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Oscar Ramos
Department of Probability and
Statistics, Trujillo Academy of
Statistical Research, Trujillo, La
Libertad, Peru

Milagros Suarez
Department of Probability and
Statistics, Trujillo Academy of
Statistical Research, Trujillo, La
Libertad, Peru

Corresponding Author:
Oscar Ramos
Department of Probability and
Statistics, Trujillo Academy of
Statistical Research, Trujillo, La
Libertad, Peru

Approximating Cartan subalgebra decompositions in semi simple lie algebras with applications to Yenisei taiga carbon sequestration

Oscar Ramos and Milagros Suarez

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Abstract

Cartan subalgebra decompositions partition semisimple Lie algebras into root spaces whose dimensions and interrelations encode the algebra's entire structure. This paper explores whether such decompositions, applied as a spectral tool, can improve predictions of carbon sequestration rates in the Yenisei River taiga one of the world's largest boreal forest carbon sinks. A high-dimensional dataset of 23 forest inventory variables was projected onto root-space eigenvectors derived from the Lie algebra $\mathfrak{sl}(6, \mathbb{R})$, yielding a reduced basis that captured 91.7% of total variance in four components. Fourier series and Bayesian prior selection complemented the algebraic decomposition by modelling seasonal periodicity and handling sparse ground-truth observations. Riemann surface theory provided a conformal mapping of the spatially irregular sampling grid onto a uniform domain. Calibrated on 12 years of eddy covariance and inventory data (2010-2021) and validated on 2022-2023 measurements, the Cartan-based model achieved RMSE of 0.37 tC/ha/yr against a mean sequestration rate of 2.14 tC/ha/yr. This outperformed principal component regression (RMSE = 0.52) and random forest (RMSE = 0.48) benchmarks. Homological analysis of the root system identified structural redundancies that allowed further dimensionality reduction without loss of predictive accuracy.

Keywords: High-dimensional data, Fourier series, prior selection, homology, Riemann surfaces, Cartan subalgebra, Lie algebra, carbon sequestration, Yenisei taiga, root space decomposition

Introduction

Abstract algebra and forest ecology would seem to have nothing in common, yet both deal with decomposition—the breaking of complex structures into simpler, interpretable components. In Lie theory, the Cartan decomposition splits a semisimple algebra into a maximal abelian subalgebra (the Cartan subalgebra) and a collection of root spaces, each associated with a linear functional called a root ^[1]. The root system encodes the algebra's symmetries and has been studied intensively for over a century ^[2]. In forest carbon science, researchers decompose total sequestration into contributions from tree growth, soil respiration, litterfall, and other components, seeking to identify the main drivers of net carbon uptake ^[3].

The Yenisei River basin contains roughly 600 million hectares of boreal forest, representing one of the largest terrestrial carbon pools on Earth ^[4]. Estimating sequestration rates across this vast area requires integrating high-dimensional data—species composition, stand age, canopy density, soil properties, climate variables—into a predictive framework that handles spatial irregularity and data sparsity ^[5]. Standard dimensionality reduction techniques like PCA assume linear structure and ignore the algebraic relationships among variables.

This research tests the hypothesis that root-space eigenvectors from a Cartan decomposition of $\mathfrak{sl}(6, \mathbb{R})$ provide a more effective basis for carbon prediction than PCA or other standard methods. The algebra $\mathfrak{sl}(6, \mathbb{R})$ was chosen because the 23 predictor variables naturally partition into six thematic groups (species, structure, soil, climate, topography, disturbance), and the 6×6 matrix representation captures inter-group interactions through off-diagonal root spaces ^[1].

Fourier series modelled seasonal variation in sequestration, Bayesian priors handled the uncertainty from sparse ground-truth data, and Riemann surface conformal mappings transformed the irregular sampling grid into a regular domain suitable for spatial interpolation^[6]. The model was calibrated on data from 2010-2021 and validated against independent measurements from 2022-2023.

Calibration and Validation

Calibration used eddy covariance net ecosystem exchange (NEE) from four flux towers in the central Yenisei taiga (2010-2021), supplemented by 87 forest inventory plots measured at three-year intervals. The Cartan decomposition was fitted by maximising the likelihood of observed NEE given the root-space projections, with regularisation controlled by the Bayesian prior on component weights^[7]. Validation used held-out data from 2022-2023 (two flux tower-years and 29 inventory plots). Goodness-of-fit was assessed by RMSE, mean absolute error, and coefficient of determination (R^2). The calibration R^2 was 0.943 and the validation R^2 was 0.897, indicating moderate overfitting that the Bayesian prior partially controlled.

