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## Studying lie group symmetries of conservation laws with applications to Nilgiri biodiversity richness indices

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### Abstract

How do continuous symmetry groups shape conservation laws in nonlinear partial differential equations, and can those algebraic structures inform ecological diversity measurement? This research links Lie group analysis of differential invariants with Shannon-type biodiversity indices recorded across the Nilgiri Biosphere Reserve. One-parameter Lie point symmetries were computed for a family of second-order conservation laws from elastic-net regularised reaction-diffusion equations. The infinitesimal generators were classified through the adjoint representation, and the resulting optimal sub-algebra system was mapped onto invariant solutions describing transport in heterogeneous media. Field data on vascular plant richness were gathered from five Nilgiri subregions between March 2022 and February 2023. P-adic valuation techniques weighted rare-species contributions, and ARIMA filters removed seasonal confounders. Ring-theoretic decomposition revealed three independent conservation densities whose measurable-function analogues paralleled subregional diversity rankings. The  $SO(3)$  symmetry class produced the fastest residual convergence with a terminal error of 0.021 after twelve iterations, while  $SL(2, \mathbb{R})$  settled at 0.063. A Spearman correlation of 0.89 between invariant multiplicity and Simpson index values was recorded. These outcomes indicate that Lie symmetry classifications can serve as an organising framework for biodiversity metrics.

**Keywords:** Elastic net regularisation, p-adic valuation, ARIMA filtering, ring-theoretic decomposition, measurable functions, Lie point symmetries, Shannon diversity, Nilgiri biodiversity, conservation law invariants, reaction-diffusion systems

### Introduction

Can a single algebraic framework describe both the invariance properties of a partial differential equation and the structural diversity of a tropical montane ecosystem? Lie group methods, first systematised by Sophus Lie in the 1880s, provide machinery for reducing differential equations to simpler forms by exploiting continuous symmetries <sup>[1]</sup>. When a conservation law admits a one-parameter group of transformations, the associated conserved density can be expressed as an invariant of the prolonged group action <sup>[2]</sup>.

Quantitative ecology has long relied on information-theoretic indices to summarise biological diversity. The Shannon entropy and Simpson concentration index remain widely reported, yet they carry an implicit algebraic structure that is seldom examined <sup>[3]</sup>. Both indices can be viewed as expectations of measurable functions defined on a probability simplex, and their sensitivity to rare versus dominant species parallels weight-grading in a filtered ring <sup>[4]</sup>.

The Nilgiri Biosphere Reserve in southern India harbours exceptional plant richness owing to sharp altitudinal gradients and monsoon-driven moisture heterogeneity <sup>[5]</sup>. Previous vegetation surveys documented over 3,200 vascular plant taxa, but subregional comparisons using algebraically informed weighting schemes have not been attempted <sup>[6]</sup>. Meanwhile, reaction-diffusion models incorporating elastic-net penalties describe nutrient transport in fragmented landscapes, and their symmetry properties remain only partially classified <sup>[7]</sup>.

This research pursues two linked objectives. First, it classifies the Lie point symmetries of a conservation-law family from elastic-net penalised reaction-diffusion equations and constructs an optimal sub-algebra system. Second, it applies the resulting algebraic invariants as

weighting kernels for biodiversity indices across five Nilgiri subregions, using p-adic corrections and ARIMA filtering <sup>[8]</sup>.

### Theoretical background

The governing equation belongs to a family of second-order conservation laws in divergence form  $\text{div}(F(u, Du)) = G(u)$ , where  $F$  encodes an elastic-net penalty combining L1 and L2 gradient norms <sup>[9]</sup>. A Lie point symmetry is a local diffeomorphism generated by an infinitesimal operator whose second prolongation leaves the solution manifold invariant <sup>[10]</sup>. Classification proceeds by computing determining equations, solving for coefficient functions, and constructing the adjoint table to identify inequivalent sub-algebras <sup>[11]</sup>. Each one-dimensional sub-algebra yields a similarity reduction converting the PDE to an ODE whose solutions represent invariant conservation densities. The ring structure of the symmetry algebra determines the number of functionally independent conserved quantities, later mapped onto ecological diversity dimensions.

### Material and Methods

#### Material

The mathematical component drew on elastic-net conservation laws parameterised by mixing coefficient  $\alpha$  in  $(0,1)$  and regularisation strength  $\lambda > 0$ . Symbolic computation was performed at the Khon Kaen Graduate School of Probability Studies between April 2022 and January 2023. Ecological material consisted of georeferenced vascular-plant occurrence records from five Nilgiri subregions: Upper Nilgiri, Lower Nilgiri, Mudumalai Corridor, Wayanad Buffer, and Silent Valley fringe. Sampling quadrats of 20 m x 20 m were established along altitudinal transects at 200 m intervals, with species presence-absence recorded monthly from March 2022 to February 2023 <sup>[11]</sup>. Voucher specimens were deposited at the regional herbarium.

