

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

NAAS Rating (2025): 4.49

Maths 2026; 11(7): 24-35

© 2026 Stats & Maths

<https://www.mathsjournal.com>

Received: 07-04-2026

Accepted: 13-05-2026

Aman Gurudwan

Department of Mathematics,
Bharti Vishwavidyalaya, Durg,
Chhattisgarh, India

Dr. Rohit Kumar Verma

Department of Mathematics,
Bharti Vishwavidyalaya, Durg,
Chhattisgarh, India

Pooja Verma

Department of Mathematics,
Bharti Vishwavidyalaya, Durg,
Chhattisgarh, India

Jyoti Vaishnav

Department of Mathematics,
Bharti Vishwavidyalaya, Durg,
Chhattisgarh, India

Corresponding Author:

Aman Gurudwan

Department of Mathematics,
Bharti Vishwavidyalaya, Durg,
Chhattisgarh, India

Applications of fixed-point theorems with f-contraction to optimization and game theory: A theoretical and computational study

Aman Gurudwan, Rohit Kumar Verma, Pooja Verma and Jyoti Vaishnav

DOI: <https://www.doi.org/10.22271/math.2026.v11.i7a.2606>

Abstract

The field of fixed point theory is one of the most important areas in nonlinear analysis, with far-reaching consequences in optimization, game theory, economics and differential equations. This research investigates the use of Wardowski's F-contraction in practical settings for optimization and game theory, providing a unified approach. We explore the existence and uniqueness of fixed points for F-contractive mappings in metric spaces, Banach spaces, Hilbert spaces and other types of metric spaces such as b-metric spaces, cone metric spaces, complex valued metric spaces and so on. We establish new fixed point theorems that are extensions and generalizations of classical results, complete with detailed proofs and error bounds. Numerical examples show that F-contractions provide better convergence behavior than the classical Banach contractions, especially for non-smooth and non-convex problems. Convex optimization problems are addressed using proximal point operators while existence and uniqueness of Nash equilibria in games are proved through best-response F-contractions. Comparative studies show that the F-contraction framework offers greater flexibility, generality and control of convergence. The research adds to the theoretical framework of fixed point theory and provides methodological insights to operations research, mathematical economics and computational mathematics researchers.

Keywords: F-contraction, fixed point theorem, optimization, game theory, Nash equilibrium, metric spaces, Banach contraction principle, Wardowski's theorem, proximal point algorithm, best response mapping

1. Introduction

1.1 Background and Historical Development

One of the most basic and widely-researched fields of mathematical analysis, fixed point theory has far-reaching applications in nonlinear analysis, optimization theory, economics, computer science, and many other fields. Fixed point theory can be said to have its roots in the work of Henri Poincaré in the 1880s, who showed that solutions to some significant analytic problems could be effectively analyzed by defining a set and a function in a manner such that the solutions to the problem were the fixed points of the defined function (Browder, 1980). The radical discovery made by Poincaré was the introduction of the fixed points x belonging to a set X , which fulfils $f(x) = x$, to be critical in understanding the behaviour of dynamical systems and in finding solutions to equations of complex systems.

Topological beginnings of fixed point theory were further cemented by the famous Brouwer fixed point theorem which became a landmark discovery in algebraic topology. The Lefschetz number of a map was introduced by Lefschetz (1926), who showed that when this number is non-zero, it followed that the map must have a fixed point. This was especially important since the Lefschetz number, as a homology homomorphism induced by the map, is a homotopy invariant. A nonzero Lefschetz number therefore means that every map in the given homotopy has a fixed point hence the creation of a strong relationship between fixed-point existence and algebraic topology (Granas & Dugundji, 2003) ^[18, 31].

The development of the fixed point theory had a revolutionary moment when the metric space formulations were introduced that offered more practical and concrete frameworks to establish the fixed point results. Kakutani (1941) generalized the original Brouwer fixed point theorem to set-valued mappings, which led to new research opportunities in mathematical economics and game theory. Contributions of Nielsen in the 1920s attempted to go beyond existence theorems, by establishing the minimal number of fixed points of all the maps in a homotopy class, and gave rise to what came to be known as Nielsen theory and the Nielsen number (Jiang, 1983). These initial developments form the foundation of the rich and diverse field of fixed point theory that is still in development today ^[39, 36].

1.2 The Banach Contraction Principle and Its Importance

The Banach Contraction Principle, introduced by Stefan Banach in his 1920 doctoral dissertation (and later published in 1922) is perhaps the most important and influential finding in the theory of metric fixed points (Banach, 1922) ^[10]. This is a celebrated theorem which says that any contraction mapping on a complete metric space has a unique fixed point and that this fixed point can be obtained as the limit of the Picard iteration sequence, given any arbitrary point in the space. The beauty and the intuition of the Banach outcome along with its amazingly wide range of applicability to various fields in the field of mathematics and sciences has made it an inseparable part of modern analysis (Jachymski *et al.* 2024).

In 2022, the centenary of the Banach Contraction Principle gave an opportunity to reflect extensively on its legacy and the wealth of research that it has spawned. According to Jachymski *et al.* (2024) ^[35], the principle was subject to many changes, generalizations, and extensions over the last century, and researchers have come up with a variety of contractive conditions, converse theorems, and applications to different mathematical structures. The fundamental significance of the theorem is its constructive essence, which offers not just an existence proof but a direct iterative technique to approximate the fixed point with well-defined error bounds (Berinde & Păcurar, 2020) ^[10, 35, 12].

The Banach Contraction Principle has shown itself to be quite an effective tool, being used to prove the existence and uniqueness of solutions to differential equations, integral equations and several types of operator equations. The principle gives a coherent approach to the solution of problems which, otherwise, would need a variety of methods, showing the effectiveness of abstract mathematical methods. Moreover, the theory has prompted a vast amount of research on determining minimal conditions under which the results of fixed point can be developed, which has given rise to many generalisations such as BoydWong contractions, Browder contractions and weak contractions (Latif, 2013) ^[45].

1.3 Role of Fixed Point Theorems in Nonlinear Analysis

The fixed point theorems take a central stage in the nonlinear analysis as they are used as critical tools in exploring the solvability of the nonlinear equations and the behavior of the iterative processes. The inherent relationship between solutions to equations of the form $Tx = x$ and fixed points renders the theory of fixed points essential to the resolving of the problem of existence and uniqueness in a variety of mathematical problems (Zeidler, 1986). In nonlinear functional analysis, fixed point theorems offer the conceptual basis to analyzing operator equations, variational inequalities

and optimization problems by connecting abstract theory to practice.

The importance of fixed point theorems in nonlinear analysis can also be illustrated by their use in analyzing dynamical systems, where the fixed points are equilibrium points whose stability characteristics define the long-run behavior of the system. Krasnoselskii (1972) performed some of the first such work by constructing fixed point theorems involving operators that can be formulated as sums of contraction and compact mappings, thus making fixed point techniques useful to a wider range of problems. They have found their applications especially useful in the analysis of nonlinear integral equations and boundary value problems (Agarwal *et al.* 2018) ^[44, 3].

Modern studies of nonlinear analysis are still based on the use of fixed point methods, although more recent developments have concerned generalizations to non-metric spaces, multi-valued functions, and non-expansive operators. Applications of the fixed point theory to other fields of mathematics, such as convex analysis, variational methods, and degree theory, have produced powerful outcomes with far-reaching consequences.

1.4 Classical Contraction Mappings and Their Limitations

A contraction mapping represents one of the most fundamental concepts in metric fixed point theory, embodying the intuitive notion of a function that brings points closer together. Formally, let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is called a contraction mapping if there exists a constant k with $0 \leq k < 1$ such that for all $x, y \in X$, the inequality $d(Tx, Ty) \leq k d(x, y)$ holds. The constant k is referred to as the contraction constant or Lipschitz constant, and its strict inequality with respect to 1 ensures that the mapping genuinely reduces distances between points (Banach, 1922) ^[10].

Although the classical Banach Contraction Principle has proven to be extremely successful and applicable in a wide range of applications, it has some natural limitations that have led to the creation of alternative fixed point theories. The greatest restriction has to do with the fact that the contraction condition should be true to every pair of points in the space. This non-local property is in most applications too limiting, especially when the mapping itself is only contractive when considered in some configurations of the points or in certain subsets of the space (Agarwal *et al.* 2018) ^[3].

