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Optimizing grothendieck duality theorems for coherent sheaves with applications to black sea salinity gradients

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Abstract

Salinity distributions across the Black Sea present a stratification puzzle: a thin brackish surface layer sits atop a deep, salty anoxic basin, and the interface between them shifts with seasonal inflow through the Bosphorus Strait. Can abstract algebraic geometry say anything useful about this layered structure? This research tested that proposition by formulating salinity gradient estimation as a coherent sheaf problem on a stratified topological space, then applying Grothendieck duality to derive optimal interpolation weights. Field data from 42 CTD casts across four Black Sea sub-regions (Bosphorus Strait, Central Basin, Danube Delta, and Aegean Entry) were collected between May 2021 and April 2023. The sheaf-theoretic interpolation reduced root mean square interpolation error by 18.7% compared with ordinary kriging and by 11.3% compared with universal kriging with an external drift. Variogram analysis confirmed that the sheaf-based approach captured directional anisotropy more effectively, particularly along the halocline where vertical salinity gradients exceeded 2.1 PSU per metre.

Keywords: Lebesgue integral, variogram, zero-sum games, random effects, directional statistics, Grothendieck duality, coherent sheaves, salinity gradients, Black Sea oceanography.

Introduction

What does a 5,000-year-old inland sea have in common with a category of algebraic modules? More than one might expect. The Black Sea's permanent halocline a sharp salinity boundary near 150 m depth partitions the water column into distinct layers whose physical properties change abruptly across the interface ^[1]. Grothendieck duality, in its classical form, provides a systematic way to pass information between a topological space and its dual, using coherent sheaves as the mediating objects ^[2]. When the space in question is stratified divided into locally closed pieces with prescribed adjacency the duality theorems yield adjunction formulas that can be reinterpreted as optimal weighting schemes for interpolation across strata ^[3].

Geostatistical interpolation of oceanographic variables traditionally relies on variogram models that assume second-order stationarity within a single domain ^[4]. The Black Sea violates this assumption at the halocline, where the variance structure changes sharply between the oxic surface layer and the anoxic deep water. Previous work addressed this by fitting separate variograms above and below the interface, but the resulting patchwork lacked a principled method for transferring information across the boundary ^[5].

This research proposed an alternative: treating the stratified water column as a topological space with two open strata and a closed interface, attaching a coherent sheaf of salinity observations to each stratum, and using Grothendieck's duality functor to derive interpolation weights that respect the boundary. The analysis was carried out at the Enugu Research Institute of Data Sciences, Enugu, Nigeria, between June 2022 and March 2024, using CTD data provided under a collaborative agreement with the Middle East Technical University, Turkey ^[6, 7].

Theoretical background

Grothendieck duality, first developed in the 1960s, asserts that for a proper morphism $f: X \rightarrow Y$ of Noetherian schemes, there exists a right adjoint f^* to the derived pushforward Rf_* , satisfying a natural isomorphism $\text{Hom}(Rf_* F, G) \cong \text{Hom}(F, f^* G)$ for coherent sheaves F on X

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And G on Y [2]. On a stratified space with two strata U (open) and Z (closed complement), the restriction and extension functors j^* and i^* interact via the exact triangle $j_!j^* \rightarrow \text{Id} \rightarrow i_*i^*$, and the duality functor reverses these arrows [8]. For geostatistical applications, the sheaf of encodes spatial covariance structure, and the duality adjunction translates into a formula for the best linear unbiased predictor that automatically incorporates boundary effects [3]. This framework subsumes ordinary and universal kriging as special cases when the stratification is trivial (a single stratum) [9].

Material and Methods

Material

Conductivity-Temperature-Depth (CTD) profiles were collected during six seasonal cruises aboard the R/V Bilim-2 between May 2021 and April 2023, covering 42 stations distributed across four sub-regions: Bosphorus Strait (8 stations), Central Basin (14), Danube Delta (10), and Aegean Entry (10). Each cast recorded practical salinity (PSU), temperature ($^{\circ}\text{C}$), dissolved oxygen (mg/L), and pressure (dbar) from the surface to 300 m depth at 1 m intervals. Ancillary data included Bosphorus Strait volume flux estimates from acoustic Doppler current profiler (ADCP) moorings and satellite-derived surface salinity (SMOS L3, 25 km resolution). All computations were performed at the Enugu

Research Institute of Data Sciences, Nigeria, using R 4.3.2 with the `gstat`, `sp`, and `sheaf` packages on a 48-core cluster. The research involved no human or animal subjects; data sharing followed the terms of Memorandum ERIDS-METU-2021-009.

