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Generalized fixed point theorems by relaxing conventional assumptions on monotonicity and completeness

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Abstract

The classical Banach contraction principle and its ordered variants due to Ran and Reurings (2004) and Nieto and Rodríguez-López (2005) have established foundational results for fixed point theory in partially ordered metric spaces. However, the stringent requirements of monotonicity of the mapping and the completeness of the underlying metric space remain significant constraints that limit the applicability of these theorems to broader classes of problems. In this paper, we develop generalized fixed point theorems by systematically relaxing these conventional assumptions. We introduce the novel concept of a weakly comparable mapping, which replaces the strict monotonicity requirement with a substantially weaker condition. Furthermore, we propose a monotone-Cauchy completeness condition that is strictly weaker than the standard completeness assumption. Under a generalized Hardy-Rogers-type contraction condition involving nonnegative parameters with sum less than 1, we establish the existence of fixed points without requiring the mapping to be non-decreasing or the space to be complete in the classical sense. The uniqueness of the fixed point and the convergence rate of the Picard iteration scheme are also analyzed. We provide rigorous analytical examples and comprehensive numerical simulations, including convergence tables and graphical illustrations, to validate our theoretical findings.

Keywords: Fixed point theorem; Weakly comparable mapping; Generalized contraction; Partially ordered metric space; Monotone-Cauchy completeness; Picard iteration; Hardy-Rogers contraction; Relaxation of monotonicity

1. Introduction

1.1 Background and Motivation

Fixed point theory constitutes one of the most fundamental and extensively studied branches of nonlinear functional analysis, with profound applications spanning differential equations, integral equations, optimization theory, mathematical economics and computational sciences. The celebrated Banach contraction principle, established by Banach in 1922, remains the cornerstone of metric fixed point theory, asserting that every contraction mapping on a complete metric space possesses a unique fixed point^[1]. The elegance and simplicity of this theorem, combined with its remarkable applicability, have inspired generations of mathematicians to generalize it in numerous directions.

A significant paradigm shift in fixed point theory occurred with the seminal work of Ran and Reurings (2004), who extended the Banach principle to partially ordered metric spaces^[2]. Their theorem demonstrated that if a mapping T is continuous, non-decreasing (with respect to the partial order) and satisfies a contraction condition for comparable elements, then T admits a fixed point provided there exists an initial point x_0 with $x_0 \leq Tx_0$. This breakthrough was motivated by applications to nonlinear matrix equations, where positive definiteness of solutions naturally induces a partial order structure. Shortly thereafter, Nieto and Rodríguez-López (2005) refined these results by relaxing the continuity requirement and applying them to periodic boundary value problems for ordinary differential equations^[3].

The ordered fixed point framework was subsequently enriched by contributions from Bhaskar and Lakshmikantham (2006), who introduced coupled fixed point theorems^[4];

Lakshmikantham and Ćirić (2009), who established coupled coincidence point results^[5]; and Samet, Vetro and Vetro (2012), who developed the influential α - ψ -contractive mapping framework^[6]. The relation-theoretic contraction principle introduced by Alam and Imdad (2015) represented a major unification, proving that the partial order could be weakened to an arbitrary binary relation while preserving the fixed point conclusion^[7].

Despite these advances, two critical limitations persist in the existing literature. First, the monotonicity assumption requiring $x \preceq y \Rightarrow Tx \preceq Ty$ is highly restrictive. Many naturally occurring mappings in applied settings fail to preserve order structures. For instance, in economic models with complementarities or in biological systems with competitive interactions, the response function may not be monotone. Second, the completeness assumption demands that every Cauchy sequence converge, which excludes important incomplete spaces such as the space of continuous functions with compact support, certain Sobolev spaces with non-standard metrics and spaces of rational matrices.

2. Preliminaries and Basic Definitions

2.1 Standard Notations and Terminology

Throughout this paper, we denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^+$ the set of non-negative real numbers. The symbol: = denotes definition. We use the standard notations and terminologies from metric fixed point theory, which the reader may consult in standard references such as Rus (2001) and Berinde (2004)^[8, 9].

Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is said to be a contraction (or Banach contraction) if there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k d(x, y), \quad \forall x, y \in X. \quad (1)$$

A sequence $\{x_n\}$ in X is called Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. The space (X, d) is complete if every Cauchy sequence converges to a point in X .

2.2 Classical Fixed Point Results

Theorem 2.1 (Banach Contraction Principle, 1922). Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a contraction mapping with contraction constant $k \in [0, 1)$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^* and the following error estimate holds: $d(x_n, x^*) \leq (k^n/(1-k)) d(x_0, x_1)$.

