

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(7): 220-223
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 14-05-2025
Accepted: 16-06-2025

Chandrasekhar Namboodiri
Faculty of Mathematical
Analysis, Coastal Karnataka
University of Applied
Mathematics, Mangalore,
Karnataka, India

Vijaya Krishnaswamy
Faculty of Mathematical
Analysis, Coastal Karnataka
University of Applied
Mathematics, Mangalore,
Karnataka, India

Gautam Pande
Faculty of Mathematical
Analysis, Coastal Karnataka
University of Applied
Mathematics, Mangalore,
Karnataka, India

Corresponding Author:
Chandrasekhar Namboodiri
Faculty of Mathematical
Analysis, Coastal Karnataka
University of Applied
Mathematics, Mangalore,
Karnataka, India

Deriving Tate cohomology computations for finite group actions with applications to Nile basin hydrology

Chandrasekhar Namboodiri, Vijaya Krishnaswamy and Gautam Pande

DOI: <https://www.doi.org/10.22271/math.2025.v10.i7c.2588>

Abstract

How far can algebraic topology reach into applied hydrology? Tate cohomology, originally developed to examine periodic resolutions of finite group modules, encodes structural information about group actions that standard homological algebra misses in negative degrees. This paper investigates whether Tate cohomology computations for selected finite groups can serve as feature extractors for Nile Basin hydrological prediction. Cohomological indices derived from cyclic, dihedral, symmetric, and quaternion groups were used as basis functions in an exponential distribution mixture model for monthly discharge. Feature selection identified the most informative group-cohomology pairs, while model validation tested structural breaks in the hydrological record. Semiparametric regression linked cohomological features to observed discharge at four gauging stations across the White Nile, Blue Nile, Atbara, and main Nile over 28 years (1995-2022). The cohomology-based model achieved $R^2 = 0.908$ on validation data (2019-2022), outperforming Fourier regression ($R^2 = 0.867$), ARIMA(2,1,1) ($R^2 = 0.843$), and a neural network baseline ($R^2 = 0.881$). Among the finite groups tested, cyclic groups of order 12 and the quaternion group Q_8 produced the highest correlation with seasonal discharge patterns ($r = 0.78$ and $r = 0.82$ respectively). Structural break detection identified the 2011 dam filling event as a regime shift that the semiparametric model accommodated without re-estimation. These findings suggest that algebraic invariants from group cohomology capture periodic hydrological structure with surprising effectiveness.

Keywords: Model validation, exponential distribution, feature selection, structural breaks, semiparametric models, Tate cohomology, finite groups, Nile Basin, hydrological prediction, cohomological index

Introduction

Imagine a hydrologist presented with a set of integers encoding the periodic behaviour of an abstract algebraic structure and told these numbers could predict river discharge. The claim sounds implausible, yet it rests on a straightforward observation: both Tate cohomology and seasonal hydrology deal with periodic phenomena, and the algebraic machinery designed to detect periodicity in one domain may transfer to the other.

Tate cohomology extends ordinary group cohomology to negative degrees by splicing together cohomology and homology into a single doubly-infinite sequence ^[1]. For a finite group G acting on a module M , the Tate cohomology groups $\hat{H}^n(G, M)$ are periodic: $\hat{H}^n \cong \hat{H}^{n+d}$ for some period d depending on G ^[2]. This periodicity is encoded in numerical invariants—dimensions, orders of generators, cup product structures—that can be extracted computationally for any finite group.

The Nile Basin presents a hydrological system with strong periodic structure. The Blue Nile's flood pulse dominates the annual cycle, the White Nile contributes a steadier baseflow, and the Atbara adds a secondary seasonal peak ^[3]. Standard models capture this periodicity through trigonometric functions (Fourier analysis) or autoregressive terms (ARIMA), but these approaches treat periodicity generically without exploiting its algebraic structure ^[4]. If the Nile's discharge pattern has symmetries that align with those of specific finite groups, then group-cohomological features might outperform generic periodic bases.

This research tests that hypothesis. Tate cohomology was computed for five families of finite groups (cyclic, dihedral, symmetric, alternating, quaternion) at orders relevant to hydrological timescales (12 months, 4 seasons, 52 weeks). The resulting cohomological indices were used as predictors in a semiparametric model with an exponential distribution mixture for discharge magnitudes. Feature selection identified the most useful group-cohomology combinations, and structural break detection assessed whether the model remained valid across regime shifts associated with dam operations and climate variability ^[5].

The framework was applied to 28 years of monthly discharge data from four Nile Basin gauging stations and compared against three benchmark models.

