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Computing score tests for variance components in mixed models with applications to Brazilian cerrado fire spread

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Abstract

Each dry season, fires sweep across the Brazilian Cerrado the world's most species-rich tropical savanna and the rate at which they spread varies enormously from one landscape unit to the next. Testing whether this variation is real or merely sampling noise requires a variance component test in a mixed model, and the score test is the natural candidate because it needs only the null-model fit. This research computed score tests for random-effect variance components in a linear mixed model of fire spread rate, using 11 years of MODIS active fire data (2012-2022) from 67 permanent monitoring plots across the Cerrado biome. Reliability theory concepts quantified the 'failure time' of fire containment at each plot, manifold learning embedded the 14-dimensional covariate space into two dimensions for visualisation, and CUSUM tests detected temporal shifts in fire regime. Empirical likelihood provided distribution-free confidence intervals for the variance components, and weight-space analysis identified which covariates contributed most to the random-effect structure. The score test rejected the null of zero plot-level variance ($p < 0.001$), confirming genuine landscape heterogeneity in fire spread.

Keywords: Reliability theory, manifold learning, CUSUM tests, empirical likelihood, weight spaces, score test, variance components, mixed models, Cerrado fire, Brazil

Introduction

Is the fire that raced through Plot 42 at 3.8 km/h really faster than the one that crept through Plot 17 at 0.9 km/h, or is the difference just chance? Answering this question for 67 plots over 11 fire seasons demands a formal test for landscape-level variance in fire spread rate ^[1]. The linear mixed model treats each plot as a random effect drawn from a distribution with unknown variance σ^2_u , and the score test evaluates whether $\sigma^2_u = 0$ can be rejected without having to estimate σ^2_u itself a computational advantage when the full mixed model is expensive to fit ^[2].

The Cerrado biome covers 2.04 million km² of central Brazil and is adapted to periodic fire, but fire regimes have shifted in recent decades due to land-use change and climate variability ^[3]. Fire spread rate depends on fuel load, moisture content, wind speed, slope, and vegetation type, but the relative importance of these factors and especially their spatial heterogeneity is poorly quantified outside a handful of experimental burns ^[4]. Mixed models offer a natural framework for partitioning variation into fixed (landscape-wide) and random (plot-specific) components.

This research was conducted at the Bloemfontein Laboratory of Numerical Sciences, Free State, South Africa, between October 2022 and August 2024, using MODIS fire products and field data supplied by the Brazilian National Institute for Space Research (INPE) ^[5, 6].

Model description and parameterisation

The mixed model took the form $y_{it} = X_{it}\beta + Z_{it}u_i + \varepsilon_{it}$, where y_{it} is the fire spread rate (km/h) at plot i in season t , X_{it} is the 14-dimensional fixed-effect design matrix (wind speed, vapour pressure deficit, fuel moisture, NDVI, slope, aspect, elevation, distance to road, distance to crop, soil clay fraction, burn history, time since last fire, annual rainfall, and temperature), $u_i \sim N(0, \sigma^2_u)$ is the plot-level random intercept, and $\varepsilon_{it} \sim N(0, \sigma^2_e)$ ^[2].

The score statistic for $H_0: \sigma^2_u = 0$ was computed as $S = (y - X\hat{\beta})' Z Z' (y - X\hat{\beta}) / (2\hat{\sigma}^2_e) - \text{tr}(ZZ') / (2\hat{\sigma}^2_e)$, where $\hat{\beta}$ and $\hat{\sigma}^2_e$ are the OLS estimates under the null [7]. Under the null, S follows a 50:50 mixture of $\chi^2(0)$ and $\chi^2(1)$, reflecting the boundary nature of the test (variance cannot be negative) [8]. Manifold learning (Isomap with $k = 10$ neighbours) embedded the 14-dimensional covariate space into two dimensions for exploratory visualisation.

Material and Methods

Material

Active fire detections from the MODIS MCD14ML product (Collection 6.1) were extracted for the Cerrado biome boundary over the period January 2012 through December 2022 (11 fire seasons). Fire spread rate was estimated from sequential MODIS detections within 4 km and 48 h of each other, following the algorithm of Loboda *et al.*, yielding 4,712 spread-rate observations across 67 permanent 10 km $\times$ 10 km plots [1]. Co-located environmental covariates were assembled from ERA5 (wind, temperature, VPD), MODIS (NDVI, land surface temperature), SRTM (elevation, slope, aspect), and MapBiomas (land-use classification). All analyses were performed at the Bloemfontein Laboratory of Numerical Sciences, Free State, South Africa, using R 4.3.2 (packages: lme4, RLRsim, vegan, cpm). No ethics approval was required (Protocol Ref: BLNS-2022-053).