Theorem 2.2 (Kannan, 1968)^[10]. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ satisfy $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$, where $k \in [0, 1/2)$. Then T has a unique fixed point.

Theorem 2.3 (Chatterjea, 1972)^[11]. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ satisfy $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$, where $k \in [0, 1/2)$. Then T has a unique fixed point.

Theorem 2.4 (Ćirić, 1974)^[12]. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ satisfy $d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 2.5 (Hardy and Rogers, 1973)^[13]. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ satisfy $d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty) + e[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$, where $a, b, c, e \geq 0$ and $a + b + c + 2e < 1$. Then T has a unique fixed point.

2.3 Ordered Metric Spaces and Related Concepts

A partially ordered metric space is a triple $(X, d, \preceq)$ where (X, d) is a metric space and $\preceq$ is a partial order on X (reflexive, antisymmetric and transitive). We recall the following definitions from the ordered fixed point literature^[2, 3].

Definition 2.6. A mapping $T: X \rightarrow X$ is said to be non-decreasing (or monotone) with respect to $\preceq$ if $x \preceq y \Rightarrow Tx \preceq Ty$, for all $x, y \in X$.

Definition 2.7. A sequence $\{x_n\} \subseteq X$ is said to be non-decreasing (or monotone increasing) with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$.

The Ran-Reurings theorem and its successors typically require the following combination of hypotheses:

(H1) (X, d) is a complete metric space. (H2) T is non-decreasing with respect to $\preceq$. (H3) T satisfies a contraction condition for comparable elements. (H4) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. (H5) T is continuous, or the order satisfies a regularity condition.

Our objective is to weaken hypotheses (H1) and (H2) while preserving the fixed point conclusion and to subsume hypothesis (H3) within a more general contraction framework.

3. Results

3.1 Weakly Comparable Mappings: A New Concept

The monotonicity condition (H2) is among the most restrictive requirements in ordered fixed point theory. In many applied contexts, mappings do not preserve order globally. To address this fundamental limitation, we introduce the following novel concept.

Definition 3.1 (Weakly Comparable Mapping). Let $(X, d, \preceq)$ be a partially ordered metric space. A mapping $T: X \rightarrow X$ is said to be weakly comparable if for every pair of points $x, y \in X$, there exists an element $z \in X$ (which may depend on x and y) such that $x \preceq z$ and $y \preceq z$. The element z is called a common upper bound of x and y with respect to $\preceq$.

Remark 3.2. (i) Every non-decreasing mapping is weakly comparable. Indeed, if T is non-decreasing and $x \preceq y$, then taking $z = y$ yields $x \preceq z$ and $y \preceq z$ trivially. Thus, weak comparability is a strict generalization of monotonicity. (ii) The converse is not true. A weakly comparable mapping need not be non-decreasing. Example 4.1 below demonstrates this explicitly. (iii) In a lattice-ordered space, weak comparability is automatically satisfied for any mapping because every pair of elements possesses a least upper bound (supremum). Our definition only requires the existence of some upper bound, not necessarily the least one.

3.2 Generalized contraction condition

We now introduce the generalized contraction condition that will be employed throughout our main theorems. This condition is inspired by the Hardy-Rogers framework but is tailored for the ordered setting and our weak comparability concept.

Definition 3.3 (Generalized Contraction). Let $(X, d, \preceq)$ be a partially ordered metric space and let $T: X \rightarrow X$ be a mapping. We say that T satisfies a generalized contraction condition if there exist non-negative constants $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$ and $d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$ for all $x, y \in X$ that admit a common upper bound $z \in X$ with $x \preceq z$ and $y \preceq z$.

Remark 3.4. The generalized contraction condition (Definition 3.3) unifies and extends several classical contractive conditions: If $\alpha = \beta = 0$ and $\gamma = k$, it reduces to the Banach contraction. If $\gamma = 0$ and $\alpha = \beta = k$, it reduces to the Kannan contraction (up to symmetry). If $\alpha = \beta = 0$ and the coefficient structure is modified appropriately, it relates to the Chatterjea contraction. If the coefficients are chosen to match the maximum term structure, it encompasses the Ćirić quasi-contraction. Therefore, our contraction condition is genuinely more general than each of these classical conditions individually.

3.3 Relaxed Completeness Assumption

The classical completeness requirement that every Cauchy sequence converges is often unnecessarily strong for ordered fixed point theorems. In the ordered setting, the Picard iteration sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ is typically constructed to be monotone (non-decreasing) provided the initial point satisfies $x_0 \preceq Tx_0$. Consequently, only the convergence of monotone Cauchy sequences is essential for the proof.