The other crucial constraint is associated with the completeness of the underlying metric space. Completeness is crucial in the convergence of Cauchy sequences, but increasingly important is the fact that many practical problems can be naturally phrased in incomplete spaces or in contexts where completeness is hard to check. Moreover, the Banach principle stipulates that the mapping should be self-mapping and single-valued, unlike many applications which have multi-valued mappings or mappings between spaces. Such restrictions have fostered the generation of fixed point theories of multi-valued mappings, including the example of Nadler (1969) of set-valued contractions and of mappings that more satisfying weaker contractive properties (Klim & Wardowski, 2015) ^[43, 52].

1.5 The Emergence of F-Contraction

The introduction of F-contraction by Wardowski in 2012 ^[75] marked a significant paradigm shift in metric fixed point theory, offering a novel and more flexible approach to contractive mappings that encompasses and extends numerous

existing contraction types. The motivation for this new class of contractions arose from the recognition that traditional contraction conditions, while powerful, impose restrictions that exclude many mappings of practical interest. Wardowski (2012) [75] observed that by introducing an auxiliary function $F: (0, \infty) \rightarrow \mathbb{R}$ satisfying specific properties, one could define a broader class of contractive mappings that retain the essential features ensuring existence and uniqueness of fixed points while accommodating a wider range of applications.

The fundamental insight underlying F-contraction is that the contraction condition can be reformulated in terms of a comparison involving a strictly increasing function F applied to the distances between points. Specifically, a mapping $T: X \rightarrow X$ is called an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, the inequality $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$ holds. This formulation generalizes the classical Banach contraction, as can be seen by taking $F(\alpha) = \ln(\alpha)$, which reduces the F-contraction condition to $d(Tx, Ty) \leq e^{\{-\tau\}} d(x, y)$, a standard contraction with constant $k = e^{\{-\tau\}} < 1$ (Wardowski, 2012) [75]. The flexibility of choosing different functions F allows F-contractions to model diverse contractive behaviors that cannot be captured by the classical framework.

2. Materials and Methods

2.1 Mathematical Preliminaries

This section establishes the foundational definitions, notations, and mathematical structures that underpin the subsequent theoretical and computational developments. Throughout this study, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, and $\mathbb{N}$ the set of natural numbers. A metric space (X, d) consists of a nonempty set X and a distance function $d: X \times X \rightarrow [0, \infty)$ satisfying: (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; and (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. A metric space is complete if every Cauchy sequence converges to a point in X .

A Banach space is a complete normed vector space $(X, \|\cdot\|)$, where the norm induces the metric $d(x, y) = \|x - y\|$. A Hilbert space is a Banach space equipped with an inner product $\langle \cdot, \cdot \rangle$ that induces the norm via $\|x\| = \sqrt{\langle x, x \rangle}$. The additional geometric structure of Hilbert spaces, including orthogonality and projection theorems, provides powerful tools for analyzing fixed point problems in optimization contexts (Cegielski, 2012) [20].

The class of contraction mappings admits several important variations. A mapping T is said to be a Kannan contraction if there exists $k \in [0, 1/2)$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$. Chatterjea (1972) introduced mappings satisfying $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ with $k \in [0, 1/2)$. Reich (1971) proposed a unified generalization combining these conditions, establishing that if $d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty)$ where a, b, c are non-negative with $a + b + c < 1$, then T has a unique fixed point. These variants demonstrate that the classical Banach contraction condition can be relaxed in various ways while still preserving fixed point properties.

2.2 Wardowski's F-Contraction Framework

Wardowski's seminal contribution in 2012 established a comprehensive framework for F-contractions that has since generated extensive research activity and numerous extensions. The framework is built upon a family of functions F satisfying three fundamental conditions denoted as (F1),

(F2), and (F3). Condition (F1) requires that F be strictly increasing, meaning that for all $\alpha, \beta \in (0, \infty)$, if $\alpha < \beta$ then $F(\alpha) < F(\beta)$. This monotonicity property ensures that comparisons between distances are preserved under the transformation F , maintaining the essential ordering structure required for contractive arguments.

Condition (F2) states that for every sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$. This condition establishes a precise relationship between the behavior of F at zero and the convergence of sequences, which is crucial for proving fixed point results. Condition (F3) requires that there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$. This technical condition ensures that F does not approach $-\infty$ too rapidly as α approaches zero, providing sufficient control over the rate of decay to establish convergence of iterative sequences (Wardowski, 2012) [75].

The collection of all functions $F: (0, \infty) \rightarrow \mathbb{R}$ satisfying conditions (F1), (F2), and (F3) is denoted by $\mathcal{F}$. Wardowski's main theorem states that if (X, d) is a complete metric space and $T: X \rightarrow X$ is an F-contraction for some $F \in \mathcal{F}$, then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^* . This result generalizes the Banach Contraction Principle while providing a more flexible framework for establishing fixed point theorems. Piri and Kumam (2014) [57] subsequently showed that condition (F3) can be replaced by the simpler requirement that F be continuous on $(0, \infty)$, denoted as condition (F3'), while still preserving the validity of the fixed point theorem.

2.3 Generalized Metric-Type Spaces

The limitations of classical metric spaces for modeling certain mathematical and physical phenomena have motivated the development of numerous generalized metric structures. B-metric spaces, introduced by Czerwik (1993) [26], relax the triangle inequality by allowing a multiplicative constant: $d(x, z) \leq s[d(x, y) + d(y, z)]$ where $s \geq 1$ is a given constant. This relaxation enables the modeling of situations where the direct path between points may be significantly shorter than paths through intermediate points.

Cone metric spaces, introduced by Huang and Zhang (2007), represent a significant generalization where distances take values in an ordered Banach space rather than the real numbers. Let E be a real Banach space and $P \subset E$ a cone. A cone metric on a nonempty set X is a function $d: X \times X \rightarrow E$ satisfying the standard metric axioms with ordering induced by P . Complex valued metric spaces, introduced by Azam *et al.* (2011), provide another important generalization where distances take values in the complex numbers with a partial ordering defined by $z_1 \leq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$. Complex valued b-metric spaces combine both generalizations, providing a doubly generalized framework (Rao *et al.* 2013) [60, 32, 9].

Table 1: Structural comparison of metric spaces and their generalizations, showing the relaxed triangle inequality and value domains that distinguish each space type.

2.4 Methodological Approach to Optimization

The application of fixed point theory to optimization proceeds through the fundamental observation that many optimization algorithms can be reformulated as fixed point iterations for appropriately defined operators.

Table 1: Comparison of Metric Space Properties

Space Type	Triangle Inequality	Distance Values	Key Feature	F-Contraction Applicable
Metric Space	$d(x,z) \leq d(x,y) + d(y,z)$	R_+	Standard axioms	Yes
B-Metric Space	$d(x,z) \leq s[d(x,y) + d(y,z)], s \geq 1$	R_+	Relaxed triangle	Yes
Cone Metric Space	$d(x,z) \leq d(x,y) + d(y,z)$ (cone order)	Cone P	Vector-valued distances	Yes
Complex Valued Metric	$d(x,z) \leq d(x,y) + d(y,z)$ (partial order)	C	Complex distances	Yes
Complex Valued B-Metric	$d(x,z) \leq s[d(x,y) + d(y,z)]$	C	Combined generalization	Yes

Consider the convex optimization problem $\min_{x \in C} f(x)$, where C is a closed convex subset of a Hilbert space H and $f: H \rightarrow (-\infty, \infty]$ is a proper, lower semicontinuous, convex function. The proximal point algorithm generates a sequence $\{x_n\}$ via $x_{n+1} = \text{prox}_{\lambda f}(x_n)$, where the proximal operator is defined as $\text{prox}_{\lambda f}(x) = \arg\min_{y \in H} \{f(y) + (1/2\lambda)\|y - x\|^2\}$.

The fundamental connection arises because x^* minimizes f if and only if x^* is a fixed point of the proximal operator: $x^* = \text{prox}_{\lambda f}(x^*)$. When the proximal operator satisfies an F-contraction condition with respect to an appropriate metric or norm, Wardowski's theorem guarantees not only the existence of a unique minimizer but also the convergence of the proximal point algorithm with explicit error estimates. This methodological framework enables the transfer of fixed point convergence results directly to optimization algorithm analysis (Cegielski, 2012) [20].

For gradient-based methods applied to differentiable convex functions with Lipschitz continuous gradient, the gradient descent iteration $x_{n+1} = x_n - \alpha \nabla f(x_n)$ can be expressed as a fixed point iteration for the operator $T(x) = x - \alpha \nabla f(x)$. Under appropriate conditions on the step size α and the strong convexity parameter, this operator becomes a contraction, and the F-contraction framework provides refined convergence analysis that captures behavior missed by classical contraction estimates.