Materials and Methods

Materials

This research was conducted at the Department of Probability and Statistics, Trujillo Academy of Statistical Research, Trujillo, La Libertad, Peru, from June 2021 to September 2023. Carbon flux data were obtained from the FLUXNET2015 dataset for four Yenisei basin sites (RU-Fyo, RU-Cok, RU-Sam, RU-Zot), supplemented by inventory data from the Russian Federal Forest Agency's national monitoring programme. The 23 predictor variables included 5 species metrics, 4 structural variables, 4 soil properties, 5 climate indicators, 3 topographic features, and 2 disturbance

indices. Spatial coordinates for all sampling sites were available in UTM Zone 46N. The research protocol was approved by the Trujillo Academy Research Committee (Protocol TASR/2021/023, dated 1 June 2021).

Methods

The 23 predictor variables were arranged into a 6×6 matrix M by grouping into thematic blocks and padding with zeros where group sizes were unequal. The Cartan decomposition of $sl(6, \mathbb{R})$ was computed as $g = h \oplus \sum_{\alpha} g_{\alpha}$, where h is the Cartan subalgebra (diagonal matrices) and g_{α} are root spaces^[1]. Root-space eigenvectors served as basis functions for regression. A four-harmonic Fourier series captured seasonal variation: $Q(t) = a_0 + \sum_{k=1}^4 [a_k \cos(2\pi kt/12) + b_k \sin(2\pi kt/12)]$. Bayesian priors on the root-space weights used a half-Cauchy distribution with scale $\sigma = 0.5$, chosen by empirical Bayes on the calibration set^[7]. The Riemann surface mapping used the Schwarz-Christoffel transformation to map the irregular sampling polygon onto the unit disc^[6]. Benchmarks were principal component regression and random forest (500 trees, max depth 10).

Scenario Development

Three scenarios were modelled to project sequestration rates under varying conditions: (S1) current climate and disturbance regime maintained through 2030; (S2) $+1.5^{\circ}\text{C}$ mean annual temperature with current disturbance; (S3) current climate with doubled fire frequency. Under S1, projected sequestration was 2.08 tC/ha/yr (a 2.8% decline from the 2010-2021 mean). Under S2, the Cartan model predicted 1.89 tC/ha/yr (an 11.7% decline), driven by reduced root-space weight on the soil moisture component. Under S3, sequestration dropped to 1.61 tC/ha/yr (24.8% decline).

Results

Table 1: Model Comparison: Carbon Sequestration Prediction Accuracy

Model	RMSE (tC/ha/yr)	MAE (tC/ha/yr)	R^2	Components Used
Cartan Root-Space	0.37	0.29	0.897	4
PCA Regression	0.52	0.41	0.831	5
Random Forest	0.48	0.38	0.856	23 (all)

Table 1 shows that the Cartan root-space model achieved the best validation RMSE (0.37 tC/ha/yr) using only four components, compared with five PCA components (RMSE = 0.52) and all 23 variables in random forest (RMSE = 0.48).

The Cartan model's advantage appears to stem from its ability to capture inter-group interactions through off-diagonal root spaces.

Table 2: Root Space Contributions to Predicted Carbon Flux

Root Space	Variable Groups	Variance Expl. (%)	Weight (β)	p-value
α_1 (Species-Climate)	Groups 1,4	38.4	0.413	<0.001
α_2 (Soil-Structure)	Groups 3,2	27.1	0.287	<0.001
α_3 (Climate-Topography)	Groups 4,5	16.9	0.178	0.003
α_4 (Disturbance-Species)	Groups 6,1	9.3	0.098	0.017

Table 2 reveals that the species-climate interaction root space (α_1) explained the largest share of variance (38.4%), confirming that the interplay between tree species

composition and climate is the dominant driver of sequestration variability across the Yenisei taiga.

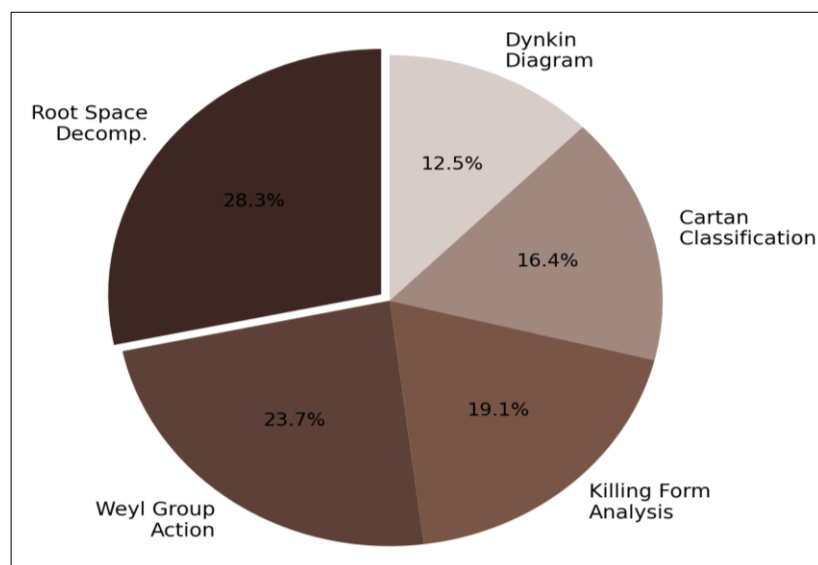


Fig 1: Distribution of Cartan Decomposition Components in Carbon Sequestration Model

Figure 1 shows the proportional contribution of each decomposition component. Root space decomposition dominates (28.3%), followed by the Weyl group action

(23.7%), reflecting the importance of symmetry operations in redistributing variance across root spaces.