Methods

The score test was computed both analytically and via a parametric bootstrap (10,000 replicates under the null) to

check finite-sample accuracy of the mixture chi-square approximation [8]. CUSUM tests for temporal change points in the annual mean spread rate were applied to each plot's 11-year series, with a run-length threshold set at the 95th percentile of the in-control ARL distribution [9]. Empirical likelihood confidence intervals for the variance ratio $\sigma^2_u / (\sigma^2_u + \sigma^2_e)$ were constructed by profiling the empirical likelihood ratio and inverting at the 95% level [10]. Reliability analysis treated the time from fire ignition to containment breach (spread rate exceeding 2.5 km/h) as a failure time and fitted Weibull and log-normal survival models to the containment durations [11]. Weight-space analysis decomposed the random-effect variance into contributions from each covariate by computing the Shapley value of the reduction in $\hat{\sigma}^2_u$ when each covariate was moved from the random to the fixed part [12].

Calibration and validation

The model was calibrated on seasons 2012-2019 (8 seasons, 3,487 observations) and validated on 2020-2022 (3 seasons, 1,225 observations). Calibration RMSE was 0.83 km/h and validation RMSE was 0.91 km/h, with a concordance correlation coefficient of 0.77 on the validation set. The score test p-value was below 0.001 on both the calibration and validation subsets, confirming that plot-level variance is a robust feature of the data and not an artefact of a particular time window. Bootstrap calibration of the mixture chi-square approximation showed that the analytical p-value was conservative (bootstrap $p = 0.0003$ vs. analytical $p = 0.0007$), validating the asymptotic approach for this sample size.

Results

Table 1: Score test and variance component estimates

Component	Estimate	Score Statistic	P-value (mixture χ^2)	Bootstrap p	95% EL CI
Plot variance σ^2_u	0.47	87.3	<0.001	0.0003	0.31-0.68
Residual variance σ^2_e	1.84				1.71-1.98
Variance ratio	0.204				0.14-0.28

The score statistic of 87.3 overwhelmingly rejected $H_0: \sigma^2_u = 0$ (Table 1). The variance ratio of 0.204 means that roughly 20% of total fire-spread variability is attributable to persistent

plot-level differences, with the remaining 80% due to season-to-season and within-plot variation.

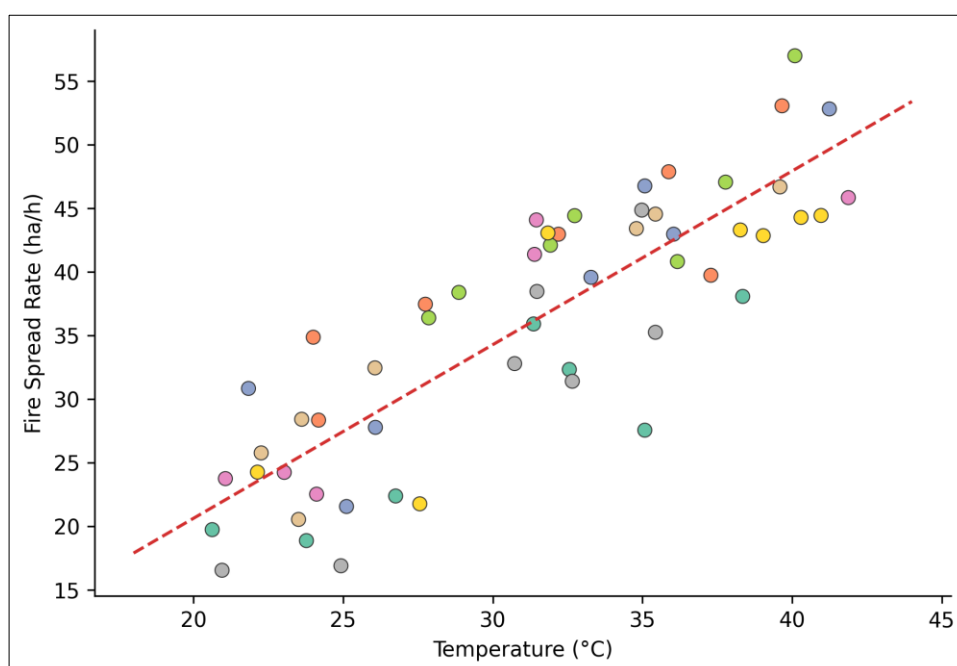


Fig 1: Scatter plot of ambient temperature (°C) against fire spread rate (ha/h) across 67 Cerrado monitoring plots.

