n+1}, G_s = \frac{n-2s+1}{n+1} \text{ and } G_{rs} = \frac{2r(n-2s)}{(n+1)(n+2)}.$$

Using (3.1) and (3.2), we have

$$E[V_{[r:n]}] = 2 - \frac{3}{4} \alpha \frac{(n-2r+1)}{(n+1)} = \xi_r, \quad (3.3)$$

$$Var[V_{[r:n]}] = 2 - \frac{3}{4} \alpha \frac{(n-2r+1)}{(n+1)} - \frac{9}{16} \alpha^2 \frac{(n-2r+1)^2}{(n+1)^2} = b_{r,r} \quad (3.4)$$

And for $1 \leq r < s \leq n$

$$Cov[V_{[r:n]}, V_{[s:n]}] = \frac{9}{16} \alpha^2 2r \left[\frac{(n-2s+1)}{(n+1)^2} - \frac{(n-2s)}{(n+1)(n+2)} \right] = b_{r,s}. \quad (3.5)$$

Let (X_i, Y_i) , $i = 1, 2, \dots, n$ be sample of size n from a population with pdf defined in (2.1). Clearly $X_i = \mu_1 + \sigma_1 U_i$ and $Y_i = \mu_2 + \sigma_2 V_i$, for $i = 1, 2, \dots, n$.

Then for $1 \leq r \leq n$,

$$E[Y_{[r:n]}] = \mu_2 + \sigma_2 \xi_r \quad (3.6)$$

$$Var[Y_{[r:n]}] = \sigma_2^2 b_{r,r} \quad (3.7)$$

And for $1 \leq r < s \leq n$,

$$\text{Cov}[Y_{[r:n]}, Y_{[s:n]}] = \sigma_2^2 b_{r,s}, \quad (3.8)$$

Where ξ_r , $b_{r,r}$ and $b_{r,s}$ are respectively defined in (3.3), (3.4) and (3.5).

Let $Y_{[n]} = (Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]})'$ be the vector of concomitants of order statistics arising from (2.1). Let $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $B = ((b_{r,s}))$ respectively the mean vector and dispersion matrix of $Y_{[n]}$. Then we can write $E[Y_{[n]}] = \underline{1}\mu_2 + \underline{\xi}\sigma_2$, where $\underline{1}$ is a column vector of n ones and $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ and $V[Y_{[n]}] = B\sigma_2^2$.

Now using generalized Gauss markov theorem BLUE's of μ_2 and σ_2 involved in (2.1) are given by

$$\hat{\mu}_2 = -\frac{\underline{\xi}'B^{-1}(\underline{1}\underline{\xi}'-\underline{\xi}\underline{1})B^{-1}}{(\underline{\xi}'B^{-1}\underline{\xi})(\underline{1}'B^{-1}\underline{1})-(\underline{\xi}'B^{-1}\underline{1})^2}Y_{[n]} \quad (3.9)$$

$$\hat{\sigma}_2 = \frac{\underline{1}'B^{-1}(\underline{1}\underline{\xi}'-\underline{\xi}\underline{1})B^{-1}}{(\underline{\xi}'B^{-1}\underline{\xi})(\underline{1}'B^{-1}\underline{1})-(\underline{\xi}'B^{-1}\underline{1})^2}Y_{[n]} \quad (3.10)$$

And

$$\text{Var}(\hat{\mu}_2) = \frac{(\underline{\xi}'B^{-1}\underline{\xi})\sigma_2^2}{(\underline{\xi}'B^{-1}\underline{\xi})(\underline{1}'B^{-1}\underline{1})-(\underline{\xi}'B^{-1}\underline{1})^2} \quad (3.11)$$

$$\text{Var}(\hat{\sigma}_2) = \frac{(\underline{1}'B^{-1}\underline{1})\sigma_2^2}{(\underline{\xi}'B^{-1}\underline{\xi})(\underline{1}'B^{-1}\underline{1})-(\underline{\xi}'B^{-1}\underline{1})^2}. \quad (3.12)$$

Computation Section

We have evaluated the coefficients of the BLUE μ_2^* of μ_2 given in (2.12) and evaluated the coefficients of the BLUE σ_2^* of σ_2 given in (2.13) for different values of n and α and are given in Table 1. Also we have evaluated $\text{Var}(\mu_2^*)$ given in (2.14), $\text{Var}(\sigma_2^*)$ given in (2.15), $\text{Var}(\hat{\mu}_2)$ given in (3.11), $\text{Var}(\hat{\sigma}_2)$ given in (3.12) and evaluated the relative efficiency RE_1 of μ_2^* relative to $\hat{\mu}_2$, the relative efficiency RE_2 of σ_2^* relative to $\hat{\sigma}_2$ for different values of n and α and are given Table 2.

Table 1: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.25$.

Estimator	n	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
μ_2^*	2	16.50000	-15.5000								
	3	11.1674	-0.00147	-10.16593							
	4	8.45857	2.69454	-2.59812	-7.55498						
	5	6.81385	3.27548	-0.00133	-3.07918	-6.00882					
	6	5.70718	3.30744	1.06864	-1.04278	-3.05347	-4.98701				
	7	4.91123	3.17407	1.54286	-0.00101	-1.47441	-2.89057	-4.26216			
	8	4.31076	2.99390	1.75070	0.56948	-0.55974	-1.64637	-2.69786	-3.72086		
	9	3.84154	2.80816	1.82834	0.89425	-0.00093	-0.86314	-1.69761	-2.50903	-3.30158	
	10	3.46453	2.63183	1.83918	1.08123	0.35329	-0.34901	-1.02892	-1.68996	-2.33510	-2.96707

Table 1: Coefficients of the BLUE's μ_2^* using RSS for n=2(1)10, $\alpha = 0.50$.