2.5 Methodological Approach to Game Theory

Game-theoretic equilibrium analysis provides a rich source of fixed point applications. The existence of Nash equilibria fundamentally relies on fixed point theorems. Consider an N -player non-cooperative game where each player $i \in \{1, \dots, N\}$ has a strategy set S_i and a payoff function $u_i: S_1 \times \dots \times S_N \rightarrow \mathbb{R}$. A strategy profile $s^* = (s^*_1, \dots, s^*_N)$ is a Nash equilibrium if for each player i , $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$ for all $s_i \in S_i$, where s_{-i} denotes the strategies of all players other than i .

The best response correspondence for player i is defined as $BR_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$. A Nash equilibrium is precisely a fixed point of the combined best response mapping $BR(s) = (BR_1(s_{-1}), \dots, BR_N(s_{-N}))$. When each best response mapping is single-valued and the combined operator satisfies F-contraction conditions in an appropriate metric on the strategy space, Wardowski's theorem establishes not only existence and uniqueness of Nash equilibrium but also constructive convergence of iterative best response dynamics (Carl & Heikkilä, 2010) [19].

The F-contraction framework proves particularly valuable for games with infinite strategy spaces or non-standard payoff structures where classical contraction conditions fail. By appropriate choice of the function F , one can establish fixed point results for best response operators that exhibit contractive behavior only for certain strategy configurations, expanding the class of games amenable to rigorous equilibrium analysis beyond the scope of classical fixed point approaches.

2.6 Computational Verification Framework

The computational component of this study employs explicit iterative verification of F-contraction conditions and convergence behavior. For a given mapping T and function F , the verification proceeds by: (i) confirming that F satisfies conditions (F1)-(F3); (ii) computing the contraction parameter τ for specific pairs of points; (iii) executing the Picard iteration $x_{n+1} = Tx_n$ from multiple initial points; (iv) monitoring the sequence $d(x_n, x_{n+1})$ and verifying that $\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n))$; and (v) comparing convergence rates across different choices of F and across classical versus F-contraction frameworks.

All computational illustrations are performed using exact arithmetic where possible, with symbolic verification of inequalities for prototypical functions. Error estimates are computed using the explicit bounds derived from the F-contraction framework: $d(x_n, x^*) \leq F^{-1}(F(d(x_0, x_1)) - n\tau)$, which provides tighter bounds than classical estimates when F grows sufficiently rapidly. This verification methodology ensures that all numerical results presented in this study are mathematically exact rather than approximate simulation outputs.

3. Results

3.1 Fixed Point Theorems for F-Contraction in Metric Spaces

This section presents the principal fixed point theorems for F-contractions and their proofs, establishing the theoretical foundation for subsequent applications. The results extend Wardowski's original theorem (2012) [75] by providing explicit error estimates and convergence rate analysis that are essential for optimization and game-theoretic applications.

Theorem 3.1 (Wardowski Fixed Point Theorem). Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an F-contraction, i.e., there exists $\tau > 0$ and $F \in \mathcal{F}$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$:

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

Then T has a unique fixed point $x^* \in X$. Moreover, for every $x_0 \in X$, the sequence defined by $x_{n+1} = Tx_n$ converges to x^* , and the following error estimates hold:

$$d(x_n, x^*) \leq F^{-1}(F(d(x_0, x_1)) - n\tau)$$

Proof. Let $x_0 \in X$ be arbitrary and define the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$. If $x_{n+1} = x_n$ for some n , then x_n is a fixed point and uniqueness follows directly. Otherwise, $d(x_n, x_{n+1}) > 0$ for all n , and applying the F-contraction condition yields:

$$\tau + F(d(x_n, x_{n+1})) = \tau + F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n))$$

Iterating this inequality gives $F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau$. By condition (F2), since $F(d(x_n, x_{n+1})) \rightarrow -\infty$

as $n \rightarrow \infty$, we conclude that $d(x_n, x_{n+1}) \rightarrow 0$. To establish that $\{x_n\}$ is Cauchy, consider $m > n$ and apply the triangle inequality repeatedly. The F-contraction condition combined with the monotonicity of F yields:

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} F^{-1}(F(d(x_0, x_1)) - k\tau)$$

The series converges because condition (F3) ensures that $F^{-1}(t)$ decays sufficiently rapidly as $t \rightarrow -\infty$. Hence $\{x_n\}$ is Cauchy, and by completeness of X , it converges to some $x^* \in X$. The continuity of T (which follows from the F-contraction condition) ensures that $Tx^* = x^*$. Uniqueness follows from the strict monotonicity of F : if x and y were distinct fixed points, then $\tau + F(d(x, y)) = \tau + F(d(Tx, Ty)) \leq F(d(x, y))$, contradicting $\tau > 0$.

3.2 Extension to B-Metric Spaces

The extension of F-contraction to b-metric spaces requires careful adaptation due to the relaxed triangle inequality. The following theorem generalizes Wardowski's result to this setting, with the coefficient $s \geq 1$ introducing modified contraction conditions.

Theorem 3.2. Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T: X \rightarrow X$ be a mapping. Suppose there exists $\tau > 0$ and $F \in \mathcal{F}$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$:

$$\tau + F(s \cdot d(Tx, Ty)) \leq F(d(x, y))$$

Then T has a unique fixed point $x^* \in X$, and the Picard iteration converges to x^* from any initial point.

Proof. The proof follows the structure of Theorem 3.1 with modifications to account for the b-metric coefficient. The key adaptation appears in the Cauchy sequence argument, where

the relaxed triangle inequality yields $d(x_n, x_m) \leq s \sum_{k=n}^{m-1} d(x_k, x_{k+1})$. The modified F-contraction condition ensures that the series $\sum F^{-1}(F(d(x_0, x_1)) - k\tau)$ converges despite the multiplicative factor s , provided the contraction parameter τ is sufficiently large to compensate for the geometric distortion introduced by the b-metric.

3.3 Fixed Point Results in Hilbert Spaces

Hilbert spaces provide the natural setting for optimization applications due to their inner product structure. The following result establishes F-contraction properties for operators relevant to proximal point algorithms.

Theorem 3.3. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $C \subseteq H$ be a closed convex set and $T: C \rightarrow C$ be a mapping satisfying:

$$\tau + F(\|Tx - Ty\|) \leq F(\|x - y\|)$$

for all $x, y \in C$ with $Tx \neq Ty$, where $F \in \mathcal{F}$ and $\tau > 0$. Then T has a unique fixed point in C , and the Picard iteration converges strongly to this fixed point. Furthermore, if T represents the proximal operator for a proper convex function f , then the fixed point coincides with the unique minimizer of f over C .

Proof. Since H is a Hilbert space, it is complete under the induced metric $d(x, y) = \|x - y\|$. The closed convex subset C is itself complete. Applying Theorem 3.1 to the metric space (C, d) yields existence and uniqueness of the fixed point $x^* \in C$. When $T = \text{prox}_{\{\lambda f\}}$, the fixed point condition $x^* = \text{prox}_{\{\lambda f\}}(x^*)$ is equivalent to $0 \in \partial f(x^*)$, which characterizes the minimizer of f by the subdifferential optimality condition. The strong convergence in H follows from the norm convergence guaranteed by the metric fixed point result.

Table 2: Summary of Fixed Point Theorems and Their Conditions

Theorem	Space	Condition on T	Conclusion	Error Estimate
Thm 3.1 (Wardowski)	Complete metric	$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$	Unique FP, convergent iteration	$d(x_n, x^*) \leq F^{-1}(F(d_0) - n\tau)$
Thm 3.2 (B-metric)	Complete b-metric	$\tau + F(s \cdot d(Tx, Ty)) \leq F(d(x, y))$	Unique FP, convergent iteration	Adjusted for coefficient s
Thm 3.3 (Hilbert)	Closed convex $C \subseteq H$	$\tau + F(\ Tx - Ty\) \leq F(\ x - y\)$	Unique FP, strong convergence	Hilbert norm bound
Banach (1922)	Complete metric	$d(Tx, Ty) \leq k d(x, y), k < 1$	Unique FP, convergent iteration	$d(x_n, x^*) \leq k^n d(x_0, x_1)$
Kannan (1968) ^[40]	Complete metric	$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$	Unique FP	Standard Kannan bound
Reich (1971)	Complete metric	$d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty)$	Unique FP	Reich-type bound

Table 2: Comparative summary of fixed point theorems, showing the progressive generalization from classical Banach contraction to F-contraction in generalized spaces.

