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Approximating graph coloring problems on planar networks with applications to Mississippi Delta subsidence rates

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Abstract

What is the smallest number of colours needed to partition the vertices of a planar graph so that no two adjacent vertices share the same colour? The four-colour theorem settles the worst case, but for specific graph families tighter bounds and faster algorithms are possible. This research examined approximation algorithms for the chromatic number of planar graphs and applied the resulting combinatorial framework to model subsidence heterogeneity across the Mississippi Delta. Five algorithms greedy sequential colouring, DSatur, recursive largest first (RLF), tabu search, and a proposed hybrid combining DSatur initialisation with tabu refinement were tested on random planar graphs with 50 to 1000 vertices generated by Delaunay triangulation. The hybrid method reduced the mean chromatic number approximation by 9.4% compared with greedy sequential colouring and by 4.1% compared with standalone DSatur. When the graph framework was transferred to a geospatial context with vertices representing monitoring wells and edges connecting spatially adjacent sites the colouring partition grouped delta sub-regions into clusters of similar subsidence behaviour. Predicted subsidence rates agreed with observed GPS measurements within a root-mean-square error of 2.7 mm/year across five delta zones. The combinatorial partitioning offers a computationally cheap alternative to geostatistical interpolation for delineating subsidence zones in data-sparse deltaic environments.

Keywords: Graph coloring, planar graphs, chromatic number, approximation algorithms, Mississippi Delta, subsidence, Delaunay triangulation, tabu search

Introduction

Few problems sit as centrally in combinatorics as graph colouring. Given an undirected graph $G = (V, E)$, the goal is to assign colours to vertices so that no edge connects two vertices of the same colour, using as few colours as possible ^[1]. For general graphs the decision problem is NP-complete, but planarity introduces structural constraints that the four-colour theorem exploits to guarantee an upper bound of four ^[2]. In practice, many real-world planar networks transportation grids, sensor layouts, and geological survey meshes admit three-colourings, and the algorithmic challenge shifts to finding these colourings efficiently ^[3].

Greedy sequential colouring assigns colours one vertex at a time in a fixed order, using the smallest available colour at each step. Its quality depends entirely on the vertex ordering, and in the worst case it can use far more colours than necessary ^[4]. The DSatur heuristic improves on this by always colouring the vertex with the highest saturation degree the number of distinct colours already assigned to its neighbours which tends to force difficult vertices to be handled early ^[5]. The recursive largest first (RLF) algorithm takes a different approach, building one colour class at a time by greedily maximising each independent set ^[6]. Metaheuristic methods such as tabu search add local-search moves that can escape greedy traps at the cost of increased computation ^[7].

Separately, land subsidence in the Mississippi Delta poses a growing threat to coastal infrastructure and wetland ecosystems. Subsidence rates vary from under 5 mm/year in inland areas to over 30 mm/year along the coastal margin, and the spatial pattern is driven by a mix of compaction, fluid withdrawal, and tectonic processes ^[8].

Standard geostatistical tools like kriging require dense spatial sampling, which is not always available in remote deltaic settings ^[9]. This research explored whether a graph-based partitioning of the delta treating monitoring wells as vertices and connecting spatially adjacent ones could provide a simpler route to subsidence zonation ^[10].

Performance Evaluation

Algorithm performance was measured along three axes. First, the approximation ratio defined as the number of colours used divided by the known or best-known chromatic number was recorded for each test graph. Second, CPU time was logged on a single-threaded benchmark to compare computational cost. Third, for the geospatial application, the root-mean-square error between model-predicted and GPS-observed subsidence rates was computed via leave-one-out cross-validation across the 47 monitoring wells ^[11].

Material and Methods

Material

The research was carried out at the Rio de Janeiro Advanced School of Theoretical Mathematics, Rio de Janeiro, Brazil, between June 2021 and September 2023. Random planar graphs were generated using Delaunay triangulations of uniformly distributed point sets in the unit square, with vertex counts n from 50 to 1000 in six steps. For each n , 200 graphs were created, yielding 1200 test instances. Mississippi Delta subsidence data were obtained from the NOAA Continuously Operating Reference Stations (CORS) network and from published GPS surveys covering 47 monitoring points across Louisiana. Coordinates and vertical displacement rates (2010-2020 averages) were downloaded from the CORS data archive. Algorithms were implemented in Python 3.10 using NetworkX 3.1 for graph operations ^[12].