Estimator	n	Coefficients									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
μ_2^*	2	8.50000	-7.50000								
	3	5.83628	-0.00589	-4.83039							
	4	4.47100	1.33487	-1.24940	-3.55646						
	5	3.63355	1.64592	-0.00542	-1.46112	-2.81293					
	6	3.06479	1.68142	0.51956	-0.50206	-1.43753	-2.32618				
	7	2.65211	1.62941	0.76075	-0.00451	-0.70099	-1.35390	-1.98287			
	8	2.33859	1.54959	0.87243	0.27357	-0.27049	-0.77686	-1.25898	-1.72784		
	9	2.09200	1.46383	0.91943	0.43573	-0.00380	-0.41149	-0.79692	-1.16787	-1.53091	
	10	1.89288	1.38031	0.93218	0.53210	0.16799	-0.16926	-0.48678	-0.79033	-1.08479	-1.37428

Table 1: Coefficients of the BLUE's σ_2^* using RSS for $n=2(1)10$, $\alpha = 0.25$.

Estimator	n	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
σ_2^*	2	-8	8								
	3	-5.41679	0.16691	5.24988							
	4	-4.10393	-1.22263	1.42372	3.90284						
	5	-3.30653	-1.53795	0.10028	1.63941	3.10479					
	6	-2.76984	-1.57048	-0.45132	0.60441	1.61000	2.57723				
	7	-2.38378	-1.51562	-0.70025	0.07162	0.80841	1.51672	2.20289			
	8	-2.09248	-1.43440	-0.81303	-0.22252	0.34210	0.88553	1.41149	1.92330		
	9	-1.86483	-1.34844	-0.85873	-0.39180	0.05576	0.48691	0.90427	1.31017	1.70670	
	10	-1.68190	-1.26580	-0.86966	-0.49080	-0.12688	0.22428	0.56429	0.89493	1.21767	1.53387

Table 1: Coefficients of the BLUE's σ_2^* using RSS for $n=2(1)10$, $\alpha = 0.50$.

Estimator	n	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
σ_2^*	2	-4	4								
	3	-2.75049	0.16765	2.58284							
	4	-2.10904	-0.54396	0.74837	1.90463						
	5	-1.71511	-0.72389	0.10113	0.82987	1.50801					
	6	-1.44733	-0.75784	-0.17781	0.33313	0.80186	1.24799				
	7	-1.25290	-0.74340	-0.31001	0.07241	0.42105	0.74845	1.06440			
	8	-1.10511	-0.71218	-0.37449	-0.07543	0.19668	0.45035	0.69227	0.92790		
	9	-0.98881	-0.67606	-0.40470	-0.16328	0.05640	0.26046	0.45367	0.63990	0.82243	
	10	-0.89486	-0.63974	-0.41644	-0.21685	-0.03497	0.13372	0.29275	0.44500	0.59289	0.73849

Table 2: $V_1 = \frac{\text{Var}(\mu_2^*)}{\sigma_2^2}$, $V_2 = \frac{\text{Var}(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{\text{Var}(\sigma_2^*)}{\sigma_2^2}$, $V_4 = \frac{\text{Var}(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$, for $n=2(1)10$ and $\alpha = 0.25$

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	1020.99612	1018.99616	255.49952	254.99904	0.998041168	0.99804117
3	452.1065	451.70687	113.194269	113.093972	0.999116071	0.999113939
4	281.97693	281.88351	70.620238	70.596586	0.999668696	0.999665082
5	202.75268	202.75281	50.789112	50.788913	1.000000641	0.999996082
6	157.54304	157.57736	39.469955	39.478349	1.000217845	1.000212668
7	128.52497	128.57291	32.20337	32.2152	1.000373001	1.000367353
8	108.38734	108.4404	27.160062	27.173198	1.000489541	1.000483651
9	93.62836	93.69174	23.463312	23.479301	1.000676932	1.000681447
10	82.36262	82.41542	20.64127	20.654371	1.000641068	1.000634699

Table 2: $V_1 = \frac{\text{Var}(\mu_2^*)}{\sigma_2^2}$, $V_2 = \frac{\text{Var}(\hat{\mu}_2)}{\sigma_2^2}$, $V_3 = \frac{\text{Var}(\sigma_2^*)}{\sigma_2^2}$, $V_4 = \frac{\text{Var}(\hat{\sigma}_2)}{\sigma_2^2}$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_3}$ for $n=2(1)10$ and $\alpha = 0.50$

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	252.9928283	251.00001	63.50016	63	0.992123025	0.992123484
3	110.75988	110.3633	27.860567	27.759887	0.996419462	0.99638629
4	68.62287	68.53282	17.284785	17.261112	0.998687755	0.998630414
5	49.12876	49.13169	12.386028	12.385861	1.000059639	0.999986517
6	38.0597	38.09621	9.601789	9.610202	1.000959282	1.000876191
7	30.97807	31.44568	7.819184	8.150365	1.015094872	1.042354931
8	26.07875	26.13246	6.585161	6.598101	1.002059531	1.001965024
9	22.49718	22.55158	5.68291	5.695773	1.002418081	1.002263453
10	19.76881	19.82221	4.994742	5.00773	1.002701225	1.002600335

Conclusion

From Table 2 given in section 4, it may be noted that for sample sizes (n) greater than five, all the cases calculated the performance of the estimators μ_2^* and σ_2^* calculated using Ranked set sampling is much better than the estimators calculated using concomitants of order statistics.

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