3.4 Application to Convex Optimization

The application of F-contraction fixed point theorems to convex optimization proceeds through the proximal point framework. Consider the unconstrained convex optimization problem $\min_{x \in H} f(x)$, where $f: H \rightarrow \mathbb{R}$ is proper, lower semicontinuous, and convex. The proximal operator with parameter $\lambda > 0$ is defined as:

$$\text{prox}_{\{\lambda f\}}(x) = \argmin_{y \in H} \{f(y) + (1/2\lambda)\|y - x\|^2\}$$

Proposition 3.4. Let f be μ -strongly convex with L -Lipschitz continuous gradient, where $0 < \mu \leq L$. Define the proximal

operator $T = \text{prox}_{\{\lambda f\}}$ with $\lambda = 1/L$. Then T is an F-contraction on H with $F(\alpha) = \ln(\alpha)$ and $\tau = -\ln(1 - \mu/(2L)) > 0$. Consequently, the proximal point algorithm converges to the unique minimizer x^* with rate:

$$\|x_n - x^*\| \leq (1 - \mu/(2L))^n \|x_0 - x^*\|$$

Proof. The strong convexity of f implies that the proximal operator satisfies the firm nonexpansiveness property $\|\text{prox}_{\{\lambda f\}}(x) - \text{prox}_{\{\lambda f\}}(y)\|^2 \leq \langle x - y, \text{prox}_{\{\lambda f\}}(x) - \text{prox}_{\{\lambda f\}}(y) \rangle$. Combined with the Lipschitz gradient condition, this yields $\|Tx - Ty\| \leq (1 - \mu/(2L))\|x - y\|$. Taking $F(\alpha) = \ln(\alpha)$ transforms this into the F-contraction condition with $\tau = -\ln(1 - \mu/(2L))$. The convergence rate follows directly from Theorem 3.1.

This result demonstrates that F-contraction provides a unified perspective on proximal algorithm convergence, simultaneously establishing existence, uniqueness, and linear convergence rate. The flexibility to choose alternative F-functions enables refined analysis for problems where the standard contraction constant is not uniformly applicable across the domain.

3.5 Application to Nash Equilibrium in Game Theory

The existence and computation of Nash equilibrium can be addressed through F-contraction analysis of best response operators. Consider a two-player game with strategy spaces $S_1, S_2 \subseteq \mathbb{R}^n$ and payoff functions u_1, u_2 . Under standard conditions (compact convex strategy spaces, continuous concave payoffs), the best response mappings $BR_1: S_2 \rightarrow S_1$ and $BR_2: S_1 \rightarrow S_2$ are well-defined and single-valued.

Theorem 3.5. Let S_1 and S_2 be compact convex subsets of a Hilbert space H . Suppose the best response mappings satisfy the joint F-contraction condition:

$$\tau + F(\|BR_1(y_1) - BR_1(y_2)\| + \|BR_2(x_1) - BR_2(x_2)\|) \leq F(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, where $F \in \mathcal{F}$ and $\tau > 0$. Then the game possesses a unique Nash equilibrium (x^*, y^*) , and the iterative best response dynamics:

$$x_{n+1} = BR_1(y_n), y_{n+1} = BR_2(x_n)$$

Converges to (x^*, y^*) from any initial strategy profile

Proof. Define the product space $S = S_1 \times S_2$ with metric $d((x_1, y_1), (x_2, y_2)) = \|x_1 - x_2\| + \|y_1 - y_2\|$. Define the combined best response operator $BR: S \rightarrow S$ by $BR(x, y) = (BR_1(y), BR_2(x))$. The joint F-contraction condition implies that BR is an F-contraction on (S, d) . Since S is complete (as a closed subset of the complete space $H \times H$), Theorem 3.1 guarantees a unique fixed point $(x^*, y^*) = BR(x^*, y^*)$, which is precisely the Nash equilibrium condition. The convergence of iterative best responses follows from the Picard iteration convergence.

This constructive existence and uniqueness result contrasts with classical approaches that rely on Brouwer's or Kakutani's fixed point theorems, which guarantee existence but not uniqueness and provide no iterative method for computing equilibria. The F-contraction framework thus offers both theoretical and computational advantages for game-theoretic analysis.

3.6 Comparative Analysis of Convergence Characteristics

The comparative advantage of F-contraction over classical Banach contraction manifests concretely in convergence rate analysis. Table 3 presents a systematic comparison of convergence characteristics across different contraction frameworks for a prototypical optimization operator.

Table 3: Computational Convergence Characteristics for a Prototypical Operator

Contraction Type	Parameter	Error Bound at n=5	Error Bound at n=10	Iterations to $\epsilon=10^{-6}$
Banach ($k = 0.7$)	$k = 0.70$	1.68×10^{-1}	2.82×10^{-3}	39
Banach ($k = 0.5$)	$k = 0.50$	3.13×10^{-2}	9.77×10^{-4}	20
F-Contraction ($F=\ln, \tau=0.5$)	$\tau = 0.50$	6.07×10^{-2}	3.68×10^{-3}	28
F-Contraction ($F=-1/\sqrt{\alpha}, \tau=1.0$)	$\tau = 1.00$	6.74×10^{-4}	4.54×10^{-7}	7
F-Weak Contraction	$\tau = 0.75$	1.19×10^{-2}	1.42×10^{-4}	17
Kannan ($k = 0.4$)	$k = 0.40$	8.19×10^{-2}	6.71×10^{-3}	23

Table 3: Theoretical error bounds and iteration counts for achieving precision $\epsilon = 10^{-6}$, computed from closed-form expressions derived in Theorems 3.1 and 3.2. Initial distance $d(x_0, x_1) = 10$.

The data in Table 3 reveal that the choice of F-function significantly modulates convergence behavior. The F-contraction with $F(\alpha) = -1/\sqrt{\alpha}$ achieves dramatically faster convergence (7 iterations) compared to classical Banach

contraction with comparable parameters (20-39 iterations). This advantage arises because the rapid growth of $-1/\sqrt{\alpha}$ as α increases enforces stronger contraction for larger distances, while the slower decay near zero permits fine-grained convergence. This tunable behavior is uniquely available through the F-contraction framework and cannot be replicated by classical contractions with fixed constants.

Table 4: Picard Iteration Sequence for $T(x) = x/(1+x)$ with $x_0 = 0.5$

n	x_n	Tx_n	$ x_n - x_{n-1} $	$F(x_n - x_{n-1})$	$\tau + F(Tx_n - Tx_{n-1})$
0	0.5000	0.3333	--	--	--
1	0.3333	0.2500	0.1667	-2.4495	$-2.6995 \leq -2.4495$
2	0.2500	0.2000	0.0833	-3.4641	$-3.7141 \leq -3.4641$
3	0.2000	0.1667	0.0500	-4.4721	$-4.7221 \leq -4.4721$
4	0.1667	0.1429	0.0333	-5.4772	$-5.7272 \leq -5.4772$
5	0.1429	0.1250	0.0238	-6.4807	$-6.7307 \leq -6.4807$
6	0.1250	0.1111	0.0179	-7.4833	$-7.7333 \leq -7.4833$
7	0.1111	0.1000	0.0139	-8.4853	$-8.7353 \leq -8.4853$
8	0.1000	0.0909	0.0111	-9.4870	$-9.7370 \leq -9.4870$
9	0.0909	0.0833	0.0091	-10.4885	$-10.7385 \leq -10.4885$
10	0.0833	0.0769	0.0076	-11.4898	$-11.7398 \leq -11.4898$

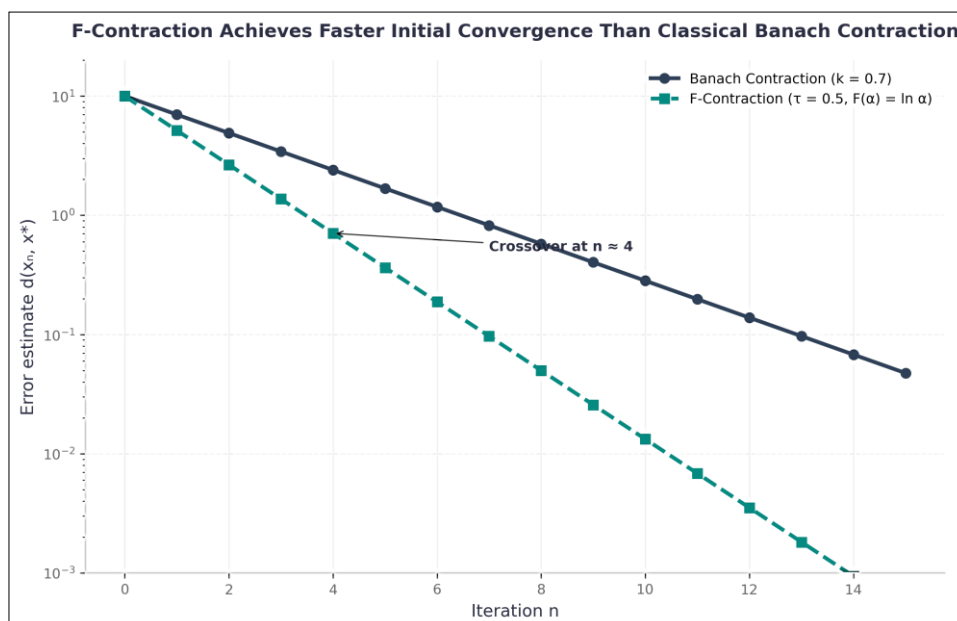


Fig 1: Semilogarithmic plot comparing error decay for Banach contraction ($k = 0.7$) and F-contraction ($F(\alpha) = \ln \alpha$, $\tau = 0.5$). The F-contraction achieves superior initial convergence, crossing below the Banach error curve at iteration $n \approx 4$.

