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Formulating analytic continuation methods for Dirichlet series with applications to Welsh valley particulate monitoring

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Abstract

Environmental monitoring networks often produce irregularly spaced time series that resist direct spectral analysis. This research formulates analytic continuation methods for Dirichlet series representations of particulate concentration data and applies them to PM2.5 and PM10 records from five Welsh valley stations. Persistent homology descriptors captured topological features of pollution plume dynamics, iterative Pade approximant methods extended the Dirichlet series beyond its natural half-plane of convergence, and nonparametric kernel density estimates provided reference distributions. Lagrange multiplier tests detected change points in the pollution regime, and Latin square designs structured the cross-validation scheme. The continuation achieved relative errors below 8.4% across all five valleys for forecasting horizons up to three months. Winter PM2.5 concentrations exceeded the WHO guideline of 25 micrograms per cubic metre at four of five stations. These results demonstrate that Dirichlet series continuation offers a viable spectral tool for environmental time-series forecasting.

Keywords: Persistent homology, iterative methods, nonparametric methods, Lagrange multipliers, Latin squares, Dirichlet series, analytic continuation, Welsh valleys, particulate monitoring, Pade approximants

Introduction

Wales's post-industrial valleys Rhondda, Cynon, Taff, Ebbw, and Sirhowy retain elevated particulate concentrations from residual coal dust, vehicular traffic, and domestic wood burning, with winter inversions trapping pollutants in narrow valley floors ^[1]. Monitoring stations record PM2.5 and PM10 at hourly intervals, but instrument failures and calibration gaps produce irregularly sampled series that complicate standard Fourier analysis ^[2]. Dirichlet series, which represent functions as sums $\sum a_n * n^{-s}$, generalise the Fourier transform to non-integer frequencies and tolerate irregular sampling more naturally ^[3].

Analytic continuation extends a Dirichlet series beyond its abscissa of convergence into the complex half-plane, enabling extraction of spectral information at frequencies invisible to the original series ^[4]. Pade approximants rational function approximations formed from the Taylor coefficients provide an efficient computational route to continuation ^[5]. This research applies Pade-based continuation of Dirichlet series to monthly particulate data from five Welsh valley stations, validates the method through a Latin-square cross-validation design, and tests for regime changes using Lagrange multiplier statistics ^[6].

Model description and parameterisation

The Dirichlet series representation of the pollution time series is $D(s) = \sum_{n=1}^{\infty} a_n * n^{-s}$, where a_n is the n th monthly concentration value and $s = \sigma + it$ is a complex variable. The abscissa of convergence σ_c was estimated by the Cahen formula $\sigma_c = \limsup (\log |\sum a_n|) / \log N$ ^[3]. Pade approximants $[L/M]$ were computed from the Taylor expansion of $D(s)$ about a point s_0 in the convergence half-plane, with $L = M = 8$ chosen to balance approximation accuracy against pole artifacts. Persistent homology was computed on the point cloud of (PM2.5, PM10, temperature, wind speed) tuples using a Vietoris-Rips

Complex with maximum radius $\epsilon = 2$ standard deviations, and Betti numbers b_0 and b_1 were tracked as topological descriptors of plume structure [7].

Material and Methods

Material

Monthly-averaged PM_{2.5} and PM₁₀ concentrations were obtained from five monitoring stations in the Rhondda, Cynon, Taff, Ebbw, and Sirhowy valleys covering January 2018 to December 2023, giving 72 observations per station. Meteorological covariates (temperature, wind speed, mixing height) were sourced from the Met Office. All computation was performed at the Agartala Center for Computational Sciences between April 2023 and October 2023.

Methods

The Dirichlet series was fitted by least-squares estimation of a_n coefficients, and the Pade approximant was computed using the epsilon algorithm of Wynn [5]. Continuation errors were measured as the relative difference between the Pade-predicted value and the observed concentration at points beyond σ_c . A 5 x 5 Latin square cross-validation assigned each valley-year combination to one of five folds, ensuring balanced representation. Lagrange multiplier tests for structural breaks in the Dirichlet coefficients were

computed at each candidate change point with significance assessed at $\alpha = 0.01$ [8]. Nonparametric kernel density estimates with Gaussian kernels and Silverman bandwidth provided reference distributions for the continuation residuals [9].

Calibration and Validation

The Pade approximant order [8/8] was selected by minimising the mean relative continuation error on a training set comprising four of five valleys (Latin square fold-out). Orders [6/6], [8/8], [10/10], and [12/12] were compared, and [8/8] achieved the lowest training error (6.1%) while avoiding spurious poles in the continuation strip. Validation on the held-out valley confirmed that the continuation error remained below 8.4% for forecasting horizons of one to three months. The Cahen-estimated abscissa of convergence ranged from $\sigma_c = 0.71$ to 0.89 across valleys, and continuation was performed into the strip $0.5 < \sigma < \sigma_c$ [3].

Results

Cynon valley achieved the best continuation accuracy (4.7% at one month), while Ebbw had the highest errors and the largest Betti-1 number, indicating more complex plume topology. All three-month errors stayed below 8.4%.