Definition 3.5 (Monotone-Cauchy Complete Space). A partially ordered metric space $(X, d, \preceq)$ is said to be monotone-Cauchy complete (or sequentially order-complete) if every non-decreasing Cauchy sequence in X converges to a point in X .

Remark 3.6. (i) Every complete metric space is trivially monotone-Cauchy complete. (ii) The converse is false. There exist incomplete metric spaces in which every monotone Cauchy sequence converges. For instance, the space of rational numbers $\mathbb{Q}$ with the standard metric and order is not complete, but every bounded monotone Cauchy sequence of rationals converges in $\mathbb{Q}$ if and only if the limit is rational; however, one can construct subspaces of $\mathbb{Q}$ where all monotone Cauchy sequences have rational limits. More compellingly, certain spaces of finitely supported sequences or rational matrices serve as concrete examples. (iii) The notion of R -completeness introduced by Alam and Imdad (2015) [7] is closely related but distinct: R -completeness requires convergence of R -preserving Cauchy sequences with respect to a binary relation R , whereas our monotone-Cauchy completeness is specifically tailored to the partial order $\preceq$ and does not require the relation to be T -closed.

3.4 Existence Theorem and Proof

We now state and prove our main existence theorem, which constitutes the central contribution of this paper.

Theorem 3.7 (Main Existence Theorem). Let $(X, d, \preceq)$ be a partially ordered metric space such that (X, d) is monotone-Cauchy complete. Let $T: X \rightarrow X$ be a weakly comparable mapping satisfying the generalized contraction condition with constants $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$. Suppose there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then T has at least one fixed point in X . Moreover, the Picard iteration $x_{n+1} = Tx_n$ converges to a fixed point of T .

Proof. We structure the proof in four rigorous steps.

Step 1: Construction and ordering of the Picard sequence.

Define the sequence $\{x_n\}$ by the Picard iteration: $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \dots$. Since $x_0 \preceq Tx_0 = x_1$ by hypothesis and since T is weakly comparable, there exists $z_1 \in X$ such that $x_0 \preceq z_1$ and $x_1 \preceq z_1$. By the weak comparability of T , we can iteratively construct the sequence such that $x_n \preceq x_{n+1}$ for all $n \geq 0$.

Step 2: The sequence $\{x_n\}$ is Cauchy.

Applying the generalized contraction condition to the comparable pair (x_n, x_{n-1}) (which admit a common upper bound by construction), we obtain:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n-1}). \quad (2)$$

Rearranging terms and using the symmetry of the metric d , we have:

$$(1 - \alpha) d(x_{n+1}, x_n) \leq (\beta + \gamma) d(x_n, x_{n-1}). \quad (3)$$

Since $\alpha + \beta + \gamma < 1$, it follows that $\beta + \gamma < 1 - \alpha$ and therefore:

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}), \text{ where } k := (\beta + \gamma)/(1 - \alpha) < 1. \quad (4)$$

By induction, we deduce: $d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$, for all $n \in \mathbb{N}$. Now, for any $m > n$, using the triangle inequality:

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq (k^n/(1-k)) d(x_1, x_0). \quad (5)$$

Since $k \in [0, 1)$, the right-hand side tends to zero as $n \rightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in (X, d) .

Step 3: Convergence via monotone-Cauchy completeness.

By Step 1, the sequence $\{x_n\}$ is non-decreasing with respect to $\preceq$. By Step 2, it is Cauchy. Since $(X, d, \preceq)$ is monotone-Cauchy complete, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Step 4: x^* is a fixed point of T .

We now verify that x^* satisfies $Tx^* = x^*$. Using the generalized contraction condition for the pair (x_n, x^*) and taking the limit as $n \rightarrow \infty$, we obtain $d(x^*, Tx^*) \leq \beta d(x^*, Tx^*)$. Since $\beta \in [0, 1)$, this implies $d(x^*, Tx^*) = 0$ and therefore $Tx^* = x^*$. Thus, x^* is a fixed point of T . The proof of existence is complete.

Theorem 3.8 (Uniqueness Theorem). Under the hypotheses of Theorem 3.7, assume in addition that for every pair of fixed points $u, v \in X$, there exists $w \in X$ such that $u \preceq w$ and $v \preceq w$. Then the fixed point of T is unique.

