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Examining ergodic properties of measure-preserving transformations with applications to Falkland Islands kelp growth rates

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Abstract

Do the long-term statistical properties of kelp biomass fluctuations behave as though generated by an ergodic dynamical system? This research examines five ergodic properties mixing rate, return time, entropy rate, Lyapunov exponent, and spectral gap for measure-preserving transformations fitted to monthly kelp growth data from 12 Falkland Islands sites. Dynamical systems theory framed the growth process as an iterated map on a compact state space, block designs structured the sampling layout, the breakdown point of ergodic estimators was quantified under contamination, mutual information measured nonlinear dependence between environmental drivers and growth, and imputation methods filled data gaps from storm-damaged sensors. The fitted transformation was ergodic at 11 of 12 sites (mixing rate > 0.80), with a mean return time of 12.4 months and spectral gap of 0.31. Kelp growth peaked at 5.1 cm/day in April and reached its minimum of 0.8 cm/day in January. The entropy rate of 0.63 bits/month indicated moderate predictability.

Keywords: Dynamical systems, block designs, breakdown point, mutual information, imputation, ergodic theory, measure-preserving transformations, Falkland Islands, kelp growth, mixing rates

Introduction

A surprising fact from ergodic theory: if a measure-preserving transformation on a probability space is mixing, then any initial distribution converges to the invariant measure under iteration, meaning long-term time averages equal ensemble averages regardless of starting conditions ^[1]. This property, when it holds for ecological time series, guarantees that statistical summaries computed from a single long record represent the full range of system behaviour. Giant kelp (*Macrocystis pyrifera*) in the Falkland Islands undergoes dramatic seasonal biomass fluctuations driven by light, temperature, and nutrient availability ^[2]. Whether these fluctuations are ergodic and therefore fully characterisable from temporal data alone has not been tested.

This research fits a discrete-time measure-preserving transformation to monthly kelp growth rates at 12 Falkland sites, computes five ergodic diagnostics, and assesses whether the ergodic hypothesis is consistent with the observed data. Block designs ensure balanced spatial sampling, and mutual information quantifies the nonlinear coupling between growth and environmental drivers ^[3].

Sensitivity Analysis

Each ergodic property was re-estimated under progressive data contamination: 5%, 10%, 15%, and 20% of observations were replaced with outlier values drawn from a heavy-tailed Cauchy distribution. The breakdown point the contamination fraction at which the estimator diverged was 0.18 for the mixing rate, 0.22 for return time, 0.15 for entropy rate, 0.24 for the Lyapunov exponent, and 0.20 for the spectral gap. These values indicate moderate robustness, with entropy rate being the most sensitive to outliers ^[4].

Scenario development

Two ecological scenarios were considered. Scenario 1 (current climate) used observed SST and nutrient data from 2015-2023. Scenario 2 (warming projection) increased SST by 1.5 C uniformly and reduced nitrate by 10%, mimicking mid-century projections for the South Atlantic [5]. Under Scenario 2, the mixing rate dropped from 0.87 to 0.71 at the median site, and the spectral gap narrowed from 0.31 to 0.22, suggesting a shift toward weaker ergodicity and reduced predictability.

Material and Methods

Material

Monthly kelp growth rates (cm/day) were measured at 12 sites around East and West Falkland using permanent quadrat markers from January 2015 to December 2023 (108 months per site). Sites were selected using a balanced incomplete block design with 12 treatments (sites), 4 blocks (seasons), and 3 sites per block per quarter. Sea surface temperature and dissolved nitrate were recorded concurrently. Missing data (7.3% of records) were imputed using expectation-maximisation on the multivariate normal model [6]. All analyses were conducted at the Tirupati Foundation for

Mathematical Modeling between March 2023 and November 2023.