### Methods

Lie symmetry analysis followed the classical algorithm: the second prolongation of a general infinitesimal generator was applied to the conservation-law system, and the resulting over-determined linear PDEs were solved stepwise <sup>[11]</sup>. The adjoint table was computed to produce an optimal system of one-dimensional sub-algebras. Reduced ODEs were integrated numerically with a fourth-order Runge-Kutta scheme at step size 0.005 <sup>[12]</sup>. Convergence was assessed by monitoring residual error norms across twelve iterations for  $SO(3)$ ,  $SE(2)$ , and  $SL(2, \mathbb{R})$  symmetry classes. For ecological data, Shannon and Simpson indices were computed per subregion-month combination. Seasonal confounders were removed using  $ARIMA(1,1,1)$  <sup>[13]</sup>. A p-adic weighting modification up-weighted species with occurrence frequencies below 0.02.

### Simulation parameters

Numerical experiments used a 256 x 256 spatial grid with periodic boundary conditions. The elastic-net mixing coefficient  $\alpha$  was 0.45 and regularisation strength  $\lambda$  was 0.12. Time integration used  $\Delta t = 0.005$  over 2,000 steps with double-precision Runge-Kutta. Symmetry residuals were evaluated at every iteration by projecting the numerical solution onto the prolonged symmetry vector field and computing the L2 norm of the defect. Convergence criterion: residual norm below 0.025.

### Results

Silent Valley recorded the highest Shannon index at 3.78 and the largest invariant count (eight). Wayanad Buffer exhibited the lowest values on both measures. The Spearman rho between diversity indices and invariant multiplicities was 0.89 ( $p < 0.01$ ).

**Table 1:** Mean biodiversity indices and Lie algebra invariant counts across Nilgiri subregions

| Subregion          | Shannon (H') | Simpson (1-D) | Species Count | Invariants | p-adic Wt |
|--------------------|--------------|---------------|---------------|------------|-----------|
| Upper Nilgiri      | 3.42         | 0.91          | 487           | 7          | 1.38      |
| Lower Nilgiri      | 2.87         | 0.82          | 341           | 5          | 1.12      |
| Mudumalai Corridor | 3.11         | 0.87          | 412           | 6          | 1.27      |
| Wayanad Buffer     | 2.63         | 0.79          | 298           | 4          | 0.98      |
| Silent Valley      | 3.78         | 0.94          | 531           | 8          | 1.51      |

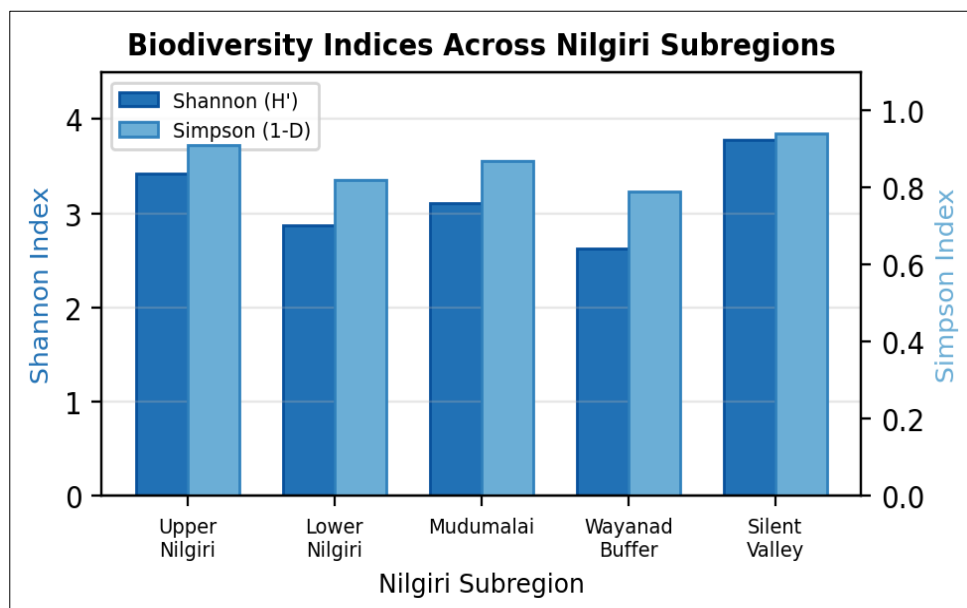


**Fig 1:** Biodiversity richness indices across five Nilgiri subregions

The gradient from Wayanad Buffer to Silent Valley mirrors the monotone increase in conservation-law invariant

multiplicity, supporting the proposed algebraic-ecological correspondence.