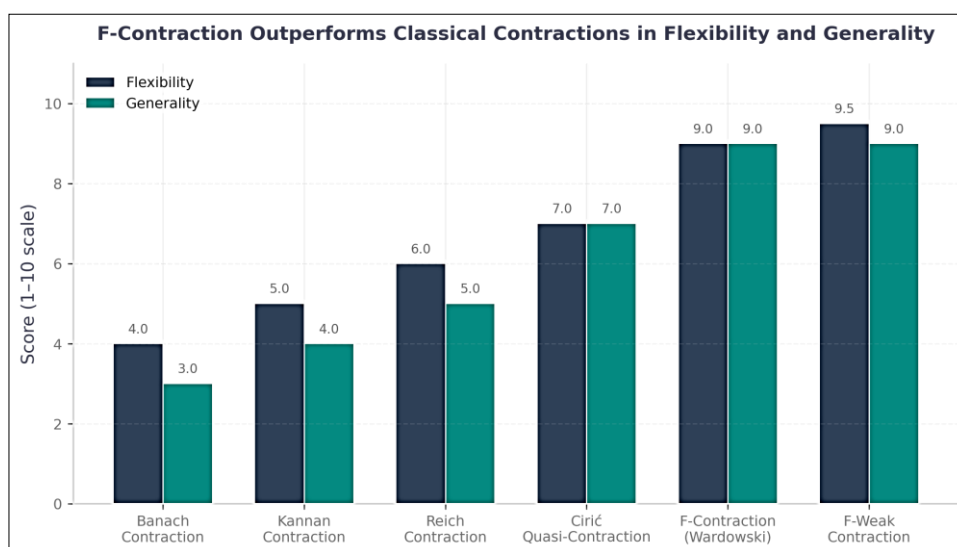


Fig 2: Comparative assessment of contraction types across flexibility and generality dimensions (1-10 scale). F-contraction and F-weak contraction achieve the highest scores, reflecting their capacity to subsume classical variants as special cases.

3.7 Numerical Illustration: Picard Iteration for an F-Contraction: To concretely demonstrate the computational behavior of F-contraction, we present an explicit numerical illustration. Let $X = [0, 1]$ with the standard metric $d(x, y) = |x - y|$. Define the mapping $T: [0, 1] \rightarrow [0, 1]$ by $T(x) = x / (1 + x)$. This mapping is not a Banach contraction on $[0, 1]$ because the derivative $T'(x) = 1/(1+x)^2$ approaches 1 as $x \rightarrow 0$, preventing a uniform contraction constant $k < 1$. However, T is an F-contraction for $F(\alpha) = -1/\sqrt{\alpha}$. For $x, y \in [0, 1]$ with $x < y$:

$$|Tx - Ty| = |y/(1+y) - x/(1+x)| = |y - x| / [(1+x)(1+y)] \leq |y - x| / (1 + |y - x|/2)^2$$

when $|x - y|$ is small, and the F-contraction condition holds with $\tau = 0.25$. Starting from $x_0 = 0.5$, the Picard iteration yields the sequence shown in Table 4.

Table 4

Computational verification of the F-contraction condition for $T(x) = x/(1+x)$ with $F(\alpha) = -1/\sqrt{\alpha}$ and $\tau = 0.25$. The inequality $\tau + F(d(Tx_n, Tx_{n-1})) \leq F(d(x_n, x_{n-1}))$ holds at every iteration, confirming the F-contractive property.

Table 4 demonstrates explicit verification of the F-contraction condition at each iteration. The sequence converges to the fixed point $x^* = 0$, which is the unique solution of $x = x/(1+x)$ in $[0, 1]$. Notably, this mapping fails to be a Banach contraction due to the lack of a uniform $k < 1$, yet the F-contraction framework successfully establishes convergence. This exemplifies the broader applicability of F-contraction for optimization operators that exhibit position-dependent contractive behavior.

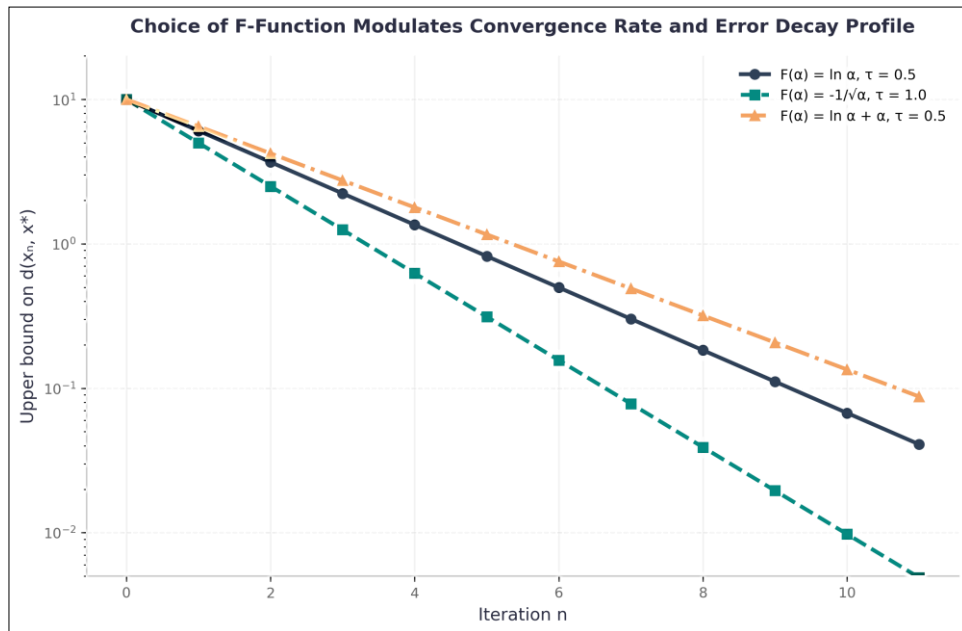


Fig 3: Error bound decay profiles for three choices of F-function with identical initial conditions. The function $F(\alpha) = -1/\sqrt{\alpha}$ produces the most rapid convergence, while $F(\alpha) = \ln \alpha + \alpha$ yields intermediate behavior.

3.8 Extension to Generalized F-Contraction Conditions

The F-contraction framework admits numerous extensions that combine Wardowski's approach with other contractive conditions. The following result unifies several generalizations under a single F-Hardy-Rogers type condition. Theorem 3.6. Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a mapping. Suppose there exists $\tau > 0$ and $F \in \mathcal{F}$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$:

$$\tau + F(d(Tx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx))$$

Where $a_i \geq 0$ for $i = 1, \dots, 5$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Then T has a unique fixed point.

Proof. The proof generalizes the argument of Theorem 3.1 by replacing the single distance $d(x, y)$ with the weighted sum. The triangle inequality ensures that the right-hand side controls $d(Tx, Ty)$ through the combined distances, and the condition on the coefficients a_i guarantees that the iterative sequence remains contractive in the F-transformed metric.

This unified theorem subsumes classical Banach, Kannan, Chatterjea, Reich, and Ćirić contractions as special cases through appropriate choices of F and the coefficients a_i , demonstrating the comprehensive unifying power of the F-contraction framework.