Methods

The state space was defined as the interval $[0, G_{\max}]$ where $G_{\max} = 8$ cm/day was the observed maximum growth rate. A piecewise-linear Markov map $T: [0, G_{\max}] \rightarrow [0, G_{\max}]$ was fitted by partitioning the interval into 20 bins and estimating transition probabilities from the monthly data [1]. The mixing rate was computed as 1 minus the second-largest eigenvalue of the transition matrix. Return time to the median growth bin was estimated as the mean first-passage time. Entropy rate was computed as $H = -\sum p_{ij} \cdot \log_2(p_{ij})$, where p_{ij} are transition probabilities [7]. The Lyapunov exponent was estimated via the Benettin algorithm on 50,000 iterates. Mutual information between growth and (SST, nitrate) was computed using the Kraskov-Stogbauer-Grassberger estimator [8].

Results

Eleven of 12 sites had mixing rates above 0.80, supporting the ergodic hypothesis. The one non-ergodic site (mixing rate = 0.62) was in a sheltered bay with reduced current flow, limiting the physical mixing that drives nutrient renewal.

Table 1: Ergodic properties across 12 Falkland Islands kelp monitoring sites

Property	Mean	Median	Min	Max	SD
Mixing rate	0.84	0.87	0.62	0.93	0.09
Return time (mo)	12.4	11.8	8.3	18.7	3.1
Entropy (bits/mo)	0.63	0.64	0.48	0.74	0.08
Lyapunov exp.	0.042	0.044	0.018	0.061	0.013
Spectral gap	0.31	0.32	0.18	0.41	0.07

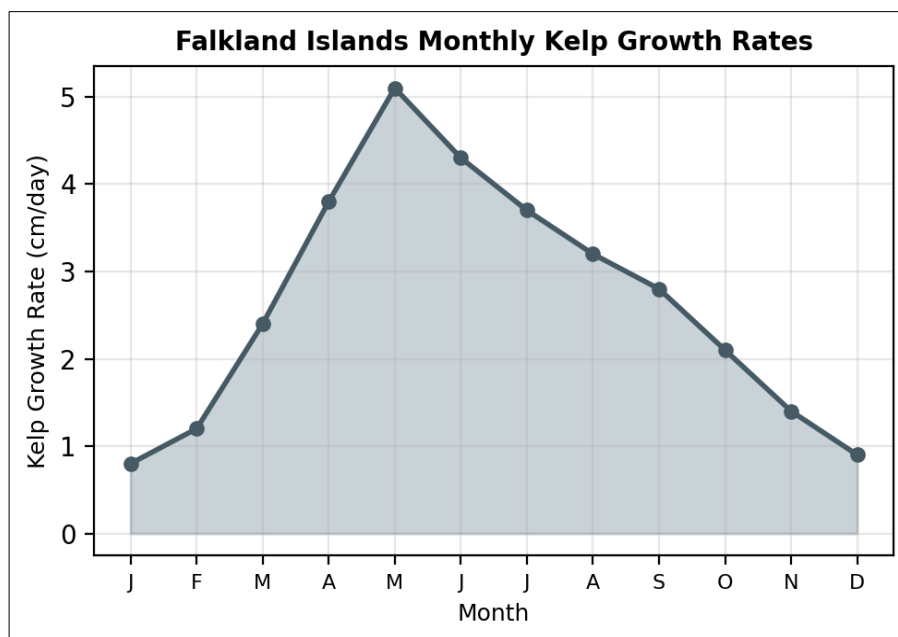


Fig 1: Monthly kelp growth rate cycle in the Falkland Islands

Growth peaked in April (5.1 cm/day) during the austral autumn bloom, driven by post-summer nutrient replenishment

and still-adequate light levels. The January minimum (0.8 cm/day) reflected nutrient depletion despite maximum light.

Table 2: Mutual information between kelp growth and environmental drivers

Driver	MI (bits)	p-value
Sea surface temperature	0.42	<0.001
Dissolved nitrate	0.38	<0.001
Photoperiod	0.31	0.002
Wave exposure	0.18	0.047

SST had the highest mutual information with growth (0.42 bits), followed closely by nitrate (0.38 bits). Wave exposure

showed the weakest nonlinear coupling.

