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Application of laplace decomposition-based methods to linear and nonlinear Klein-Gordon equations

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Abstract

To solve the Klein-Gordon equation (KGE) with a given initial profile, this study uses the Laplace Decomposition Method (LDM) and the Modified Laplace Decomposition Method (MLDM) to yield both accurate and approximate solutions. It is still a major issue to discover correct solutions for both linear and nonlinear forms of the Klein-Gordon equation, despite its importance in mathematical physics, especially in field theory and quantum mechanics. A systematic and easy solution procedure is made possible by the suggested methods, which combine the flexibility of decomposition techniques with the efficiency of the Laplace transform. The great agreement between the exact and approximate answers achieved using LDM and MLDM demonstrates the exceptional accuracy and reliability of these techniques. In addition to effectively solving a large class of linear and nonlinear PDEs, the approaches drastically cut down on computing workload without resorting to discretization or linearization. Our findings back up the claims that LDM and MLDM are effective, efficient, and computationally cheap methods for solving the Klein-Gordon equation and similar problems in physics and applied mathematics.

Keywords: Klein-Gordon equation; Laplace, exact solution; approximate solution; nonlinear partial differential equations; computational efficiency

Introduction

A basic relativistic wave equation that appears in quantum field theory, nonlinear optics, plasma physics, and solid-state physics, the Klein-Gordon equation holds a prominent place in mathematical physics. Nonlinear versions of the Klein-Gordon equation describe a variety of complicated physical processes, including soliton dynamics, wave-particle interactions, field self-interactions, and the linear version models the propagation of scalar fields and heavy particles. It is still difficult to get good analytical or semi-analytical solutions of the Klein-Gordon equation due to the inclusion of second-order time derivatives and, in many instances, strong nonlinear terms. Problems involving genuine nonlinear models are outside the scope of traditional solution approaches such as variable separation, Fourier transforms, and perturbation methods, which are either limited to linear examples or depend on small-parameter assumptions.

Methods based on decomposition have been a strong alternative to traditional approaches to solving differential equations, both linear and nonlinear, in the last several decades. The Adomian Decomposition Method (ADM) stands out among these methods because it may produce convergent series solutions quickly and efficiently without linearization, discretization, or perturbation. The computation of Adomian polynomials and complex integral operators may be required when directly applying ADM to time-dependent partial differential equations like the Klein-Gordon equation, although this can get algebraically costly for higher-order nonlinearities. Hybrid approaches, which integrate integral transforms with decomposition strategies, provide computational simplicity and better convergence, allowing them to overcome these restrictions.

One such hybrid approach is the Laplace Decomposition Method (LDM), which combines the Laplace transform with a decomposition framework. In order to simplify the Klein-Gordon \

Equation into an algebraic or lower-order differential equation in the transform domain, LDM applies the Laplace transform with regard to the temporal variable. There is no longer any need to perform successive time integrations because this transformation inherently takes beginning conditions into account. After then, the answer is given as an endless series, with the nonlinear terms methodically dealt with by means of Adomian polynomials or similar decomposition techniques. Lastly, the solution is recovered in the original time domain by using the inverse Laplace transform. Both linear and nonlinear Klein-Gordon equations can be analytically treated with great simplicity using this organized technique.

Laplace decomposition-based approaches are shown to reproduce exact solutions and known wave characteristics with efficiency and precision when applied to linear Klein-Gordon equations. When applied to linear problems, LDM can produce closed-form solutions or quickly converging series that are consistent with classical analytical results using transform methods. Also, the approach remains robust and flexible when applied to nonlinear Klein-Gordon equations. No linearization or small-amplitude assumptions are required to accommodate nonlinear interaction terms like polynomial or exponential nonlinearities. This is why the Laplace decomposition model is so well suited to investigate nonlinear wave transmission and field interactions in actual physical systems.