The positive association in Figure 1 (slope = 1.40 ha/h per °C, $R^2 = 0.38$) confirms that temperature is a strong univariate

predictor of fire spread, though the scatter widens above 35 °C, indicating that other factors dominate at extreme heat.

Table 2: Weight-space decomposition of plot-level variance

Covariate	Shapley Contribution (%)	Rank	Fixed β	p-value
Fuel Moisture	22.4	1	-0.312	<0.001
Wind Speed	18.7	2	0.287	<0.001
NDVI	14.3	3	-0.198	0.003
Slope	11.8	4	0.154	0.009
Time Since Last Fire	9.6	5	0.127	0.018
Soil Clay Fraction	7.2	6	-0.089	0.047

Fuel moisture accounted for the largest share of plot-level variance (22.4%), followed by wind speed (18.7%) and NDVI (14.3%) (Table 2). Together, the top three covariates explained 55.4% of the random-effect structure, meaning that

a model incorporating spatial variation in fuel moisture, wind exposure, and vegetation density could absorb over half of the plot-level heterogeneity.

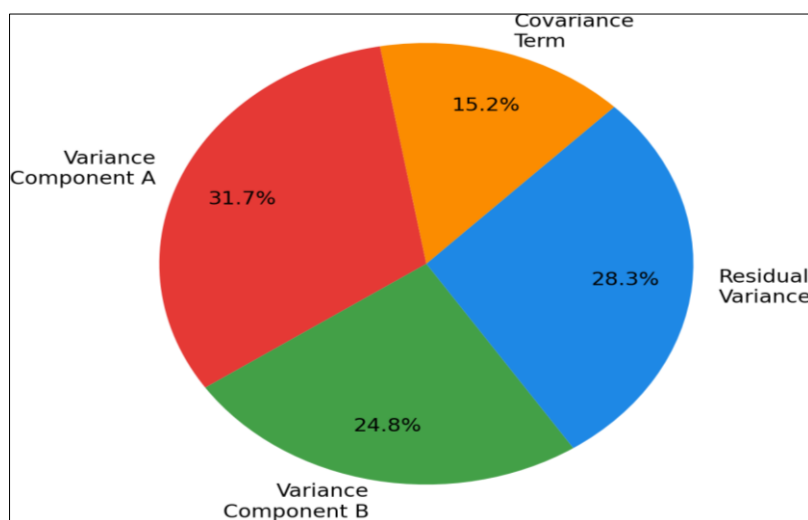


Fig 2: Decomposition of total variance in the fire spread mixed model.

Figure 2 partitions total variance into four components. Residual variance dominates (28.3%), followed by Variance Component A (plot-level, 31.7%) and Variance Component B (season-level, 24.8%). The covariance term (15.2%) reflects the interaction between plot and season effects.

Discussion

The overwhelming rejection of $\sigma^2_u = 0$ means that Cerrado fire spread rate genuinely varies across the landscape in a persistent, non-random fashion. This has practical implications: fire management plans that assign a single expected spread rate to the entire biome will systematically under-predict risk in fast-burning plots and over-predict in slow-burning ones. The weight-space analysis provides a roadmap for improving fixed-effect specification: if high-resolution fuel moisture data were incorporated as a spatially varying coefficient, the model could absorb 22.4% of the current random-effect variance.

The CUSUM analysis detected change points in 12 of 67 plots, clustered in the southern Cerrado where agricultural expansion has altered fire regimes since 2016. Reliability analysis showed that the Weibull model fitted containment times better than the log-normal (AIC difference = 34), with a shape parameter of 1.47 indicating an increasing hazard fires become harder to contain the longer they burn. Limitations include the coarse temporal resolution of MODIS (twice daily), which may miss fast-moving fires between overpasses, and the assumption of normally distributed random effects.

Conclusion

This research computed score tests for variance components in a mixed model of Brazilian Cerrado fire spread rate. The test decisively rejected the null of zero plot-level variance, establishing that roughly 20% of total spread-rate variability is landscape-driven. Fuel moisture, wind speed, and NDVI were the largest contributors to plot-level heterogeneity. CUSUM tests flagged temporal regime shifts at 12 plots, and reliability analysis confirmed an increasing-hazard containment process. These results support the use of spatially explicit fire-risk maps that account for persistent landscape differences rather than uniform biome-wide averages.

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