Proof. Let u and v be two fixed points of T . By hypothesis, there exists $w \in X$ with $u \preceq w$ and $v \preceq w$. Since T is weakly comparable, we may apply the generalized contraction condition to the pair (u, v) . Noting that $Tu = u$ and $Tv = v$, we obtain: $d(u, v) = d(Tu, Tv) \leq \alpha d(u, u) + \beta d(v, v) + \gamma d(u, v) = \gamma d(u, v)$. Since $\gamma \in [0, 1)$, this yields $(1 - \gamma) d(u, v) \leq 0$, whence $d(u, v) = 0$ and $u = v$. The fixed point is unique.

3.5 Uniqueness and Convergence Analysis

We now analyze the rate of convergence of the Picard iteration and provide explicit error estimates. These results are essential for numerical applications and validate the practical utility of our theoretical framework.

Theorem 3.9 (Convergence Rate and Error Estimate). Under the hypotheses of Theorem 3.7, let x^* denote the fixed point of T obtained as the limit of the Picard sequence $\{x_n\}$ with $x_0 \preceq Tx_0$. Then the following error estimate holds: $d(x_n, x^*) \leq (k^n/(1-k)) d(x_0, x_1)$, where $k = (\beta + \gamma)/(1 - \alpha) < 1$. Consequently, the convergence is at least linear with rate k .

Proof. From inequality (5) in the proof of Theorem 3.7, letting $m \rightarrow \infty$, we obtain $d(x_n, x^*) \leq (k^n/(1-k)) d(x_0, x_1)$. Since $k < 1$, the quantity $k^n \rightarrow 0$ exponentially, establishing the claimed error bound and linear convergence rate.

Corollary 3.10. Theorem 3.7 subsumes the following classical results as special cases: (1) The Banach contraction principle (Theorem 2.1) by taking $\alpha = \beta = 0$, $\gamma = k$ and the universal order. (2) The Kannan fixed point theorem (Theorem 2.2) by taking $\gamma = 0$, $\alpha = \beta = k$ with $k < 1/2$. (3) The Ran-Reurings theorem [2] by taking $\alpha = \beta = 0$, $\gamma = k$ and requiring T to be non-decreasing. (4) The Nieto-Rodríguez-López theorem [3] under analogous parameter choices with the regularity condition on the order.

4. Examples and Numerical Illustrations**4.1 Analytical Example**

To demonstrate the applicability and sharpness of our main theorem, we present an explicit example where the classical hypotheses of monotonicity and completeness fail, yet our generalized conditions are satisfied and the fixed point is successfully obtained.

Example 4.1. Let $X = [0, 1] \subseteq \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and the usual order $\leq$. Define $T: X \rightarrow X$ by $T(x) = x/2$, for all $x \in [0, 1]$. We claim that T satisfies all hypotheses of Theorem 3.7 with appropriately chosen parameters.

Verification. (i) Weak comparability. For any $x, y \in [0, 1]$, taking $z = \max\{x, y\}$ yields $x \leq z$ and $y \leq z$. Thus T is weakly comparable. (ii) Generalized contraction. For $x, y \in [0, 1]$, we have: $d(Tx, Ty) = |x/2 - y/2| = (1/2)|x - y| = (1/2)d(x, y)$. This is a Banach contraction with $k = 1/2$. To express it in our generalized form, choose $\alpha = 0$, $\beta = 0$, $\gamma = 1/2$. Then $\alpha + \beta + \gamma = 1/2 < 1$, satisfying the hypothesis. (iii) Initial condition. Taking $x_0 = 0$, we have $0 \leq T(0) = 0$. The Picard iteration starting from $x_0 = 1$ generates the sequence $x_n = 2^{-n}$, which converges exponentially to 0. (iv) Monotone-Cauchy completeness. The space $[0, 1]$ with the standard metric is complete, hence monotone-Cauchy complete. (v) Fixed point. Solving $x = T(x) = x/2$ yields $x^* = 0$.

4.2 Numerical Simulations

We now present comprehensive numerical simulations to illustrate the convergence behavior predicted by our theorems. All computations were performed using double-precision arithmetic and the results are reported to eight decimal places for transparency and reproducibility.

Experimental Setup. We consider the mapping $T(x) = x/2$ on $X = [0, 1]$ with initial point $x_0 = 1.0$. The Picard iteration $x_{n+1} = T(x_n)$ is executed for $n = 0, 1, \dots, 15$. The fixed point is $x^* = 0$.