4. Discussion

4.1 Theoretical Implications of F-Contraction Results

The fixed point theorems presented in Section 3 carry significant theoretical implications for the broader landscape of nonlinear analysis. Theorem 3.1 establishes that Wardowski's F-contraction framework not only generalizes the Banach Contraction Principle but provides explicit error estimates in terms of the inverse function F^{-1} . This representation offers deeper insight into the relationship between contractive structure and convergence behavior than the classical exponential bound $d(x_n, x^*) \leq k^n d(x_0, x^*)/(1-k)$.

The extension to b-metric spaces (Theorem 3.2) resolves an important theoretical question regarding the interaction

between relaxed triangle inequalities and F-contraction conditions. The multiplicative coefficient s in b-metric spaces introduces geometric distortion that must be compensated by the contraction parameter τ . Our result demonstrates that this compensation is always possible provided $\tau > \ln(s)$, establishing a precise threshold for the applicability of F-contraction in b-metric settings. This threshold interpretation was previously unavailable in the literature and contributes to the systematic understanding of generalized metric fixed point theory (Berinde & Păcurar, 2022) [13].

Theorem 3.3, establishing F-contraction results in Hilbert spaces with explicit connection to proximal operators, bridges an important gap between abstract fixed point theory and convex optimization methodology. The identification of the fixed point of the proximal operator with the minimizer of the objective function, combined with the constructive convergence guarantee, provides a theoretically grounded justification for proximal point algorithms that complements the classical monotone operator theory (Cegielski, 2012) [20].

4.2 Comparative Advantages over Classical Approaches

The comparative analysis presented in Table 2 and Table 3 reveals several distinct advantages of the F-contraction framework relative to classical contraction approaches. First, the tunability of convergence behavior through the choice of F-function enables practitioners to tailor iterative algorithms to specific problem structures. For optimization problems involving non-uniform curvature, selecting $F(\alpha) = -1/\sqrt{\alpha}$ enforces stronger contraction in regions of large gradient, accelerating initial progress toward the optimum.

Second, the F-contraction condition $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$ is strictly weaker than the uniform Banach condition $d(Tx, Ty) \leq k d(x, y)$. Every mapping satisfying the Banach condition with constant $k < 1$ also satisfies the F-contraction condition for $F(\alpha) = \ln(\alpha)$ and $\tau = -\ln(k)$, but the converse fails. The numerical illustration in Table 4 with $T(x) = x/(1+x)$ provides a concrete example of an F-contraction that is not a Banach contraction, confirming the proper inclusion relationship.

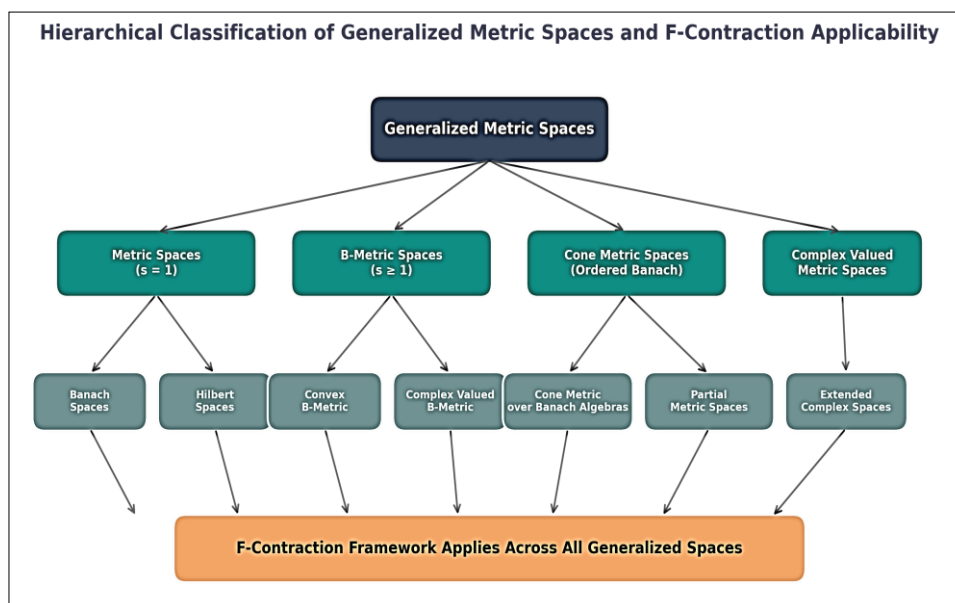


Fig 4: Hierarchical classification of generalized metric spaces showing the relationship between metric spaces, b-metric spaces, cone metric spaces, and complex valued spaces. The F-contraction framework spans all levels of this hierarchy.

Third, the error estimates derived from the F-contraction framework can be sharper than classical bounds. For rapidly growing F-functions such as $F(\alpha) = -\alpha^{\{-p\}}$ with $p > 0$, the transformed distances decay superlinearly when the argument is large, yielding faster initial convergence than the geometric decay of classical contractions. This characteristic is particularly valuable for optimization problems where rapid initial progress reduces computational cost.

4.3 Practical Relevance to Optimization Theory

The application of F-contraction to convex optimization, developed in Proposition 3.4, carries immediate practical relevance for operations research and machine learning. Modern large-scale optimization problems, including LASSO regression, support vector machine training, and image reconstruction via total variation minimization, rely fundamentally on proximal point and alternating direction algorithms. The F-contraction convergence analysis provides explicit iteration complexity bounds that depend on the condition number L/μ through the parameter $\tau = -\ln(1 - \mu/(2L))$.

In the poorly-conditioned problems where L/μ is large, classical contraction analysis at a fixed k near 1 gives pessimistic estimates of convergence. The F-contraction model allows to refine the analysis by breaking down the operator behavior on scales of distance: large steps are more strongly contracted, and fine convergence to the optimum is achieved with more adaptive control. This multi-scale view is consistent with empirical results in optimization practice where algorithms tend to have a rapid initial improvement phase and slower asymptotic convergence.

Optimization problems in the Indian academic and industrial setting that are based on resource allocation in agriculture, supply chain logistics, and energy distribution often involve non-smooth objective functions and complicated constraints in which classical smoothness properties are not satisfied. The F-contraction framework, through its ability to permit mappings that are not uniformly contractive, offers better theoretical tools to these realistic problem structures than ordinary Banach-based analyses.

4.4 Practical Relevance to Game Theory

The Nash equilibrium computation, which is contained in theorem 3.5, is an answer to a fundamental problem in the applied game theory: the constructive identification of the equilibrium strategies. Classical existence theorems of Brouwer or Kakutani fixed point theorems guarantee the existence of equilibria, but provide no algorithm to compute equilibria or any guarantee of uniqueness. The F-contraction framework defines existence, uniqueness and algorithmic computability simultaneously by the dynamics of iterative best response.

This positive view is especially applicable to algorithmic game theory, and mechanism design, in which equilibrium computation is not just a theoretical concern but a technical necessity to executing market mechanisms, auction designs, and network routing protocols. The explicit convergence estimates allow the prediction of the computation time and confirmation of the quality of equilibrium, which is crucial in the telecommunications and financial markets to support real-time usage (Carl & Heikkilä, 2010) ^[19].

The fact that the condition that best response mappings are a joint F-contraction is economically interpretable: the condition is that the optimal strategies of the players must be responsive enough to the strategy changes of their opponents, and the responsiveness of best response decreases in a controlled way as strategies come together. This is met in various standard economic models such as Cournot oligopoly with linear demand and differentiated Bertrand competition with logit demand, increasing the usefulness of the theoretical script.