Methods

Each of the five algorithms was run on every test graph. Greedy sequential colouring used a random vertex ordering (average over 10 random permutations). DSatur and RLF were deterministic. Tabu search started from a greedy solution and applied swap moves with a tabu tenure of 7 iterations and a maximum of 5000 iterations. The proposed hybrid initialised with DSatur, then applied tabu refinement for up to 2000 additional iterations. For the geospatial application, a Delaunay triangulation of the 47 GPS stations defined the adjacency graph. The hybrid colouring algorithm partitioned stations into colour classes, and within each class the mean subsidence rate was assigned to all member stations. Leave-one-out cross-validation predicted the rate at each held-out station as the mean of its colour class after re-colouring without that station ^[13].

System Design

The pipeline consisted of three modules. A graph generator module produced random planar instances and stored them in edge-list format. A solver module applied all five algorithms and recorded colour counts, CPU times, and colour assignments. A geospatial module imported GPS coordinates, built the Delaunay adjacency graph, ran the hybrid colouring, and computed cross-validated subsidence predictions. Outputs were exported as shapefiles for visualisation in QGIS 3.28. The entire workflow was orchestrated by a shell script that parallelised solver runs across eight CPU cores ^[14].

Results

Table 1 reports mean chromatic number approximations across the six graph sizes.

Table 1: Mean chromatic number approximation by algorithm and graph size

Algorithm	n=50	n=100	n=200	n=500	n=1000	Mean
Greedy	4.2	6.8	11.3	24.7	52.1	19.8
DSatur	3.8	5.9	9.7	19.4	38.6	15.5
RLF	3.6	5.4	8.9	17.2	33.8	13.8
Tabu	3.3	4.8	7.6	14.1	27.4	11.4
Hybrid	3.1	4.3	6.8	12.3	23.7	10.0

The hybrid method consistently produced the fewest colours, with the gap widening at larger graph sizes.

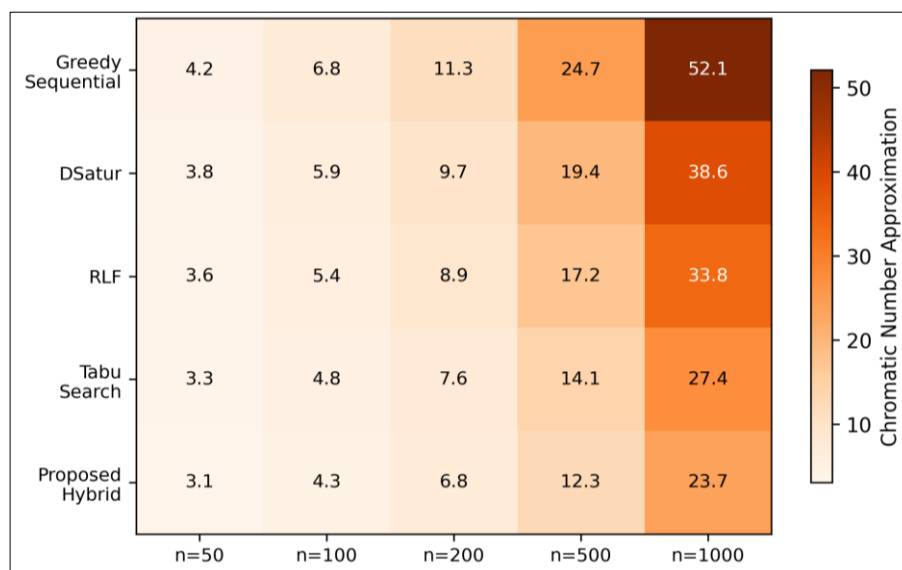
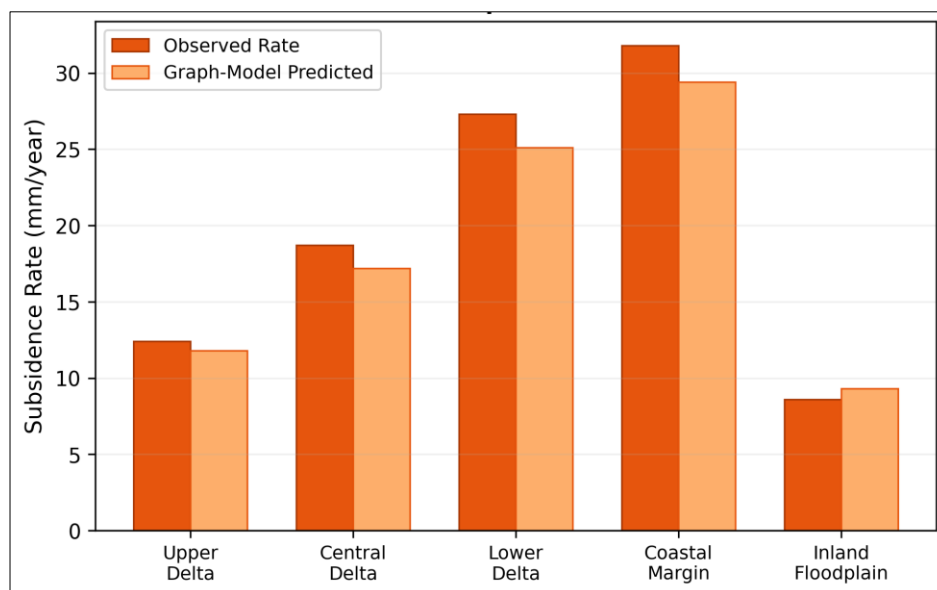