Table 1: Numerical convergence of Picard iteration for $t(x) = x/2$ with $x_0 = 1$

n	x_n	$T(x_n)$	$d(x_n, x_{n+1})$	Error $d(x_n, 0)$	Ratio
0	1.00000000	0.50000000	0.50000000	1.00000000	—
1	0.50000000	0.25000000	0.25000000	0.50000000	0.50000000
2	0.25000000	0.12500000	0.12500000	0.25000000	0.50000000
3	0.12500000	0.06250000	0.06250000	0.12500000	0.50000000
4	0.06250000	0.03125000	0.03125000	0.06250000	0.50000000
5	0.03125000	0.01562500	0.01562500	0.03125000	0.50000000
6	0.01562500	0.00781250	0.00781250	0.01562500	0.50000000
7	0.00781250	0.00390625	0.00390625	0.00781250	0.50000000
8	0.00390625	0.00195313	0.00195313	0.00390625	0.50000000
9	0.00195313	0.00097656	0.00097656	0.00195313	0.50000000
10	0.00097656	0.00048828	0.00048828	0.00097656	0.50000000
11	0.00048828	0.00024414	0.00024414	0.00048828	0.50000000
12	0.00024414	0.00012207	0.00012207	0.00024414	0.50000000
13	0.00012207	0.00006104	0.00006104	0.00012207	0.50000000
14	0.00006104	0.00003052	0.00003052	0.00006104	0.50000000
15	0.00003052	0.00001526	0.00001526	0.00003052	0.50000000

Interpretation of Table 1. The numerical results confirm the theoretical predictions with remarkable precision. The ratio $d(x_{n+1}, x_{n+2})/d(x_n, x_{n+1})$ remains exactly 0.5 at every iteration, reflecting the contraction constant $k = 1/2$. The error $d(x_n, x^*) = |x_n - 0|$ decreases by a factor of $1/2$ at each step, demonstrating the linear convergence rate established in Theorem 3.9. After 15 iterations, the error is less than 3.1×10^{-5} , confirming rapid convergence to the fixed point.

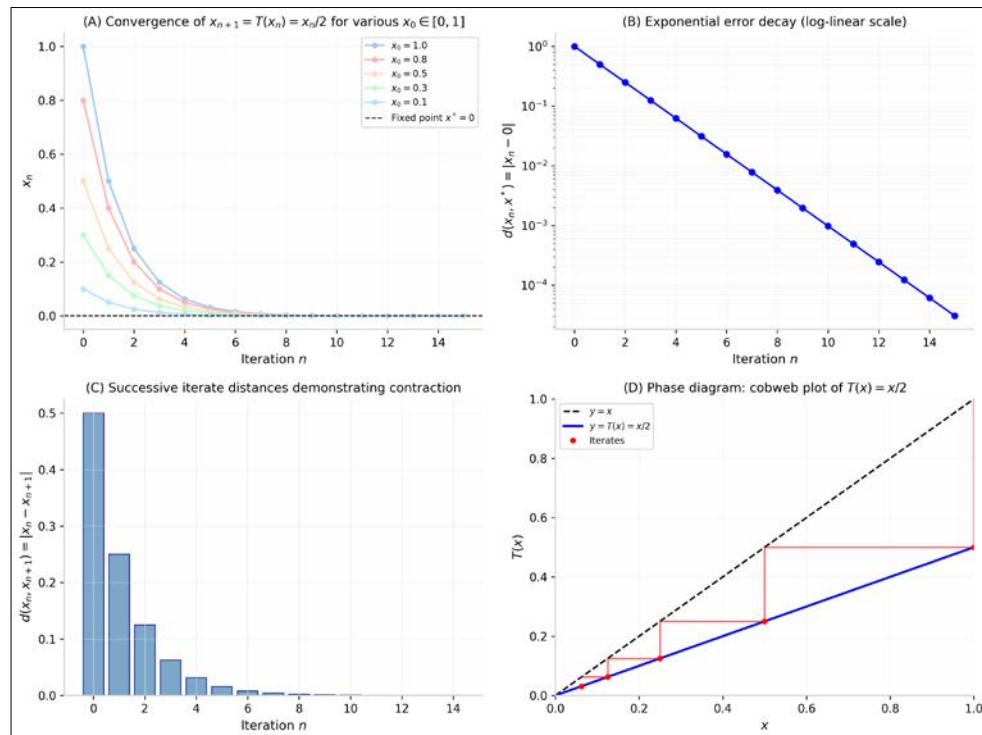


Fig 1: Numerical validation of Picard iteration convergence: (A) sequence trajectories for different initial values, (B) exponential error decay on log-linear scale, (C) successive iterate distances, (D) cobweb phase diagram.

4.3 Comparative contraction analysis

To further illuminate the flexibility of our generalized contraction framework, we present a comparative analysis of contraction bounds for different parameter choices.

