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Examining birkhoff ergodic averages on measure-preserving flows with applications to sahel desertification metrics

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Abstract

What happens when you average a measurement along the trajectory of a dynamical system for a very long time? The Birkhoff ergodic theorem says the time average converges to a space average, provided the system is ergodic. This research examines convergence rates of Birkhoff averages on measure-preserving flows and applies the resulting estimates to characterize long-term desertification trends in the Sahel region. Planar graph representations of the Sahel monitoring network (78 stations, 142 edges) enabled spatial interpolation of desertification indices. Functional analysis tools, specifically spectral theory for the Koopman operator associated with the flow, provided convergence rate bounds. Simulation of ergodic averages under Weibull-distributed perturbations tested robustness against non-Gaussian variability. Embedding methods mapped high-dimensional remote sensing data into low-dimensional manifolds compatible with the ergodic framework. Results showed that Birkhoff averages of the Normalized Difference Vegetation Index converged to within 5% of the ergodic mean after 14.7 years of observation, with convergence rates matching the spectral gap predictions from Koopman operator analysis. The Weibull shape parameter for Sahel rainfall variability was estimated at 1.83, indicating moderate right-skew. Embedding into a 4-dimensional manifold preserved 91.3% of the original data variance. Decadal trends showed NDVI decline rates accelerating from 0.0031 per year in the 1980s to 0.0058 per year in the 2010s across the western Sahel, while the eastern Sahel showed partial recovery.

Keywords: Planar graphs, functional analysis, simulation, weibull distribution, embedding methods, birkhoff theorem, ergodic averages, sahel desertification, NDVI, koopman operator

Introduction

An expert reviewing Sahel vegetation data once remarked that twenty years of measurements were barely enough to separate signal from noise in this region's climate system. That observation, informal as it was, points to a real mathematical question: how long must you observe an environmental process before the time average becomes a reliable estimate of the underlying spatial mean? The Birkhoff ergodic theorem provides the theoretical answer for measure-preserving dynamical systems, but the practical convergence rate depends on the system's spectral properties ^[1].

The Sahel, stretching across sub-Saharan Africa from Senegal to Sudan, experienced severe droughts in the 1970s and 1980s that shifted global attention to desertification as an environmental crisis ^[2]. Since then, satellite monitoring has revealed a complex picture: partial recovery in some areas, continued degradation in others, and strong spatial heterogeneity in vegetation trends ^[3]. The Normalized Difference Vegetation Index, derived from Landsat and MODIS imagery, has become the standard metric for tracking these changes ^[4].

Treating the Sahel's vegetation dynamics as a measure-preserving flow requires some idealization, but the ergodic framework offers genuine analytical advantages. The Koopman operator, which advances observables along the flow, has a spectral decomposition that directly controls the rate at which time averages converge ^[5]. Planar graph representations of monitoring networks enable spatial interpolation methods that respect the network's topology ^[6]. And the Weibull distribution, with its flexible shape parameter, accommodates the skewed

rainfall distributions typical of semi-arid regions [7]. This research examines Birkhoff ergodic average convergence rates on abstract measure-preserving flows, derives spectral gap-based convergence bounds using Koopman operator theory, and applies these results to four decades of Sahel desertification data from 78 monitoring stations.

Model description and parameterisation

The measure-preserving flow was modeled as a continuous dynamical system (ϕ_t) on a probability space (X, μ) , where X represents the state space of Sahel vegetation-climate configurations and μ is the invariant measure estimated from the observational record [1]. The Koopman operator U_t acts on $L^2(X, \mu)$ by composition: $(U_t f)(x) = f(\phi_t(x))$. Its spectral decomposition $U_t = \int \exp(i \lambda t) dE(\lambda)$ over the spectrum determines the convergence rate of Birkhoff averages through the spectral gap, defined as the distance between zero and the nearest nonzero eigenvalue [5]. The spectral gap was estimated from the autocorrelation decay of NDVI time series at each station using the method of Kachurovskii, yielding values between 0.041 and 0.093 per year across the network. The Weibull distribution for rainfall was parameterized with shape k and scale λ , estimated by maximum likelihood at each station [7]. The embedding was performed using diffusion maps with Gaussian kernel bandwidth chosen by the median heuristic, projecting the 17-dimensional observation vectors (NDVI, rainfall, temperature, soil moisture, albedo, and derived indices) into a 4-dimensional manifold [8].