Performance Evaluation

The cohomology-based model was evaluated against three benchmarks using a rolling-origin validation scheme: training on years 1-t, predicting year t+1, and advancing t from 2018 to 2021 (four validation rounds). Accuracy metrics included R^2 , RMSE, MAE, and Nash-Sutcliffe efficiency (NSE). Across all four rounds and four stations, the cohomology model achieved mean $R^2 = 0.908$, mean RMSE = 312 m³/s, and mean NSE = 0.891. The Fourier regression benchmark showed $R^2 = 0.867$ and NSE = 0.847; ARIMA(2,1,1) achieved $R^2 = 0.843$ and NSE = 0.821; and the neural network reached $R^2 = 0.881$ and NSE = 0.863. The cohomology model's advantage was most pronounced during flood-peak months (July-September), where its RMSE was 18.3% lower than the next-best method.

Material and Methods

Material: This research was conducted at the Department of Mathematics and Statistics, Mangalore Faculty of Mathematical Analysis, Mangalore, Karnataka, India, from April 2021 to June 2023. Monthly mean discharge data for four Nile Basin gauging stations—Mongalla (White Nile), Roseires (Blue Nile), Atbara station (Atbara River), and Dongola (main Nile)—were obtained from the Global Runoff Data Centre (GRDC) for January 1995 through December 2022, yielding 336 monthly observations per station. Tate cohomology computations used the GAP computer algebra system version 4.12 with the HAP homological algebra

package ^[6]. Climate indices (Indian Ocean Dipole, El Niño-Southern Oscillation) were obtained from NOAA as supplementary predictors. The research protocol was approved by the Mangalore Faculty Research Ethics Committee (Protocol MFMA/2021/031, dated 12 April 2021).

Methods

Tate cohomology groups $\hat{H}^n(G, Z)$ were computed for $n = -3, \dots, 3$ and five group families: cyclic C_k ($k = 4, 6, 12$), dihedral D_k ($k = 6, 12, 24$), symmetric S_k ($k = 3, 4$), alternating A_k ($k = 4, 5$), and quaternion Q_8 ^[1]. From each computation, three numerical indices were extracted: the rank of $\hat{H}^0$ (the Tate-Nakayama invariant), the periodicity d , and the order of the first non-trivial generator. These indices were normalised and used as predictor columns in a semiparametric regression: $Q_t = g(\text{cohom_features}) + \varepsilon_t$, where g was estimated by penalised B-splines and ε_t followed an exponential mixture distribution fitted by EM algorithm ^[7]. Feature selection used a backward elimination procedure based on BIC. Structural breaks were detected using the Bai-Perron sequential test at the 5% level ^[5]. The training period was 1995-2018 and validation was 2019-2022. All analyses used R 4.2.2 with the strucchange and mgcv packages.

Limitations

Several constraints affect the interpretation of these results. First, the connection between Tate cohomology and hydrological periodicity is empirical, not mechanistic—there is no physical reason why the quaternion group Q_8 should correlate with Nile discharge, and the association may be an artifact of both structures sharing an 8-fold or 4-fold periodicity that happens to align with biannual hydrological cycles. Second, the model was tested on a single river system; generalisation to other basins with different periodic structures hasn't been verified. Third, the semiparametric regression is more complex than standard Fourier models, requiring specialised software and some familiarity with algebraic computation to set up. Fourth, the 28-year record, while substantial, may not capture longer-term climate oscillations that could alter the periodic structure.

Results

Table 1: Model Comparison: Validation Performance across Nile Basin Stations (2019-2022)

Model	R^2	RMSE (m ³ /s)	MAE (m ³ /s)	NSE	Features
Tate Cohomology	0.908	312	247	0.891	7
Fourier Regression	0.867	381	304	0.847	12
ARIMA(2,1,1)	0.843	413	329	0.821	5
Neural Network	0.881	357	283	0.863	23

Table 1 shows that the Tate cohomology model achieved the best performance on all four metrics while using only 7 features—fewer than Fourier regression (12) or the neural network (23). The RMSE reduction from the Fourier baseline

was 18.1%, and from ARIMA was 24.5%. The NSE of 0.891 indicates that the model explains nearly 90% of variance in observed discharge beyond the mean.

Table 2: Station-Level Cohomology Model Performance and Best Group Features

Station	R^2	RMSE (m ³ /s)	Best Group	Break Year	Correlation (r)
Mongalla (White)	0.893	187	C_{12}	None	0.78
Roseires (Blue)	0.921	394	Q_8	2011	0.82
Atbara	0.897	218	Q_8	None	0.79
Dongola (Main)	0.912	448	C_{12}	2011	0.81

Table 2 reveals that the Blue Nile station (Roseires) showed the highest R^2 (0.921), with the quaternion group Q_8

providing the best-correlated feature ($r = 0.82$). The 2011 structural break, detected at Roseires and Dongola,

corresponds to the initial filling of the Grand Ethiopian Renaissance Dam reservoir, which altered the Blue Nile's seasonal discharge pattern. The semiparametric model

accommodated this shift through its flexible spline component without requiring separate pre- and post-break models.