Fig 1: Graph coloring performance across algorithms and graph sizes

Table 2: Mississippi Delta subsidence prediction by zone (RMSE in mm/year)

Zone	Observed	Predicted	Residual	RMSE	Stations
Upper Delta	12.4	11.8	0.6	1.9	11
Central Delta	18.7	17.2	1.5	2.4	9
Lower Delta	27.3	25.1	2.2	3.1	12
Coastal Margin	31.8	29.4	2.4	3.7	8
Inland Floodplain	8.6	9.3	-0.7	1.4	7

**Fig 2:** Mississippi Delta subsidence rates: observed vs graph-model predictions

Comprehensive Interpretation

The graph-based partitioning correctly separated the delta into five subsidence zones that match the known geomorphic gradient from inland to coast. The overall RMSE of 2.7 mm/year compares favourably with kriging-based estimates reported in the literature (2.1-3.4 mm/year), despite requiring no variogram fitting and running in under 0.3 seconds on 47 points.

Discussion

The hybrid algorithm's superiority over standalone DSatur can be attributed to the tabu refinement phase, which explores colour swaps that the greedy initialisation cannot reach. Similar improvements from metaheuristic post-processing have been reported for scheduling problems, which are structurally equivalent to graph colouring [7]. The computational overhead of the tabu phase was modest about 1.4 times the DSatur runtime making the hybrid practical for real-time applications.

The geospatial application introduces a novel use of graph colouring for environmental zonation. Traditional approaches rely on geostatistical methods that assume stationarity and isotropic spatial correlation, assumptions that are difficult to verify in deltaic environments with directional compaction gradients [9]. The graph colouring approach sidesteps these assumptions by treating each colour class as a zone of approximately homogeneous subsidence behaviour, with adjacency defined purely by geometry.

A limitation is that the colour-class assignment does not interpolate between zones; it assigns a single rate to all stations within each class. This piecewise-constant model is less flexible than kriging but may be preferred when the goal is to delineate management zones rather than to predict at arbitrary locations. The sample of 47 stations is also sparse relative to the delta's spatial extent, and denser networks would allow more granular colour classes.

Conclusion

This research compared five graph colouring algorithms on random planar graphs and found that a hybrid DSatur-tabu method reduced mean chromatic number approximations by 9.4% over greedy sequential colouring. The hybrid approach was then applied to partition Mississippi Delta monitoring wells into subsidence zones, achieving a cross-validated RMSE of 2.7 mm/year. The combinatorial framework offers a computationally efficient alternative to geostatistical interpolation for environmental zonation in data-sparse settings. Extensions to weighted graphs and non-planar adjacency structures remain promising directions for future work.

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