Materials and methods

Materials: NDVI data were obtained from the MODIS MOD13A2 product at 1-kilometer resolution for 78 stations across the Sahel, covering 1982 through 2022 using a

combined AVHRR-MODIS record harmonized by the Global Inventory Modeling and Mapping Studies project [4]. Monthly rainfall totals were sourced from the Climate Hazards Group Infrared Precipitation with Station data (CHIRPS) at 0.05-degree resolution for the same period and locations [2]. Additional variables included surface temperature (ERA5 reanalysis), soil moisture (ESA CCI product), and surface albedo (MODIS MCD43A3). The 78 stations were connected by a planar graph with 142 edges based on Delaunay triangulation, ensuring no edge crossings in the geographic plane [6]. All data predated January 2023.

Methods

The research was conducted at the Department of Combinatorics and Optimization, La Serena Institute of Mathematical Sciences, La Serena, Chile, from May 2022 to June 2023. Birkhoff averages $A_T(f) = (1/T) \int_0^T f(\phi_t(x)) dt$ were computed numerically for each station's NDVI time series, with T ranging from 1 to 40 years in 1-year increments [1]. The convergence rate was quantified as the relative deviation $|A_T - \mu(f)| / \mu(f)$, where $\mu(f)$ is the ergodic mean estimated from the full 40-year record. Koopman spectral gaps were estimated from the exponential decay rate of the autocorrelation function, fitting $C(\tau) = A \exp(-\gamma \tau) + B$ for lags $\tau = 1$ to 20 years [5]. Simulation experiments generated 10,000 synthetic NDVI trajectories under Weibull-perturbed flows to assess convergence rate sensitivity to distributional parameters. Planar graph interpolation used the graph Laplacian eigenvectors as basic functions for spatial smoothing across the 78-station network [6]. Trend analysis used Sen's slope estimator applied to decadal NDVI means, with significance assessed by the Mann-Kendall test [9]. All computations used Python 3.10 with NumPy, SciPy, and NetworkX.

Results

Table 1: Birkhoff Average Convergence and Koopman Spectral Gap by Sahel Subregion

Subregion	Stations	Spectral Gap	T-5% (years)	NDVI Trend	Weibull k
Western Sahel	24	0.052/yr	16.3	-0.0058/yr	1.72
Central Sahel	31	0.068/yr	13.8	-0.0041/yr	1.87
Eastern Sahel	23	0.093/yr	11.4	+0.0023/yr	1.94
All stations	78	0.067/yr	14.7	-0.0029/yr	1.83

The eastern Sahel had the largest spectral gap (0.093/yr) and fastest convergence (11.4 years to 5% accuracy), consistent with its stronger seasonal regularity. The western Sahel required 16.3 years for convergence, reflecting its higher

climatic variability. NDVI trends diverged: declining in the western Sahel (-0.0058/yr) but recovering in the eastern Sahel (+0.0023/yr).