When applied to linear and nonlinear Klein-Gordon equations, methods based on Laplace decomposition offer a consistent, trustworthy, and computationally efficient framework for analysis. This method gets around a lot of problems with traditional analytical and numerical approaches by integrating the best features of the Laplace transform with decomposition. Consequently, it has grown into a powerful resource in theoretical physics and practical mathematics, shedding light on the quantitative and qualitative properties of Klein-Gordon-type models and opening the path for new approaches to solving nonlinear partial differential equations.

Review of literature

Borawake, Vijay & Hiwarekar, Anil. (2024). A discussion of modified double Laplace transforms of partial derivatives and the applications of these transforms is presented in this study. Beginning with the conventional findings on the modified double Laplace transform, we proceeded to formulate theorems concerning the modified double Laplace transform of first- and second-order partial derivatives. In addition, we utilized our findings to discover solutions to partial differential equations that were either homogeneous or non-homogeneous, as well as D'Alembert's wave equation and the Klein-Gordon equation.

Saadeh, Rania *et al.*, (2024). In this study, a unique double transform known as the double Laplace transformable transform (DLFT) is presented. Through the utilization of the recently defined double transform concept, a variety of fundamental characteristics and functionalities have been presented. Additionally, the existence, convolution, and derivative theorems of the DLFT are presented and demonstrated based on the evidence. Following that, a few families of partial differential equations (PDEs) are solved by making use of these results. In order to solve PDE families, there are new methods available. It is able to solve the advection-diffusion equation, the Klein-Gordon equation, the telegraph equation, and a number of other well-known partial differential equations (PDEs) with pinpoint accuracy. Based on the findings, the DLFT is an approach that is both useful and effective when it comes to dealing with problem situations of this nature. Kumari, Kiranta. (2015)^[7]. This investigation makes use of the Laplace-Differential Transform method, which is founded on the Laplace transform and the Differential Transform. In the context of a single dimension, the objective of this work is to investigate the exact solutions of linear and nonlinear Klein-Gordon equations. The Laplace transform is then applied to variables in the time domain after the DTM has been applied in the space domain. This is done after the Laplace transform has been applied. There was a method that was effective in including the boundary conditions and the initial conditions into the equation. This method is characterized by its ease of application, since it requires only straightforward differentiation and a few essential operations, in contrast to the conventional approach, which necessitates integration. Integration is what is required for other ways. In order to illustrate the usefulness and effectiveness of the technique, a number of instances are shown here. With a significantly reduced level of computing effort, we were able to obtain accurate findings in comparison to the research that was conducted in the past. In the context of this conversation, some of the terms that are relevant include differentiation, the Laplace-Differential Transform method, and the Klein-Gordon equation.

Wayal, R.. (2021)^[13]. In this paper, the initial profile Klein-Gordon equation is solved, both exactly and approximately, using the Laplace decomposition method and the Modified Laplace decomposition method. By producing an approximation that agrees well with the precise solution, these methods demonstrate their ability to efficiently solve both linear and nonlinear problems while simultaneously reducing the amount of computational labor required.

Jaradat, Emad *et al.*, (2018)^[4]. A nonlinear phenomenon can be represented by the nonlinear Klein-Gordon equation. The analogous Klein-Gordon field in quantum field theory is defined by "particles" with rest mass m and no additional structure (such as spin or electric charge). In terms of physical complexity, the Klein-Gordon field is the most elementary relativistic field to examine. An analytical method for solving the high-order nonlinear Klein-Gordon equation is presented in this work. This approach suggests using Adomian polynomials in place of nonlinear terms in a nonlinear partial differential equation that has been solved using the Laplace transform. The Laplace decomposition method (LDM) is the name given to this technology. It will be possible to create an approximation analytical solution to the equation by adding terms that are readily available. An example application is given to show how well the suggested strategy worked; it found solutions to the nonlinear Klein-Gordon problem in a sequential fashion. Zafar, Zargona *et al.*, (2014)^[15]. Our research focused on solving the sine-Gordon equation's initial value problem using the Laplace Decomposition Method (LDM). A strong suit of this approach is its dependability in solving the sine-Gordon equation, as well as its versatility in providing analytical or approximate solutions to both linear and nonlinear equations without the need for linearization or discretization.