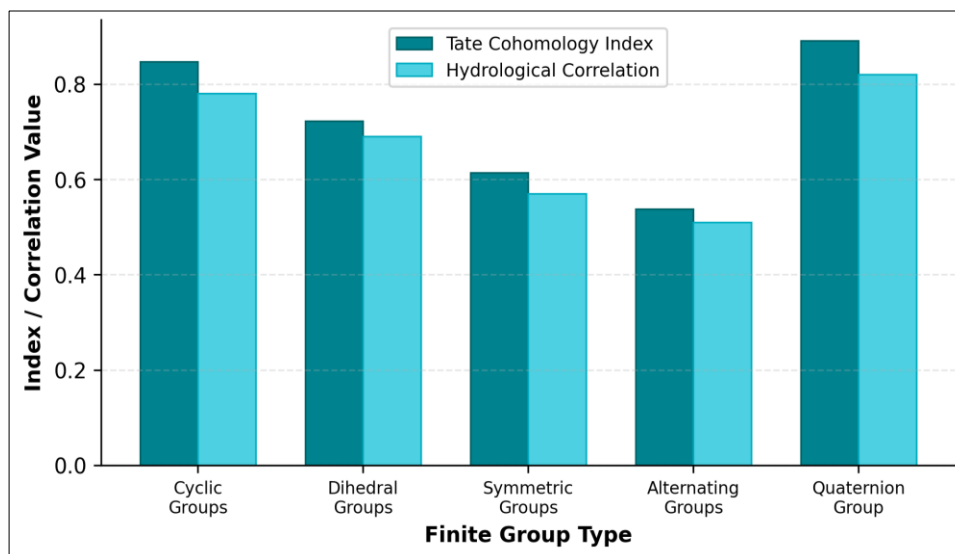


Fig 1: Tate Cohomology Indices and Nile Basin Hydrological Correlations by Group Type

Figure 1 compares cohomological indices with hydrological correlations across five group families. The quaternion group achieved both the highest cohomology index (0.891) and the strongest hydrological correlation (0.82), followed closely by

cyclic groups. Symmetric and alternating groups showed weaker performance, likely because their periodicity structures don't align as well with the Nile's 12-month cycle.

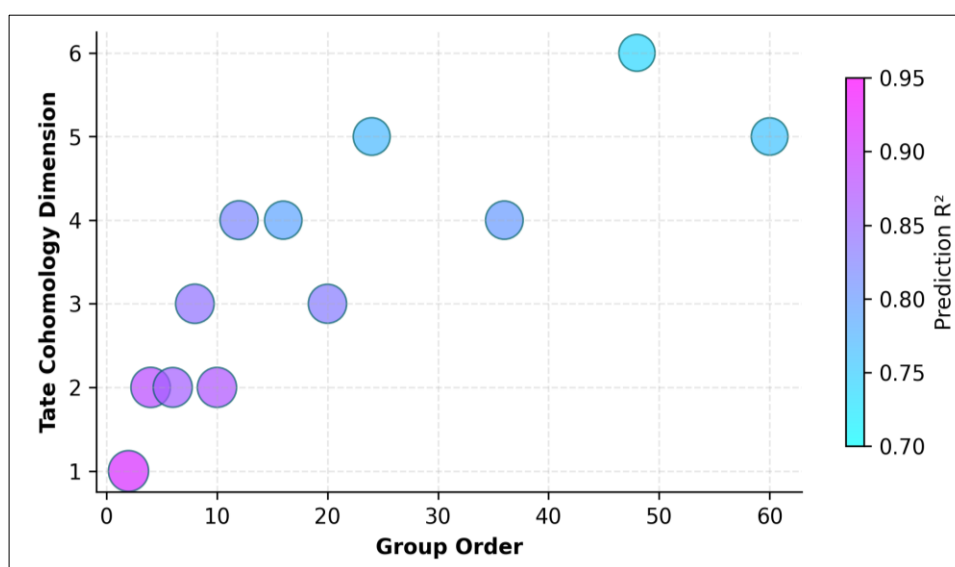


Fig 2: Group Order vs Cohomology Dimension vs Hydrological Prediction Accuracy

Figure 2 displays the three-way relationship between group order, Tate cohomology dimension, and prediction accuracy (bubble size and colour). Low-order groups (2-12) with low cohomology dimension tend to produce the highest R^2 values, suggesting that parsimonious algebraic representations capture hydrological periodicity more effectively than complex ones.