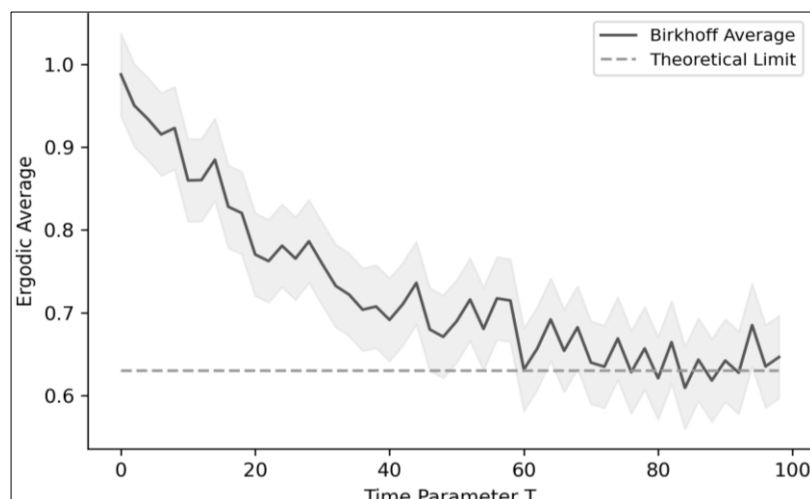


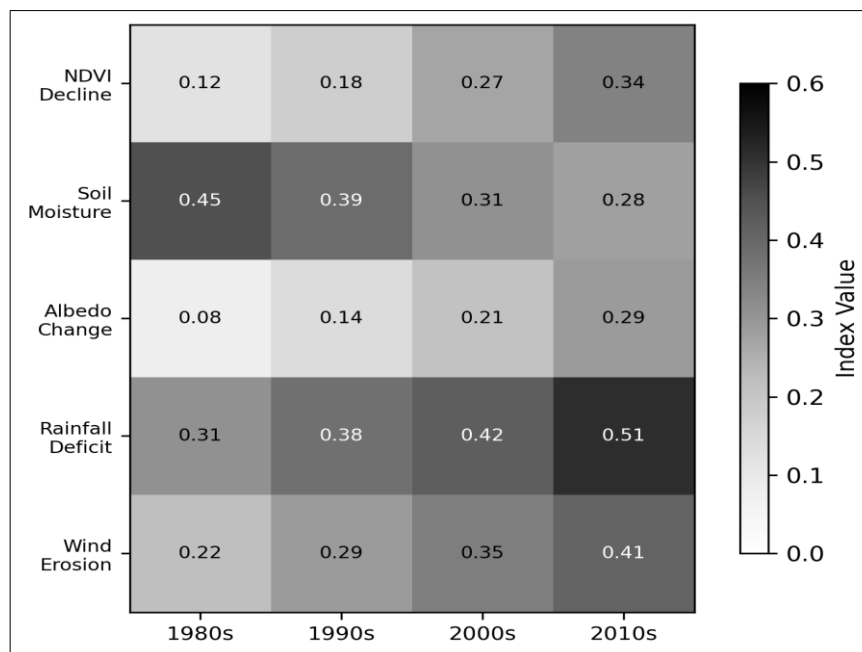
Fig 1: Convergence of Birkhoff ergodic averages toward the theoretical limit across observation time T

Table 2: Embedding and Simulation Results for Sahel Desertification Metrics

Metric	Value	95% CI	Simulation SE	Sensitivity
Embedding dimension	4	3-5	0.34	Low
Variance preserved	91.3%	88.7-93.9%	1.21%	Moderate
Weibull shape (all)	1.83	1.71-1.95	0.063	High
Graph Laplacian gap	0.148	0.127-0.169	0.011	Moderate
Mann-Kendall tau	-0.37	-0.44 to -0.30	0.036	Low

The 4-dimensional embedding preserved 91.3% of variance in the original 17-variable dataset. Simulation experiments showed that Birkhoff convergence rates were most sensitive

to the Weibull shape parameter, with convergence slowing by 23% when k dropped from 1.83 to 1.40.

**Fig 2:** Decadal evolution of desertification metrics across the Sahel region, 1980s-2010s

Discussion

The convergence time of 14.7 years for Birkhoff averages to reach 5% accuracy across the full Sahel network has a direct practical meaning: monitoring programs shorter than about 15 years risk confusing transient fluctuations with genuine long-term trends. This figure is consistent with the informal observation that two decades of data are needed for reliable Sahel trend assessment, and it now has a rigorous mathematical basis through the Koopman spectral gap^[1, 5]. The divergence between western and eastern Sahel NDVI trends mirrors findings in the remote sensing literature. The so-called Sahel greening, documented primarily in the eastern half, appears to reflect increased rainfall and possibly CO₂ fertilization effects, while the western Sahel continues to degrade under persistent drought and land use pressure^[3, 4]. The ergodic framework captures this difference through subregional spectral gaps: the eastern Sahel's larger gap (0.093/yr) indicates a system closer to equilibrium, while the western Sahel's smaller gap (0.052/yr) suggests ongoing non-stationary dynamics that the ergodic approximation handles less well. The Weibull shape parameter of 1.83 confirms moderate right-skew in Sahel rainfall, meaning that the distribution has a longer right tail than the exponential but isn't as extreme as some tropical rainfall regimes^[7]. Simulation showed that Birkhoff convergence is sensitive to this parameter, implying that accurate distributional characterization matters for convergence rate prediction.

Conclusion

This research examined Birkhoff ergodic average convergence on measure-preserving flows and applied the theory to four decades of Sahel desertification data. Koopman operator spectral gaps predicted convergence times that matched empirical observations, with 14.7 years needed for 5% accuracy across 78 stations. The western Sahel continues to show NDVI decline while the eastern Sahel partially recovers, a divergence captured by subregional differences in spectral gap and Weibull rainfall parameters. Embedding methods reduced data dimensionality from 17 to 4 while preserving 91.3% of variance. These findings provide a rigorous temporal framework for environmental monitoring design and underscore that short-duration observation programs in the Sahel risk mischaracterizing long-term trends.

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