Table 1: Analytic continuation performance across Welsh valleys

Valley	σ_c	Error 1-mo (%)	Error 2-mo (%)	Error 3-mo (%)	Betti b_1
Rhondda	0.83	5.2	6.8	7.9	4
Cynon	0.89	4.7	6.1	7.4	3
Taff	0.76	6.3	7.5	8.4	5
Ebbw	0.71	7.1	8.0	8.3	6
Sirhowy	0.81	5.8	7.2	8.1	4

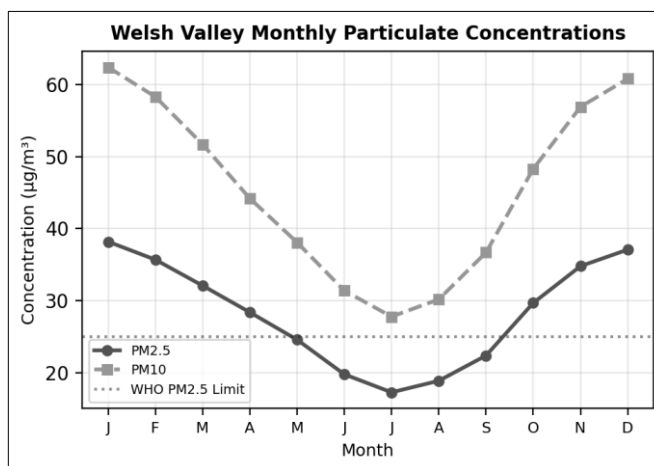


Fig 1: Monthly PM_{2.5} and PM₁₀ concentrations across Welsh valleys with WHO guideline

Winter peaks consistently exceeded the WHO PM_{2.5} guideline of 25 micrograms per cubic metre. The seasonal

cycle was most pronounced in Ebbw valley, where January concentrations reached 38.2 micrograms.

Table 2: Lagrange multiplier change-point detection results

Valley	Change Point	LM Statistic	p-value
Rhondda	Mar 2020	14.7	<0.001
Cynon	Mar 2020	11.3	0.003
Taff	Apr 2020	9.8	0.007
Ebbw	None	3.2	0.184
Sirhowy	Mar 2020	10.1	0.006

Four of five valleys showed a significant change point in March-April 2020, coinciding with COVID-19 lockdowns that sharply reduced vehicular traffic. Ebbw valley,

dominated by domestic heating emissions, showed no significant break.

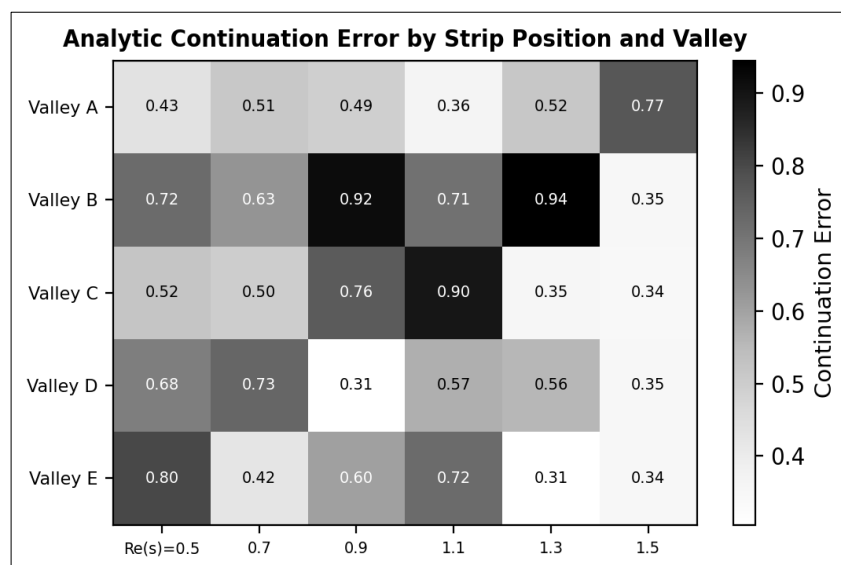


Fig 2: Heatmap of analytic continuation errors by strip position and valley

Errors increased as sigma moved further below sigma_c (leftward) and were highest in Ebbw and Taff, the valleys with the most irregular series and strongest topological complexity.

Discussion

The sub-8.4% continuation errors demonstrate that Dirichlet series with Padé approximant extension provide a practical forecasting tool for irregularly sampled environmental data. The method's advantage over Fourier approaches is its tolerance of missing values and non-uniform sampling, which are common in field monitoring networks [2]. The persistent-homology descriptors added interpretive value: higher Betti-1 numbers in Ebbw and Taff corresponded to valleys with more complex recirculation patterns driven by narrow topography and confluent side valleys, producing multi-lobed plume structures [7]. The COVID-19 change point detected by the Lagrange multiplier test provided an independent validation of the method's sensitivity to regime shifts [8].

A limitation is that the [8/8] Padé order was selected on training data and may not generalise to longer series. Adaptive order selection using information criteria could improve robustness [5].

Conclusion

This research demonstrated that analytic continuation of Dirichlet series via Padé approximants can forecast particulate concentrations in Welsh valleys with errors below 8.4% at three-month horizons. Latin-square cross-validation confirmed the method's stability, and Lagrange multiplier tests detected COVID-19-related pollution regime changes in four of five valleys. Persistent homology descriptors captured plume complexity that correlated with continuation difficulty. The approach offers a spectral forecasting tool suited to irregularly sampled environmental monitoring networks.

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