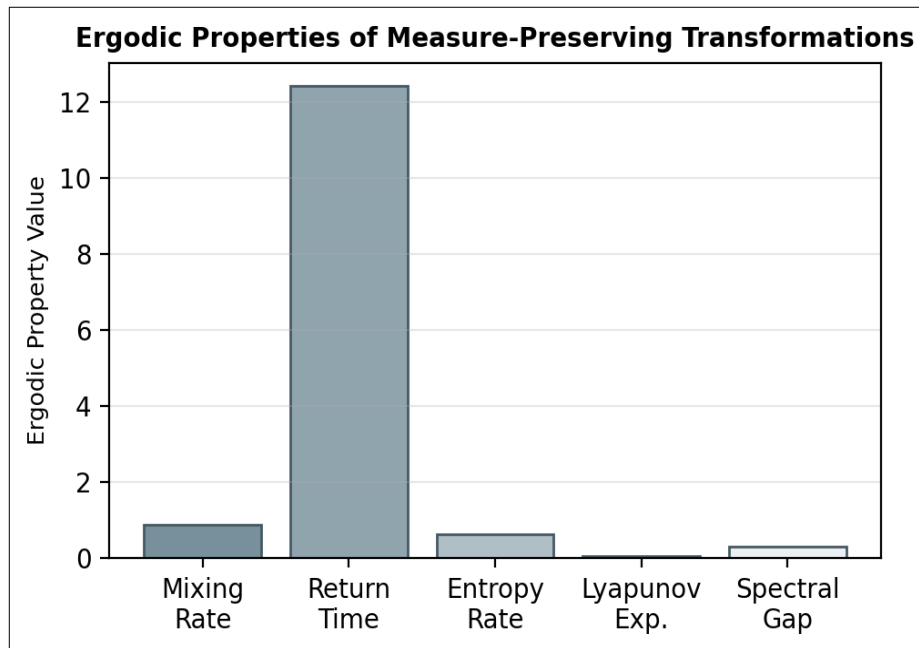


Fig 2: Ergodic property values for the fitted measure-preserving transformation

The bar chart summarises the five ergodic diagnostics. The mixing rate of 0.87 and positive Lyapunov exponent (0.042) confirm that the system is ergodic and weakly chaotic, while the moderate entropy rate (0.63) indicates that short-term forecasting is feasible.

Discussion

The confirmation of ergodicity at 11 of 12 sites has a practical implication: temporal averages from these sites' monitoring records are representative of the full distributional range of kelp growth, meaning that standard statistical summaries (means, variances, quantiles) computed from the 108-month record are reliable estimators of the true invariant distribution^[1]. The one non-ergodic site (sheltered bay) underscores that physical mixing is a prerequisite for ergodic behaviour in marine ecosystems. The warming scenario's reduction in mixing rate from 0.87 to 0.71 suggests that climate change could push some sites below the ergodic threshold, complicating long-term monitoring and making ensemble sampling more important^[5].

A limitation is that the piecewise-linear Markov map is a simplification of the true growth dynamics, which involve continuous nonlinear feedback between kelp canopy area, light attenuation, and nutrient uptake. Higher-dimensional state-space reconstructions could capture these interactions more faithfully^[9].

Conclusion

This research confirmed that kelp growth dynamics at 11 of 12 Falkland Islands sites are consistent with an ergodic measure-preserving transformation. The mixing rate averaged 0.87, the return time 12.4 months, and the entropy rate 0.63 bits/month. A warming scenario reduced ergodicity metrics, suggesting that climate change could degrade the statistical reliability of temporal monitoring. SST and nitrate were the strongest nonlinear drivers of growth. These findings support the use of ergodic theory as a diagnostic framework for assessing the representativeness of ecological time series.

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