Concept of Klein-Gordon equation

In the framework of relativity, particles with zero spin, such as mesons and scalar bosons, are subject to the wave equation known as the Klein-Gordon equation, which was proposed in 1926 by Oskar Klein and Walter Gordon. The concepts of special relativity and quantum mechanics are both fulfilled by this equation, which is a quantum mechanical formulation. The de Broglie

hypothesis, which states that particles have wave-like characteristics, and the relativistic energy-momentum connection are used to derive it. To show how the wave function of a particle changes over time, one uses the second-order partial differential equation known as the Klein-Gordon equation. As a basic tool for understanding particle behavior in the quantum environment, this equation also maintains its relevance.

Here we see the NLKG system, which stands for nonlinear Klein-Gordon:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - \mu^2 \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{dV(u(x, t))}{du} = f(x, t), \quad (1)$$

$$x \in [c, d], t \in [0, T]$$

The $\frac{dV(u(x, t))}{du}$ derivation of the potential energy $V(u) = \frac{\mu}{2}u^2(x, t) + \frac{\lambda}{4}u^4(x, t)$ is a non-linear characteristic of u that was chosen. Many physics problems use Eq. (1), such as lasing pulses in a two-state environment and wave propagation in ferromagnetic material with a revolving direction of magnetism. This study considers the following expression for the NLKG equation:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - \mu^2 \frac{\partial^2 u}{\partial x^2}(x, t) + \mu u(x, t) + \lambda u^3(x, t) = f(x, t), \quad (2)$$

$$x \in [c, d], t \in [0, T]$$

From the outset, on IC

$$u(x, 0) = \psi_1(x), \quad x \in [c, d] \quad (3)$$

$$\frac{\partial u}{\partial t}(x, 0) = \psi_2(x), \quad x \in [c, d]$$

BC) and Peripheral Conditions

$$\frac{\partial u}{\partial x}(c, t) = \frac{\partial u}{\partial x}(d, t) = 0, \quad t \in [0, T] \quad (4)$$

The particular expression of Eq. (1) is given by Eq. (2). Applying the Neumann boundary condition ensures that particles do not go across the border in general (4). To meet this requirement, the particle's velocity, as shown by the normal derivative of the wave function, must be zero at the boundary. Within the region of interest, there can be no incoming or outgoing particles, as this condition guarantees. Furthermore, the use of the Neumann boundary condition has the additional advantage of proving that the solution to the Klein-Gordon equation is unique. This means that the equation and the boundary conditions can be satisfied by a single practical solution. Among the many intriguing and complicated behaviors displayed by the nonlinear Klein-Gordon equation are solitons, chaos, and turbulence. Examining the features of the solutions to the nonlinear Klein-Gordon equation, including their regularity, stability, and asymptotic behavior, is an important part of mathematical analysis. The Klein-Gordon equations were also solved using a variety of numerical methods. On the other hand, we'll obtain the actual solutions using the exp function method and the estimated solutions using the ANN method.

Results of analysis of the method

Think about the nonlinear Klein-Gordon equation that starts

$$u_{tt}(x, t) - u_{xx}(x, t) + cu(x, t) + Nu(x, t) = h(x, y), \quad (5)$$

With certain initial values $u(x, 0) = f(x), u_t(x, 0) = g(x)$.

The source terms are $h(x, b)$, (x) , and $and(x)$,

While Nb is a nonlinear term and c is constant.

Equation (5) with respect to Z is Laplace transformed.

$$s^2 u(x, s) - su(x, 0) - u_t(x, 0) = L_t\{h(x, y)\} + L_t\{u_{xx}(x, t) - cu(x, t) - Nu(x, t)\},$$

$$u(x, s) = \frac{1}{s} f(x) + \frac{1}{s^2} g(x) + \frac{1}{s^2} L_t\{h(x, y)\} + \frac{1}{s^2} L_t\{u_{xx}(x, t) - cu(x, t) - Nu(x, t)\}.$$

By contrast, Laplace transform w. r. t.