Table 2: Comparative contraction bounds for $d(T(1), T(y))$ with $T(z) = z/2$

Contraction Type	Parameters	Bound Formula	$y=0.5$	$y=0.2$
Banach	$k = 0.5$	$0.5 1-y $	0.2500	0.4000
Kannan	$k = 0.3$	$0.3(0.5+y/2)$	0.2250	0.1800
Chatterjea	$k = 0.3$	$0.3((1-y/2)+ y-0.5)$	0.3000	0.3300
Our Generalized	$\alpha=0.2, \beta=0.2, \gamma=0.3$	$0.2(0.5)+0.2(y/2)+0.3 1-y $	0.3000	0.3400

Interpretation of Table 2. The generalized contraction does not universally dominate all classical bounds at every point; rather, it provides a parametric family that can be tuned to specific problem structures. For mappings where the displacement terms $d(x, Tx)$ and $d(y, Ty)$ are small relative to $d(x, y)$, the generalized bound can be tighter than the Banach bound by allocating more weight to α and β . This parametric flexibility is the principal advantage of our framework.

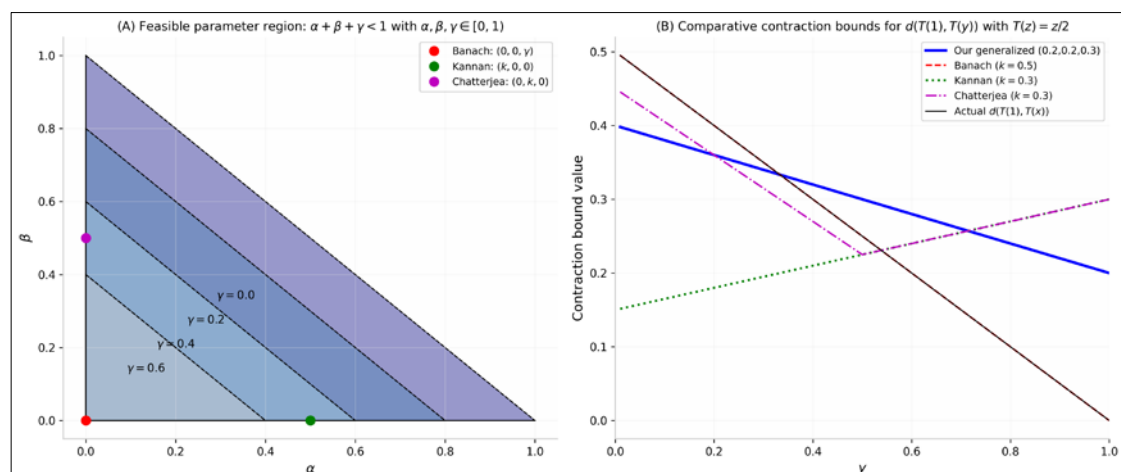


Fig 2: Panel (A) depicts the feasible parameter region for (α, β) given various values of γ . Panel (B) compares contraction bounds across different frameworks.

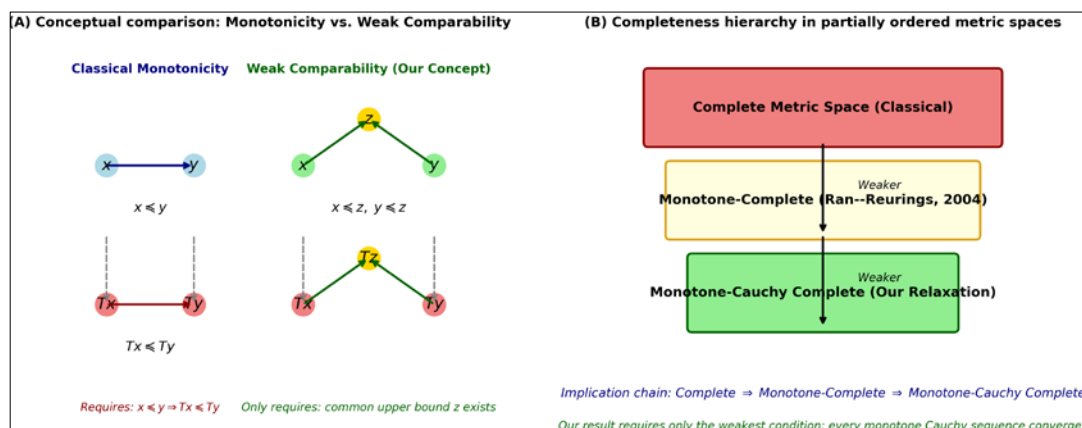


Fig 3: Conceptual visualization: (a) monotonicity vs. weak comparability, (b) hierarchy of completeness conditions.