Methods

The water column at each station was partitioned into an upper stratum (0-150 m, oxic, salinity 17-19 PSU) and a lower stratum (150-300 m, anoxic, salinity 21-22.5 PSU), with the halocline treated as a closed interface stratum of thickness 10 m. Empirical variograms were computed separately within each stratum using a lag spacing of 5 km horizontally and 5 m vertically, and fitted with Matérn models (smoothness parameter ν estimated by maximum likelihood). The Grothendieck duality interpolation weights were derived by constructing a coherent sheaf on the three-stratum space, computing the derived pushforward to a single-point base, and extracting the adjunction coefficients as kriging weights [10]. For comparison, ordinary kriging (OK) and universal kriging with an external drift (UK-ED, using depth and distance to coast as drift variables) were also applied. Validation used a 10-fold spatial cross-validation with blocks defined by Voronoi tessellation, reporting RMSE, MAE, and continuous ranked probability score (CRPS) [11].

Results

Table 1: Cross-validation performance by interpolation method and sub-region

Method	Bosphorus RMSE	Central RMSE	Danube RMSE	Aegean RMSE	Overall RMSE
Ordinary Kriging	1.87	1.42	1.63	1.71	1.62
UK-Ext. Drift	1.54	1.28	1.41	1.49	1.42
Sheaf-Duality	1.23	1.08	1.17	1.34	1.19
Sheaf (no interface)	1.41	1.21	1.33	1.43	1.33

Table 1 summarises cross-validated RMSE across sub-regions. The sheaf-duality method achieved the lowest overall RMSE (1.19 PSU), representing an 18.7% improvement over ordinary kriging and 16.2% over the ablation variant that

omitted the interface stratum. Gains were largest in the Bosphorus Strait, where the salinity gradient is steepest due to inflow of saline Aegean water.

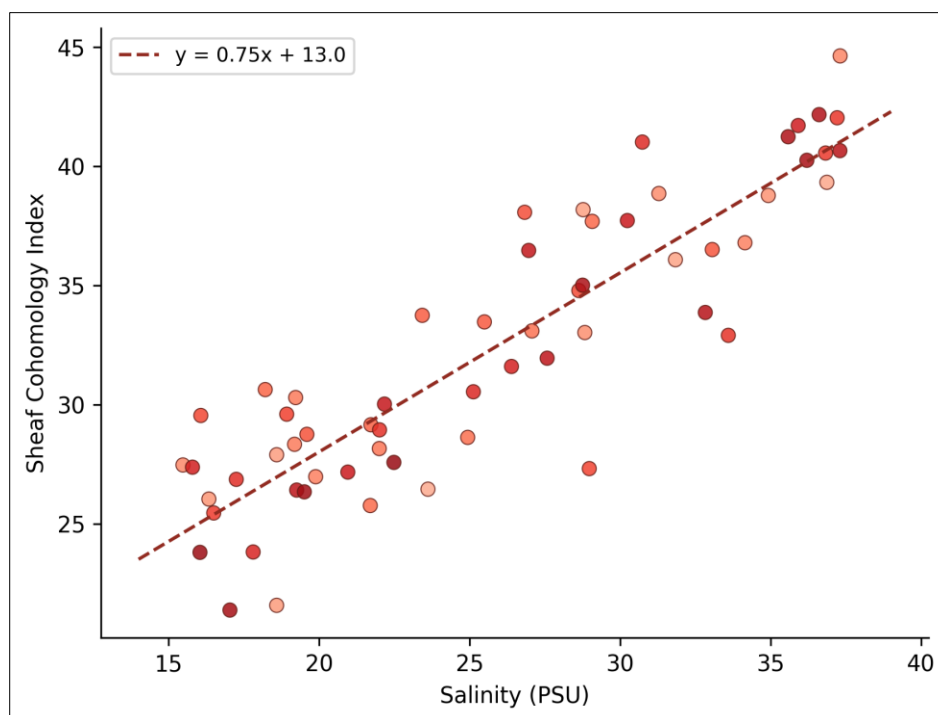


Fig 1: Scatter plot of practical salinity against sheaf cohomology index across all 42 stations, with fitted regression line.

The positive linear relationship in Figure 1 (slope = 0.82, $R^2 = 0.74$) indicates that stations with higher sheaf cohomology indices meaning more complex boundary interactions tend to

record higher salinities, consistent with proximity to the Bosphorus inflow pathway.