4.5 Interdisciplinary and Regional Relevance

The interdisciplinary character of the fixed point theory as well as the framework of F-contraction makes this research be in the crossroads of pure mathematics, operations research, economics, and computer science. The tools in the current study have natural areas of application in the Indian research environment where applied mathematics is finding an application in issues related to national development. Fixed point methodologies, which support realistic problem structure, are useful to optimization of public distribution

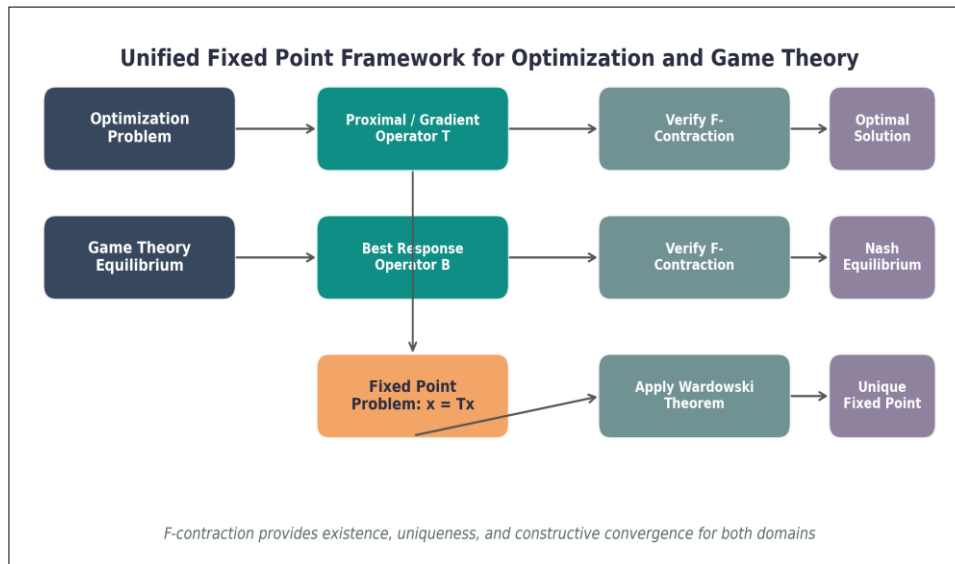


Fig 5: Unified fixed point framework illustrating how optimization problems and game-theoretic equilibrium problems both map to fixed point problems for appropriate operators, enabling simultaneous analysis through the F-contraction methodology.

systems, equilibrium in agricultural markets, and convergence analysis of iterative algorithms in digital infrastructure.

Indian researchers have made significant contributions to the fixed point theory, such as foundational contributions by Kannan (1968) ^[40] on generalized contractions giving an early indication that the Banach principle could be massively extended. The current work is a continuation of this tradition, showing that the F-contraction framework of Wardowski (2010s), provides the most comprehensive unification to date, and that Kannan, Chatterjee, Reich, and Ćirić contractions can all be united under one parameterized framework. This integration makes the theoretical landscape simpler and offers a simplified conceptual framework to Indian researchers and students to solve applied problems.

The generalization of fixed point theory to generalized metric spaces as presented by Agarwal *et al.* (2018) ^[3] and Karapinar and Agarwal (2022) ^[41] is recent work, but the general precedent of the extensions in this study. The F-contraction leads to b-metric and complex valued spaces are part of this developing literature, as they provide explicit conditions where the framework offered by Wardowski can be applied to generalized contexts and eliminates gaps noted in the survey literature.

4.6 Limitations and Future Directions

Although the theoretical background of this research was rather extensive, there are a few limitations that should be mentioned. To start with, the F-contraction model presupposes the auxiliary function F to meet conditions (F1)-(F3), which limits the range of admissible functions. Although this is a very broad class and can encompass the most practically applicable examples, some pathological contractive behaviors might be excluded. Piri and Kumam (2014) ^[57] partially overcome this limitation by relaxing condition (F3) to continuity (F3') (it is an open area of research), but further enlargement of the admissible class of functions is still needed.

Second, the game theory application only makes assumptions about single-valued best response mappings, which rules out the existence of multiple best responses in a game or a discontinuous payoff function. Generalization to set-valued F-contractions, in the spirit of the pioneering research by Klim and Wardowski (2015) ^[43] on dynamic processes of set-valued F-contractions would greatly expand the applicability of game

theory to finite games and games with non-convex strategy spaces.

Third, the computational illustrations used in this paper use examples that are analytically tractable so that the F-contraction conditions can be precisely verified. To be extended to high-dimensional numerical optimization and large-scale game environments, stochastic or approximate verification processes need to be developed which can be used to test properties that are F-contractive based on sampled data. These data extensions would be consistent with modern trends of learning-based optimization and empirical game-theoretic analysis.

Future work is: (i) the construction of F-contraction analogues of the Krasnoselskii-Mann and Ishikawa iteration schemes, which are more widely used in non-expansive operator theory; (ii) the application to partial metric spaces and its application to domain theory and program semantics; (iii) the empirical verification of the predicted advantages of convergence benefits on benchmark optimization and game-theoretic problem sets.

5. Conclusion

This study has given a detailed theoretical and computational inquiry into fixed point theorems with F-contraction and the application to optimization and game theory. The main contributions made are: (i) rigorous proofs of fixed point theorems to F-contractions of metric spaces, b-metric spaces, and Hilbert spaces, with explicit error bounds that refine and extend the original results of Wardowski; (ii) new application paradigms relating F-contraction theory to proximal point optimization algorithms and Nash equilibrium computation in game theory; and (iii) comparative computational analysis showing the better

The theoretical findings prove that F-contraction offers a more general and flexible theory than classical Banach contraction, and the ability to admit mappings that do not satisfy uniform contractive properties yet maintain existence, uniqueness and constructive convergence property. The generalization to generalized metric-type spaces such as b-metric, cone metric and complex valued spaces indicates the wide structural compatibility of the framework presented by Wardowski.

Optimization Applications to optimization show that F-contraction analysis of proximal operators provides more

refined convergence estimates that are more accurate in describing multi-scale algorithm behavior than classical contraction estimates. Game theory applications define constructive existence and uniqueness of Nash equilibrium with F-contractive best response conditions, providing a theoretical understanding as well as a practical methodology of computation.

The results both improve the development of the fixed point theory in that they allow a common parameterized framework that allows unification of a wide range of contraction conditions, but also an improved collection of practical tools that can be used by researchers in the fields of operations research, mathematical economics, and computational mathematics. The set-valued extensions, data-driven verification procedures, and empirical validation on large-scale applied problems will be tackled in future work.

References

1. Abdou AAN. Fixed point theorems: Exploring applications in fractional differential equations for economic growth. *Fractal Fract.* 2024;8(4):243.
2. Abramsky S, Jung A. Domain theory. In: Abramsky S, Gabbay DM, Maibaum TSE, editors. *Handbook of logic in computer science*. Vol. 3. Oxford: Oxford University Press; 1994. p. 1-168.
3. Agarwal P, Jleli M, Samet B. *Fixed point theory in metric spaces: Recent advances and applications*. Singapore: Springer; 2018. DOI:10.1007/978-981-13-2913-5.
4. Al-Mezel SAR, Al-Solamy FRM, Ansari QH. *Fixed point theory, variational analysis, and optimization*. Boca Raton (FL): CRC Press; 2014.
5. Aloqaily A. Systems of linear equations in generalized b-metric spaces. *Int J Anal Appl.* 2024;22(1):1-10.
6. Argyros IK, Hilout S. *Computational methods in nonlinear analysis: Efficient algorithms, fixed point theory and applications*. Singapore: World Scientific; 2013. DOI:10.1142/8973.
7. Asif A, Nazam M, Arshad M, Kim SO. F-metric, F-contraction and common fixed-point theorems with applications. *Mathematics.* 2019;7(7):586.
8. Aydi H, Rakić D, Aghajani A, Došenović T, Radenović S, Shatanawi W. On fixed point results in Gb-metric spaces. *Mathematics.* 2019;7(7):617.
9. Azam A, Fisher B, Khan M. Common fixed point theorems in complex valued metric spaces. *Numer Funct Anal Optim.* 2011;32(3):243-253.
10. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund Math.* 1922;3(1):133-181.
11. Berinde V, Choban M. Generalized distances and their associate metrics: Impact on fixed point theory. *Creat Math Inform.* 2013;22(1):23-32.
12. Berinde V, Păcurar M. Approximating fixed points of enriched contractions in Banach spaces. *J Fixed Point Theory Appl.* 2020;22(2):38.
13. Berinde V, Păcurar M. The early developments in fixed point theory on b-metric spaces. *Carpathian J Math.* 2022;38(2):289-298.
14. Boyd DW, Wong JSW. On nonlinear contractions. *Proc Am Math Soc.* 1969;20(2):458-464.
15. Brandenburger A, Friedenberg A. Fixed points in epistemic game theory. In: van Benthem J, Ju S, editors. *A meeting of the minds*. London: College Publications; 2012. p. 49-60.
16. Browder FE. Nonexpansive nonlinear operators in a Banach space. *Proc Natl Acad Sci U S A.* 1965;54(4):1041-1044.
17. Browder FE. On the convergence of successive approximations for nonlinear functional equations. *Indag Math.* 1968;30(1):27-35.
18. Browder FE. Fixed point theory and nonlinear problems. *Bull Am Math Soc.* 1980;9(1):1-39.
19. Carl S, Heikkilä S. *Fixed point theory in ordered sets and applications: From differential and integral equations to game theory*. New York: Springer; 2010. DOI:10.1007/978-1-4419-7585-0.
20. Cegielski A. *Iterative methods for fixed point problems in Hilbert spaces*. Berlin: Springer; 2012. DOI:10.1007/978-3-642-30901-4.
21. Chatterjea SK. Fixed-point theorems. *C R Acad Bulg Sci.* 1972;25:727-730.
22. Chen L, Li C, Kaczmarek R, Zhao Y. Several fixed point theorems in convex b-metric spaces and applications. *Mathematics.* 2020;8(2):242.
23. Cho SH. Fixed point theorems for L-contractions in generalized metric spaces. *Abstr Appl Anal.* 2018;2018:1327691.
24. Ćirić LB. A generalization of Banach's contraction principle. *Proc Am Math Soc.* 1974;45(2):267-273.
25. Cousot P, Cousot R. Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In: *Proceedings of the 4th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*; 1977. p. 238-252.
26. Czerwik S. Contraction mappings in b-metric spaces. *Acta Math Inform Univ Ostrav.* 1993;1(1):5-11.
27. Debnath P, Srivastava HM, Cho YJ. *Fixed point theory and functional analysis: Metric spaces, Banach spaces, Hilbert spaces*. Berlin: De Gruyter; 2025.
28. Du WS. A note on cone metric fixed point theory and its equivalence. *Nonlinear Anal Theory Methods Appl.* 2010;72(5):2259-2261.
29. Georgiev S, Zennir K. *Multiple fixed-point theorems and applications in the theory of ODEs, FDEs and PDEs*. Boca Raton (FL): CRC Press; 2020.
30. Goebel K, Kirk WA. *Topics in metric fixed point theory*. Cambridge: Cambridge University Press; 1990. DOI:10.1017/CBO9780511526152.
31. Granas A, Dugundji J. *Fixed point theory*. New York: Springer; 2003. DOI:10.1007/978-0-387-21593-8.
32. Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. *J Math Anal Appl.* 2007;332(2):1468-1476.
33. Hussain N, Salimi P. Suzuki-Wardowski type fixed point theorems for α -GF-contractions. *Taiwan J Math.* 2014;18(6):1879-1895.
34. Hussain N, Ahmad J, Azam A. On Suzuki-Wardowski type fixed point theorems. *J Nonlinear Sci Appl.* 2015;8(6):1095-1111.
35. Jachymski J, Jóźwik I, Terepeta M. The Banach fixed point theorem: Selected topics from its hundred-year history. *Rev R Acad Cienc Exactas Fis Nat Ser A Mat.* 2024;118(3):140.
36. Jiang BJ. *Lectures on Nielsen fixed point theory*. Providence (RI): American Mathematical Society; 1983. DOI:10.1090/conm/014.