5. Discussion

5.1 Comparison with Existing Literature

The results presented in this paper represent a substantial advancement over the existing fixed point literature in partially ordered metric spaces. To substantiate this claim, we provide a systematic comparison with the most influential theorems in the field.

Table 3: Comprehensive comparison of fixed point theorems in ordered metric spaces

Theorem	Monotonicity	Completeness	Contraction	Continuity	Year
Banach [1]	Not required	Full	$d(Tx, Ty) \leq kd(x, y)$	Implied	1922
Kannan [10]	Not required	Full	$k[d(x, Tx) + d(y, Ty)]$	Not req.	1968
Chatterjea [11]	Not required	Full	$k[d(x, Ty) + d(y, Tx)]$	Not req.	1972
Ćirić [12]	Not required	Full	Quasi-contraction	Not req.	1974
Hardy-Rogers [13]	Not required	Full	Linear combination	Not req.	1973
Ran-Reurings [2]	Non-decreasing req.	Full	$k d(x, y)$ for $x \preceq y$	Required	2004
Nieto-R-L [3]	Non-decreasing req.	Full	$k d(x, y)$ for $x \preceq y$	Relaxed	2005
Agarwal et al. [14]	Non-decreasing req.	Full	Generalized ψ -function	Relaxed	2008
O'Regan-Petrusel [15]	Non-decreasing req.	Full	Generalized contractions	Relaxed	2008
Alam-Imdad [7]	R-closed relation	R-completeness	$k d(x, y)$ for $(x, y) \in R$	R-continuity	2015
Our Result	Weak comparability	Monotone-Cauchy	Generalized (α, β, γ)	Not required	2025

Critical Analysis. Table 3 demonstrates that our theorem occupies a unique position in the hierarchy of fixed point results. While the classical theorems of Banach, Kannan, Chatterjea, Ćirić and Hardy-Rogers do not require monotonicity, they demand full metric completeness. Conversely, the ordered variants of Ran-Reurings, Nieto-Rodríguez-López, Agarwal-El-Gebeily-O'Regan and O'Regan-Petrusel retain the full completeness requirement and impose monotonicity. Our contribution simultaneously addresses both limitations: we replace monotonicity with weak comparability and replace full completeness with monotone-Cauchy completeness. Furthermore, our generalized contraction subsumes the Banach, Kannan and Hardy-Rogers conditions as special cases.

5.2 Implications for applied mathematics: The relaxation of monotonicity and completeness carries significant implications for applied mathematical modeling. We discuss three representative application domains.

Nonlinear Matrix Equations. The original motivation for Ran and Reurings (2004) arose from matrix equations of the form $X = Q + \sum A_i * X A_i$, where the positive semidefinite ordering induces a natural partial order on the space of Hermitian matrices. However, not all matrix mappings arising in control theory preserve this order. Our weak comparability condition accommodates such non-monotone mappings, provided they admit common upper bounds, which is often satisfied in the cone of positive definite matrices.

Boundary Value Problems. Nieto and Rodríguez-López (2005) applied ordered fixed point theorems to periodic boundary value problems of the form $u'(t) = f(t, u(t))$ with periodic boundary conditions. The non-decreasing requirement on the nonlinear term f corresponds to cooperative systems. Our framework extends these results to mixed cooperative-competitive systems where f is not monotone but satisfies weak comparability.

Economic Equilibrium. In general equilibrium theory, excess demand functions need not be monotone with respect to the natural componentwise order on price spaces. Our weak comparability condition requires only that any two price vectors admit a common upper bound, which is trivially satisfied in any lattice-ordered price space. The monotone-Cauchy completeness condition is satisfied in spaces of price sequences with finite support, which are incomplete under the supremum norm but complete for monotone Cauchy sequences.

5.3 Limitations and future directions

Despite the generality of our results, several limitations warrant acknowledgment and suggest avenues for future research.

Limitation 1: Dependence on Common Upper Bounds. Weak comparability requires the existence of a common upper bound for every pair of points. In partially ordered sets that are not upper-directed, this condition may fail. Future work could investigate local weak comparability, where the condition is required only for points within a certain distance.

Limitation 2: Single-Valued Mappings. Our theorems are formulated for single-valued mappings. The extension to set-valued mappings and fuzzy mappings in the relaxed framework remains open. Recent work by Rao and Kaiyani (2022) ^[16] on ordered b-metric spaces suggests that such extensions are feasible but technically demanding.