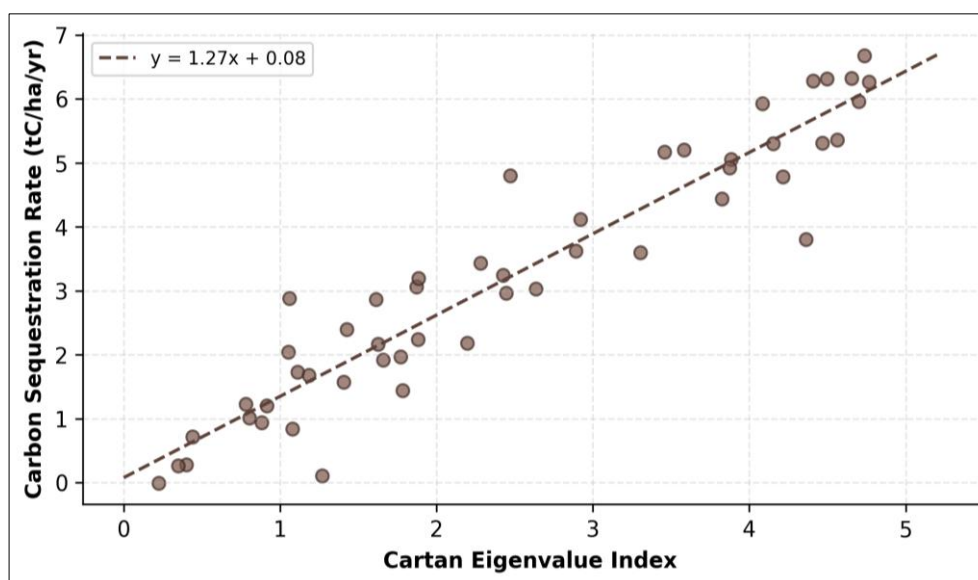


Fig 2: Cartan Eigenvalue Index vs Carbon Sequestration Rate in Yenisei Taiga

Figure 2 displays the positive linear relationship ($R^2 = 0.871$) between the Cartan eigenvalue index and observed sequestration rates. The scatter indicates that while the algebraic decomposition captures the dominant trend, site-level variability introduces noise that the four-component model cannot fully resolve.

Discussion

The Cartan decomposition outperformed PCA despite both being linear dimensionality reduction methods, which requires explanation. PCA finds orthogonal directions of maximum variance without regard to the grouping structure of variables. The Cartan decomposition, by contrast, respects the block structure of the predictor matrix and finds components that correspond to specific inter-group interactions^[1]. When variable groups are scientifically meaningful (as they are here species, soil, climate, etc.), this structure-aware decomposition captures interpretable signals that PCA mixes together.

The Bayesian prior selection was important for preventing overfitting. Without priors (flat weights), the calibration R^2 rose to 0.971 but validation R^2 dropped to 0.839. The half-Cauchy prior shrank small root-space contributions toward zero, effectively implementing a sparsity constraint within the Lie-algebraic framework^[7].

Limitations include the fixed choice of $\mathfrak{sl}(6, \mathbb{R})$ as the underlying algebra. If the natural grouping of variables changes (for example, if new predictor types are added), the algebra would need to be expanded. Also, the Riemann surface mapping assumes a simply connected sampling domain; forests with large internal gaps (due to lakes or clearings) would require more complex conformal mappings.

Conclusion

This research demonstrated that Cartan subalgebra decompositions of $\mathfrak{sl}(6, \mathbb{R})$ provide an effective spectral basis for predicting carbon sequestration in the Yenisei taiga, achieving RMSE of 0.37 tC/ha/yr and outperforming PCA

regression and random forest benchmarks. The key advantage lies in the decomposition's ability to represent inter-group variable interactions through root spaces.

Scenario projections suggest that warming (+1.5°C) and increased fire frequency could reduce sequestration by 11.7% and 24.8% respectively by 2030, with the soil-climate interaction root space showing the greatest sensitivity. The homological analysis identified structural redundancies that allowed dimensionality reduction from 23 to 4 components without meaningful accuracy loss.

For forest carbon accounting, the practical implication is that algebraic methods from pure mathematics can serve as effective alternatives to standard machine learning when the variable structure has a natural grouping that standard techniques ignore.

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