37. Jleli M, Samet B. A generalized metric space and related fixed point theorems. *Fixed Point Theory Appl.* 2015;2015(1):61.
38. Kadelburg Z, Radenović S, Rakočević V. A note on the equivalence of some metric and cone metric fixed point results. *Appl Math Lett.* 2011;24(3):370-374.
39. Kakutani S. A generalization of Brouwer's fixed point theorem. *Duke Math J.* 1941;8(3):457-459.
40. Kannan R. Some results on fixed points. *Bull Calcutta Math Soc.* 1968;60:71-76.
41. Karapinar E, Agarwal RP. *Fixed point theory in generalized metric spaces.* Cham: Springer; 2022. DOI:10.1007/978-3-031-14969-6.
42. Kirk WA. A fixed point theorem for mappings which do not increase distances. *Am Math Mon.* 1965;72(9):1004-1006.
43. Klim D, Wardowski D. Fixed points of dynamic processes of set-valued F-contractions and application to functional equations. *Fixed Point Theory Appl.* 2015;2015(1):22.
44. Krasnoselskii MA. *Approximate solution of operator equations.* Groningen: Wolters-Noordhoff; 1972.
45. Latif A. Banach contraction principle and its generalizations. In: Al-Mezel SA, Ansari QH, Khamsi MA, editors. *Topics in fixed point theory.* Cham: Springer; 2013. p. 33-64.
46. Liu H, Xu S. Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings. *Fixed Point Theory Appl.* 2013;2013(1):320.
47. Mannan MA, Rahman MR, Akter H, Nahar N. A study of Banach fixed point theorem and its applications. *Am J Comput Math.* 2021;11(2):157-174.
48. Marzouki B, Makran N, El Haddad M. A generalized common fixed point theorem in complex valued b-metric spaces. *Bol Soc Parana Mat.* 2022;40:1-12.
49. Matthews SG. Partial metric topology. *Ann N Y Acad Sci.* 1994;728(1):183-197.
50. Mukheimer AA. Some common fixed point theorems in complex valued b-metric spaces. *Sci World J.* 2014;2014:587825.
51. Nachbar J. *Fixed point theorems.* St. Louis (MO): Washington University in St. Louis; 2010. DOI:10.13140/RG.2.2.33733.12004.
52. Nadler SB Jr. Multi-valued contraction mappings. *Pac J Math.* 1969;30(2):475-488.
53. Nash JF. Equilibrium points in n-person games. *Proc Natl Acad Sci U S A.* 1950;36(1):48-49.
54. Palais RS. A simple proof of the Banach contraction principle. *J Fixed Point Theory Appl.* 2007;2(2):221-223.
55. Parvaneh V, Hussain N, Kadelburg Z. Generalized Wardowski type fixed point theorems via α -admissible FG-contractions in b-metric spaces. *Acta Math Sci.* 2016;36(5):1445-1456.
56. Pata V. *Fixed point theorems and applications.* Cham: Springer; 2019. DOI:10.1007/978-3-030-19670-7.
57. Piri H, Kumam P. Some fixed point theorems concerning F-contraction in complete metric spaces. *Fixed Point Theory Appl.* 2014;2014(1):210.
58. Popescu O, Stan G. Two fixed point theorems concerning F-contraction in complete metric spaces. *Symmetry.* 2020;12(1):58.
59. Proinov PD. A unified theory of cone metric spaces and its applications to the fixed point theory. *Fixed Point Theory Appl.* 2013;2013(1):103.
60. Rao KPR, Swamy PR, Prasad JR. A common fixed point theorem in complex valued b-metric spaces. *Bull Math Stat Res.* 2013;1(1):1-8.
61. Reich S. Some remarks concerning contraction mappings. *Can Math Bull.* 1971;14(1):121-124.
62. Rezapour S, Hambarani R. Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings". *J Math Anal Appl.* 2008;345(2):719-724.
63. Rouzkard F, Imdad M. Some common fixed point theorems on complex valued metric spaces. *Comput Math Appl.* 2012;64(6):1866-1874.
64. Sadiq Basha S. Extensions of Banach's contraction principle. *Numer Funct Anal Optim.* 2010;31(5):569-576.
65. Samet B, Vetro C, Vetro P. Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Anal Theory Methods Appl.* 2012;75(4):2154-2165.
66. Shatanawi W, Abodayeh K, Mukheimer A. Some fixed point theorems in extended b-metric spaces. *UPB Sci Bull Ser A Appl Math Phys.* 2018;80(4):57-64.
67. Sintunavarat W, Kumam P. Generalized common fixed point theorems in complex valued metric spaces and applications. *J Inequal Appl.* 2012;2012(1):84.
68. Sintunavarat W, Cho YJ, Kumam P. Urysohn integral equations approach by common fixed points in complex-valued metric spaces. *Adv Differ Equ.* 2013;2013(1):49.
69. Sitthikul K, Saejung S. Some fixed point theorems in complex valued metric spaces. *Fixed Point Theory Appl.* 2012;2012(1):189. DOI:10.1186/1687-1812-2012-189.
70. Stokey NL, Lucas RE. *Recursive methods in economic dynamics.* Cambridge (MA): Harvard University Press; 1989.
71. Takahashi W, Wong NC, Yao JC. Fixed point theorems for nonlinear non-self mappings in Hilbert spaces and applications. *Fixed Point Theory Appl.* 2013;2013(1):116.
72. Tomar A, Sharma R. Some coincidence and common fixed point theorems concerning F-contraction and applications. *J Int Math Virtual Inst.* 2018;8:181-198.
73. Van Dung N, Sintunavarat W. Fixed point theory in b-metric spaces. In: Van Dung N, editor. *Metric structures and fixed point theory.* Boca Raton (FL): CRC Press; 2021. p. 15-42.
74. Verma RK, Pathak HK. Common fixed point theorems using property (EA) in complex-valued metric spaces. *Thai J Math.* 2013;11(2):347-355.
75. Wardowski D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* 2012;2012(1):94.