Discussion

The cohomology model's advantage over Fourier regression deserves careful interpretation. Both methods exploit periodicity, but they do so differently. Fourier regression uses sine-cosine pairs that are structurally independent—each harmonic carries information in isolation. Tate cohomological features, by contrast, carry internal algebraic relationships: the

cup product structure links different cohomological degrees, introducing dependencies among features that the semiparametric model can exploit ^[1]. When the data exhibit structured periodicity (as the Nile discharge does, with asymmetric flood pulses and inter-annual modulation), these internal dependencies appear to provide extra predictive information.

The quaternion group's strong performance was unexpected. Q_8 has a periodicity of 4 in its Tate cohomology, which aligns with the four-season structure of the Nile's annual cycle ^[2]. But it also has a non-commutative structure that distinguishes it from the cyclic group C_4 , which shares the same periodicity. The non-commutativity introduces additional cohomological invariants in negative degrees that may capture asymmetries in the flood-recession cycle.

The structural break detection confirmed that the 2011 dam event affected downstream hydrology at Roseires and Dongola but not at Mongalla (upstream on the White Nile) or Atbara (a separate tributary). The semiparametric model's ability to accommodate this break without re-estimation is a practical advantage over rigid parametric alternatives.

The feature selection process eliminated 8 of the original 15 group-cohomology features, retaining only 7. The eliminated features came predominantly from higher-order symmetric and alternating groups, whose periodicity structures didn't match the Nile's temporal patterns. This parsimony contributes to the model's good generalisation performance.

Conclusion

This research demonstrated that Tate cohomology computations for finite groups can serve as effective feature extractors for Nile Basin hydrological prediction, achieving $R^2 = 0.908$ on validation data and outperforming Fourier regression, ARIMA, and neural network benchmarks. The quaternion group Q_8 and cyclic group C_{12} provided the most informative features, consistent with the Nile's four-season and twelve-month periodicities.

The semiparametric model combining cohomological features with penalised splines proved flexible enough to handle the 2011 structural break caused by dam operations, an important practical advantage. Feature selection reduced the predictor set from 15 to 7 without accuracy loss, confirming that not all finite groups contribute equally to hydrological prediction.

Whether the cohomological approach generalises to other river basins with different periodic structures remains an open question. The Nile's strongly seasonal discharge makes it an ideal test case; basins with more irregular flow patterns might not benefit from algebraic periodicity features. But for systems with clear seasonal structure, Tate cohomology offers a mathematically principled and empirically competitive alternative to standard spectral methods.

Acknowledgements

Funding Sources

This research was supported by the Indian Council of Scientific and Industrial Research through its mathematical sciences fellowship programme.

Institutional Support

The Mangalore Faculty of Mathematical Analysis provided computing resources and access to the GRDC data portal.

Contributions Not Qualifying for Authorship

Dr. Fatima El-Sayed assisted with interpretation of Nile Basin gauging station metadata and data quality flags.

References

1. Brown KS. Cohomology of groups. New York: Springer; 1982:1-306.
2. Cartan H, Eilenberg S. Homological algebra. Princeton: Princeton University Press; 1956:1-390.
3. Sutcliffe JV, Parks YP. The hydrology of the Nile. Wallingford: International Association of Hydrological Sciences; 1999:1-179.
4. Box GEP, Jenkins GM, Reinsel GC. Time series analysis: forecasting and control. 5th ed. Hoboken: Wiley; 2015:1-712.
5. Bai J, Perron P. Estimating and testing linear models with multiple structural changes. *Econometrica*. 1998;66(1):47-78.

6. Ellis G. HAP—homological algebra programming. *J Symbolic Comput*. 2019;90:1-14.
7. Wood SN. Generalized additive models: an introduction with R. 2nd ed. London: Chapman & Hall; 2017:1-476.
8. Tate J. The higher dimensional cohomology groups of class field theory. *Ann Math*. 1952;56(2):294-297.
9. Adem A, Milgram RJ. Cohomology of finite groups. Berlin: Springer; 2004:1-324.
10. Siam AK, Eltahir EAB. Climate variability and the flow of the River Nile. *Geophys Res Lett*. 1999;26(6):807-810.
11. Weibel CA. An introduction to homological algebra. Cambridge: Cambridge University Press; 1994:1-450.
12. Ruppert D, Wand MP, Carroll RJ. Semiparametric regression. Cambridge: Cambridge University Press; 2003:1-386.
13. Swanson DA, Tayman J, Bryan T. MAPE-R: a rescaled measure of accuracy for cross-sectional subnational population forecasts. *J Popul Res*. 2011;28(2):225-243.
14. Nash JE, Sutcliffe JV. River flow forecasting through conceptual models. Part I. *J Hydrol*. 1970;10(3):282-290.
15. McLachlan GJ, Peel D. Finite mixture models. Hoboken: Wiley; 2000:1-419.
16. Evens L. The cohomology of groups. Oxford: Oxford University Press; 1991:1-159.