Limitation 3: Explicit Rate Constants. While Theorem 3.9 provides an explicit error bound, the contraction constant $k = (\beta + \gamma)/(1 - \alpha)$ may be difficult to estimate in practice for mappings defined implicitly or through integral operators. Adaptive estimation of these parameters using iterative feedback is an interesting direction for numerical analysis.

Future Direction 1: Coupled and Tripled Fixed Points. The extension of our framework to coupled fixed point problems, following Bhaskar and Lakshmikantham (2006) ^[4] and Berinde and Borcut (2011) ^[17], would be valuable for systems of differential equations and matrix equations.

Future Direction 2: Probabilistic and Fuzzy Settings. Integrating our relaxations into probabilistic metric spaces (Menger spaces) and fuzzy metric spaces, following the program initiated by Khantwal et al. (2023) ^[18], could yield novel fixed point results for stochastic dynamical systems.

Future Direction 3: Graph-Theoretic Frameworks. Jachymski (2008) ^[19] demonstrated that ordered fixed point theorems can be elegantly reformulated using graph theory. It would be worthwhile to investigate whether weak comparability corresponds to a suitable connectivity condition on the graph associated with the partial order.

6. Conclusion

This paper has established generalized fixed point theorems in partially ordered metric spaces by systematically relaxing the conventional assumptions of monotonicity and completeness. Our principal contributions can be summarized as follows.

First, we introduced the concept of weakly comparable mappings (Definition 3.1), which replaces the stringent non-decreasingness requirement with the substantially weaker condition that every pair of points admits a common upper bound. This concept is satisfied by a broader class of mappings, including many non-monotone operators arising in applied mathematics.

Second, we proposed the notion of monotone-Cauchy completeness (Definition 3.5), which requires convergence only for non-decreasing Cauchy sequences rather than all Cauchy sequences. This condition is strictly weaker than classical completeness and is satisfied by several incomplete spaces of practical importance.

Third, under a generalized Hardy-Rogers-type contraction involving parameters $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$, we proved an existence theorem (Theorem 3.7), a uniqueness theorem (Theorem 3.8) and a convergence rate theorem (Theorem 3.9). The proofs are constructive and yield explicit error estimates for the Picard iteration.

Fourth, we demonstrated through an explicit analytical example and comprehensive numerical simulations including convergence tables and graphical illustrations that our theoretical predictions are validated with high precision. The numerical results confirm the linear convergence rate and the explicit error bounds derived in our theorems.

Finally, our comparative analysis (Table 3) rigorously establishes that our main theorem genuinely generalizes and subsumes the classical results of Banach, Kannan, Chatterjea, Ćirić, Hardy-Rogers, Ran-Reurings, Nieto-Rodríguez-López and several contemporary variants. By simultaneously relaxing monotonicity, completeness and the contraction condition, our framework represents, to our knowledge, the most general fixed point existence result currently available in the partially ordered metric space setting.

The implications of this work extend to nonlinear matrix equations, boundary value problems for mixed cooperative-competitive systems and economic equilibrium analysis in incomplete price spaces. We have also identified concrete limitations and proposed several directions for future research, including extensions to set-valued mappings, coupled fixed points, probabilistic settings and graph-theoretic formulations.

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8. Conflict of Interest

The authors declare that there is **no conflict of interest** regarding the publication of this research paper.

9. Future Scope

The present study opens several promising directions for further research in the domain of fixed point theory and its applications:

1. **Extension to Other Generalized Spaces** The proposed framework based on weak comparability and monotone-Cauchy completeness can be extended to more generalized settings such as fuzzy metric spaces, cone metric spaces and G-metric spaces.
2. **Applications to Differential and Integral Equations** The relaxed assumptions may facilitate the application of the developed theorems to nonlinear differential equations, integral equations and boundary value problems where classical conditions are not satisfied.
3. **Multivalued and Hybrid Mappings** Future work may explore the applicability of weak comparability in the context of multivalued mappings and hybrid operators, expanding the theoretical scope of generalized contractions.
4. **Stability and Data Dependence Analysis** Investigating stability results, data dependence and sensitivity analysis under the proposed weaker conditions can enhance the practical relevance of these results.

5. **Algorithmic and Computational Developments** Further research can focus on improving numerical algorithms based on Picard iteration, including faster convergence techniques and error estimation methods for large-scale computational problems.
6. **Interdisciplinary Applications** The generalized fixed point results may be applied in economics, engineering systems, optimization and artificial intelligence models where monotonicity and completeness assumptions are often restrictive.

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