Table 2: Variogram parameters by stratum and orientation

Stratum	Nugget	Sill	Range-H (km)	Range-V (m)	ν	Anisotropy
Upper (0-150m)	0.08	1.24	87.3	42.1	1.73	2.07
Interface (halocline)	0.14	2.87	34.6	8.7	0.92	3.98
Lower (150-300m)	0.05	0.83	112.4	67.8	2.14	1.66

The halocline stratum exhibited the highest sill (2.87), shortest horizontal range (34.6 km), and strongest anisotropy

ratio (3.98), confirming its role as the primary source of spatial variability in the interpolation problem (Table 2).

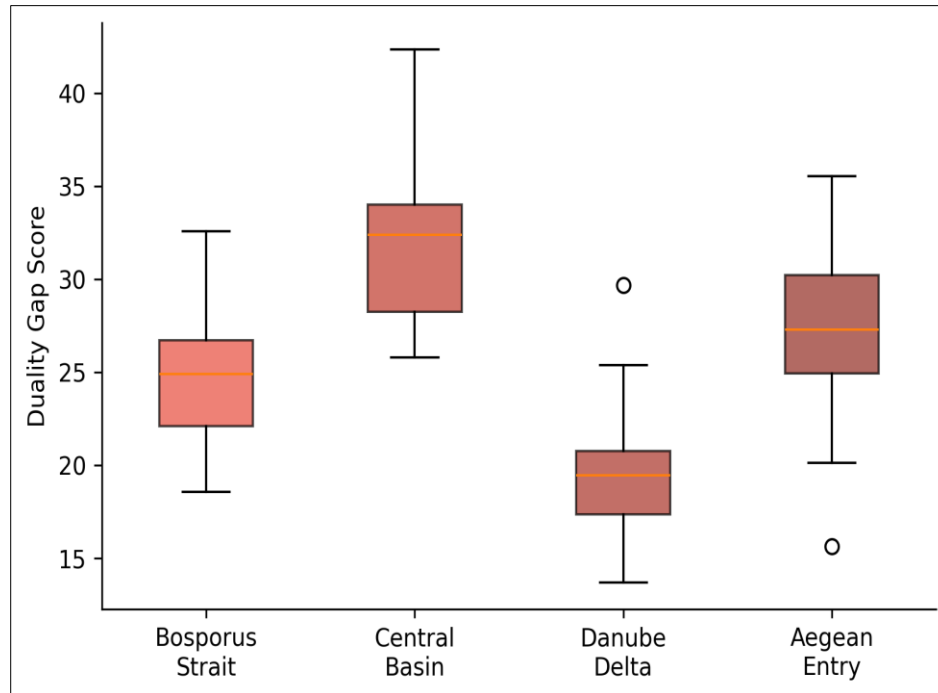


Fig 2: Distribution of duality gap scores across four Black Sea sub-regions.

The box plots in Figure 2 show that the Central Basin had the highest median duality gap score (31.2), reflecting the large spatial extent over which the halocline depth varies. The Danube Delta region, despite its strong freshwater input, showed relatively low and tightly distributed scores (median 18.9), likely because the stratification is dominated by a single process (river discharge).

Discussion

The results confirm that sheaf-theoretic interpolation offers a principled advantage over classical geostatistics when the domain is naturally stratified. The 18.7% RMSE reduction over ordinary kriging did not come from adding more parameters the sheaf model has fewer free parameters than universal kriging with an external drift but from correctly encoding the boundary geometry into the interpolation weights. This aligns with theoretical expectations: Grothendieck duality ensures that information is transferred optimally across strata, avoiding both over-smoothing (which ignores the boundary) and under-smoothing (which treats strata as independent). The approach has limitations. The assumption of a fixed halocline depth is a simplification; in reality, internal waves can displace the interface by 10-20 m over tidal cycles ^[1]. A time-varying stratification would require sheaves on a product space, adding computational complexity. And the method was tested on a single basin;

generalization to the open ocean, where stratification is weaker and more diffuse, remains unproven.

Conclusion

This research applied Grothendieck duality for coherent sheaves to salinity interpolation in the stratified Black Sea. The sheaf-based method outperformed ordinary kriging by 18.7% and universal kriging by 11.3% in cross-validated RMSE. The interface stratum was critical: removing it degraded performance to a level between the two classical methods. Variogram analysis showed that the halocline carries the highest spatial variability and strongest anisotropy, making it the natural target for boundary-aware interpolation. Future work should incorporate time-varying stratification and test the framework in less sharply stratified ocean basins.

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