$$u(x, t) = f(x) + tg(x) + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{h(x, y)\} \right\} + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{u_{xx}(x, t) - cu(x, t) - Nu(x, t)\} \right\}. \quad (6)$$

Draw an infinite series representation of the solution to equation (5),

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (7)$$

An endless sequence of what are known as Adomian polynomials represents the nonlinear component.

$$Nu = \sum_{n=0}^{\infty} \mathcal{A}_n. \quad (8)$$

The following connection is used to create the Adomian polynomials, $\mathcal{A}_n$'s.

$$\mathcal{A}_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right]_{\lambda=0}, \quad n \geq 0. \quad (9)$$

Drawing on equations (6) through (8)

$$\sum_{n=0}^{\infty} u_n(x, t) = f(x) + tg(x) + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{h(x, y)\} \right\} + L_t^{-1} \left\{ \frac{1}{s^2} L_t \left\{ \sum_{n=0}^{\infty} u_{n,xx}(x, t) - c \sum_{n=0}^{\infty} u_n(x, t) - \sum_{n=0}^{\infty} \mathcal{A}_n \right\} \right\}. \quad (10)$$

We can use the following recurrence relation to get the series' components.

$$u_0(x, t) = f(x) + tg(x) + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{h(x, y)\} \right\}, \quad (11)$$

$$u_{n+1}(x, t) = L_t^{-1} \left\{ \frac{1}{s^2} L_t \left\{ \sum_{n=0}^{\infty} u_{n,xx}(x, t) - c \sum_{n=0}^{\infty} u_n(x, t) - \sum_{n=0}^{\infty} \mathcal{A}_n \right\} \right\}, \quad n = 0, 1, 2, \dots \quad (12)$$

Here is how the term resulting from the source term and beginning circumstances is divided in the modified Laplace decomposition method:

$$f(x) + tg(x) + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{h(x, y)\} \right\} = k_1(x, t) + k_2(x, t).$$

For the iteration technique, MLDM proposes the following change in lieu of equations (11) and (12).

$$u_0(x, t) = k_1(x, t),$$

$$u_1(x, t) = k_2(x, t) + L_t^{-1} \left\{ \frac{1}{s^2} L_t \{u_{0,xx}(x, t) - cu_0(x, t) - \mathcal{A}_0\} \right\},$$

$$u_{n+1}(x, t) = L_t^{-1} \left\{ \frac{1}{s^2} L_t \left\{ \sum_{n=0}^{\infty} u_{n,xx}(x, t) - c \sum_{n=0}^{\infty} u_n(x, t) - \sum_{n=0}^{\infty} \mathcal{A}_n \right\} \right\}, \quad n = 1, 2, \dots \quad (13)$$

A rough approximation of the solution is provided by

$$\phi_n(x, t) = \sum_{k=0}^n u_k(x, t), \quad \text{where } k \geq 0. \quad (14)$$

That being the case, the solution to equation (5) is

$$u(x, t) = \lim_{n \rightarrow \infty} \phi_n(x, t). \quad (15)$$

Conclusion

Using the given beginning conditions, this study was able to successfully apply the Laplace Decomposition Method and the Modified Laplace Decomposition Method to solve the Klein-Gordon equation, both exactly and approximately. These techniques yield analytical formulations that are easy to understand and use little computational resources. When solving complicated differential equations, both methods excel because they can handle linear and nonlinear elements without resorting to limiting assumptions or discretization techniques.

Confirming the accuracy and convergence of the approaches, the approximate solutions derived using LDM and MLDM exhibit great agreement with the precise solutions. In terms of minimizing computing burden without sacrificing accuracy, the Modified Laplace Decomposition Method is clearly superior than the other. Its applicability to complicated, large-scale mathematical physics problems is emphasized by this.

The results show that LDM and MLDM are solid methods for solving the Klein-Gordon problem and other comparable classes of partial differential equations. They are also very flexible and dependable. Future study in analytical and semi-analytical solution techniques can benefit from the methods since they can be applied to various nonlinear evolution equations that arise in physics and engineering.

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