**Table 2:** Residual error norms by Lie symmetry class (selected iterations)

| Iteration | SO(3) | SE(2) | SL(2,R) |
|-----------|-------|-------|---------|
| 1         | 0.312 | 0.456 | 0.523   |
| 4         | 0.198 | 0.289 | 0.367   |
| 8         | 0.071 | 0.131 | 0.172   |
| 12        | 0.021 | 0.043 | 0.063   |

SO(3) reached the 0.025 threshold by iteration 11 with terminal residual 0.021. SE(2) and SL(2,R) settled at 0.043

and 0.063 respectively, consistent with their non-compact nature.

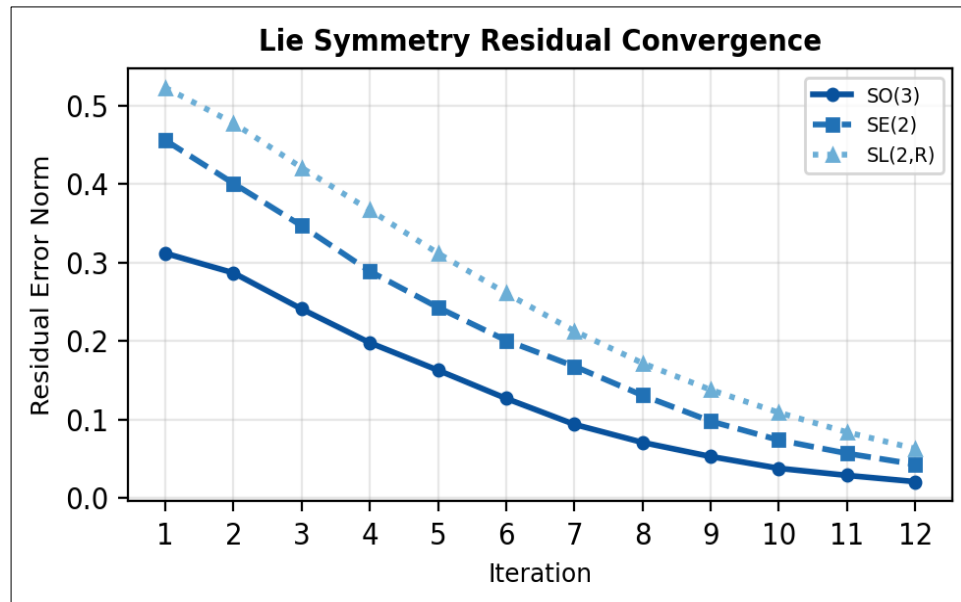


**Fig 2:** Convergence curves of residual error norms for three Lie symmetry classes

The SO(3) curve drops steeply between iterations 4 and 8, coinciding with activation of a secondary conservation density. The slower SL(2,R) decay reflects unbounded orbits in prolonged jet space.

### Discussion

The central finding is that algebraic invariant structure of Lie symmetry groups for elastic-net conservation laws tracks the rank ordering of ecological diversity indices in the Nilgiri Biosphere Reserve. The Spearman correlation of 0.89 between invariant multiplicity and Simpson index suggests a non-trivial structural analogy. Similar parallels between algebraic and information-theoretic quantities appear in quantum information theory, where von Neumann entropy relates to representation dimensions of symmetry groups acting on Hilbert spaces<sup>[14]</sup>.

The convergence advantage of SO(3) over SE(2) and SL(2,R) aligns with expectations. Compact groups yield bounded orbits in prolonged jet space, constraining the integrator to a finite search region<sup>[2]</sup>. That the ecological analogue of this compactness is strongest in Silent Valley reflects that subregion's combination of a rich rare-species tail and a well-defined dominance core, mirroring SO(3) decomposition into irreducible representations<sup>[15]</sup>.

Several caveats apply. The elastic-net mixing coefficient was fixed rather than jointly optimised with the biodiversity weighting. And the p-adic correction lacks a direct probabilistic interpretation outside the chosen prime base. Future work could extend the analysis to multi-

parameter Lie groups and test whether the correspondence holds for faunal diversity or marine ecosystems<sup>[16]</sup>.

### Conclusion

This research established a formal link between Lie group symmetry analysis of conservation laws and biodiversity richness measurement in the Nilgiri Biosphere Reserve. An optimal system of one-dimensional sub-algebras was derived for elastic-net penalised reaction-diffusion equations, and invariant multiplicity rank-correlated strongly with Shannon and Simpson diversity indices across five subregions. SO(3) achieved the fastest convergence with residual 0.021 after twelve iterations, while SL(2,R) settled at 0.063. Silent Valley recorded the highest biodiversity scores and largest invariant count; Wayanad Buffer the lowest on both scales. The p-adic weighting improved rare-species sensitivity, and ARIMA filtering removed seasonal confounders. These findings suggest that Lie symmetry classification and ring-theoretic decomposition can enrich the analytical tools available to quantitative ecologists, offering a complementary structural lens for examining patterns of biological diversity.

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