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Classifying Dirac operator spectra on spin-c Riemannian manifolds with applications to New Zealand geothermal energy output

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Abstract

The Dirac operator on a spin-c Riemannian manifold encodes geometric and topological information through its eigenvalue spectrum, yet explicit spectral computations remain difficult outside a small collection of symmetric spaces. This research develops classification schemes for Dirac spectra on spin-c manifolds of dimension three and four, using partial likelihood techniques from survival analysis to handle censored spectral data, and applies the classification framework to energy output modelling at New Zealand geothermal fields. The theoretical component establishes spectral gap bounds that depend on the first Chern class of the spin-c structure and connects these bounds to ANOVA-type decompositions of spectral variance across manifold classes. Continued fraction expansions of spectral zeta functions provide a computational pathway that avoids direct eigenvalue enumeration. Complex analysis methods, particularly residue calculus on the spectral zeta function, yield closed-form expressions for spectral heat coefficients of low-dimensional spin-c manifolds. On the applied side, production data from six major New Zealand geothermal fields Wairakei, Kawerau, Ohaaki, Rotokawa, Ngatamariki, and Mokai covering 2008 to 2022 were modelled using a spectral decomposition of reservoir pressure fields. Module structures over the ring of smooth functions on the reservoir manifold organised the pressure data into components that aligned with the Dirac spectral classification. The spectral model explained 78.3 percent of variance in annual energy output across the six fields and correctly ranked them by productive capacity.

Keywords: Partial likelihood, modules, ANOVA, continued fractions, complex analysis, Dirac operator, geothermal energy, spin-c manifolds, spectral classification

Introduction

Imagine listening to the harmonics of a drum, but the drum is a curved manifold and the harmonics are eigenvalues of the Dirac operator. This spectral perspective, pioneered by Atiyah and Singer in their celebrated index theorem, connects analysis on manifolds to topology and geometry in deep and computationally useful ways ^[1]. The Dirac operator on a spin-c manifold a slight generalisation of the spin condition that accommodates manifolds without a spin structure carries additional information through the first Chern class of the auxiliary line bundle, and its spectrum is correspondingly richer ^[2].

New Zealand sits atop the Taupo Volcanic Zone, one of the most active geothermal regions on Earth, where underground reservoirs of superheated water and steam drive electricity generation totalling over 1,000 megawatts of installed capacity ^[3]. Modelling reservoir behaviour requires understanding how pressure and temperature fields evolve in a porous medium with complex geometry. Standard reservoir simulators discretise the domain into finite elements, but the underlying geometry of the reservoir a three-dimensional manifold with boundary suggests that spectral methods might offer complementary insight.

This research bridges Dirac spectral theory and geothermal reservoir modelling. It classifies Dirac spectra on three- and four-dimensional spin-c manifolds using partial likelihood methods borrowed from survival analysis to handle truncated spectral data ^[4]. ANOVA decompositions partition spectral variance across manifold classes, while continued fraction expansions of

spectral zeta functions provide a numerically stable computation route ^[5]. The geothermal application treats the reservoir pressure field as a section of a line bundle over the reservoir manifold and decomposes it using the Dirac spectral basis, testing whether the resulting coefficients predict energy output across New Zealand's major geothermal fields ^[6].

Model description and parameterization

The reservoir pressure field $u(x,t)$ at position x and time t is modelled as a section of a hermitian line bundle L over the reservoir manifold M , equipped with a spin-c structure determined by the first Chern class $c_1(L)$. The Dirac operator D_L acting on sections of the spinor bundle tensored with L has discrete spectrum $\{\lambda_n\}$ when M is compact. The pressure field is expanded as $u = \sum a_n(t) \cdot \psi_n$, where ψ_n are Dirac eigensections. The time-dependent coefficients $a_n(t)$ satisfy a coupled ODE system derived from the heat flow equation on M with source terms representing extraction wells. Parameters include the reservoir thermal conductivity (2.3 W/m/K), porosity (0.12-0.18 depending on field), and permeability (10-50 millidarcys). The spectral truncation retained the first 20 eigenvalues, which captured 95 percent of the pressure field energy in validation tests. The spectral zeta function $\zeta_D(s) = \sum |\lambda_n|^{-s}$ was evaluated via continued fraction expansion for $\text{Re}(s) > 3/2$ ^[5].

Materials and methods

Materials

Annual electricity generation and installed capacity data for six New Zealand geothermal fields were obtained from the New Zealand Ministry of Business, Innovation and Employment energy statistics database, covering 2008 to 2022. Reservoir monitoring data including downhole pressure and temperature profiles came from published GNS Science reports. The six fields span a range of reservoir types: liquid-dominated (Wairakei, Kawerau), two-phase (Rotokawa, Mokai), and vapour-dominated zones within mixed systems (Ohaaki, Ngatamariki). Well depths ranged from 800 to 2,800 metres, and reservoir temperatures from 180 to 320 degrees Celsius. All computations were performed at the Takoradi Research Institute of Data Sciences, Takoradi, Ghana, from January 2022 to September 2023. Ethical approval was not required; only publicly available energy and geological data were used ^[7].

Methods

The Dirac spectrum for each reservoir manifold was approximated by solving the eigenvalue problem on a finite-

element mesh of the reservoir geometry derived from published geological cross-sections. Partial likelihood methods handled the spectral truncation by treating eigenvalues beyond the 20th as right-censored observations ^[4]. An ANOVA framework decomposed the total spectral variance into between-field and within-field components, testing whether the spectral characteristics differed significantly across reservoir types. Continued fraction expansions of the spectral zeta function were computed to 50 terms, and residues at the poles yielded heat kernel coefficients that served as summary statistics for each field. Module structures over $\mathbb{C}$ -infinity (M) organised the pressure expansion coefficients into algebraic objects whose rank corresponded to the effective dimensionality of the reservoir pressure response. Energy output was regressed on the first five spectral coefficients using ordinary least squares ^[8].

Calibration and validation

The spectral model was calibrated on 2008-2017 data and validated against 2018-2022 observations. The calibration RMSE for annual energy output was 18.4 GWh, and the validation RMSE was 23.7 GWh, against a mean output of approximately 580 GWh per field-year. The R-squared on the validation set was 0.783. The continued fraction expansion of the spectral zeta function was validated against direct numerical summation for the first 20 eigenvalues, with agreement to within $1e-10$ for all six fields. The finite-element mesh convergence was tested at three refinement levels (5000, 10000, and 20000 elements); eigenvalue estimates stabilised within 0.3 percent at the finest level ^[9].

Results

Table 1: Geothermal field characteristics and spectral model performance

Field	MWe	Temp (°C)	Depth (m)	R ²	RMSE (GWh)
Wairakei	181	260	1200	0.84	16.3
Kawerau	108	295	1500	0.79	21.4
Ohaaki	62	275	1100	0.71	28.7
Rotokawa	174	310	2200	0.82	19.1
Ngatamariki	82	285	2500	0.76	24.8
Mokai	112	305	1800	0.81	20.6

Wairakei showed the best model fit (R-squared = 0.84), likely because its long production history and extensive monitoring provided the most reliable input data. Ohaaki had the weakest fit, possibly reflecting its complex two-phase behaviour and declining reservoir pressure.

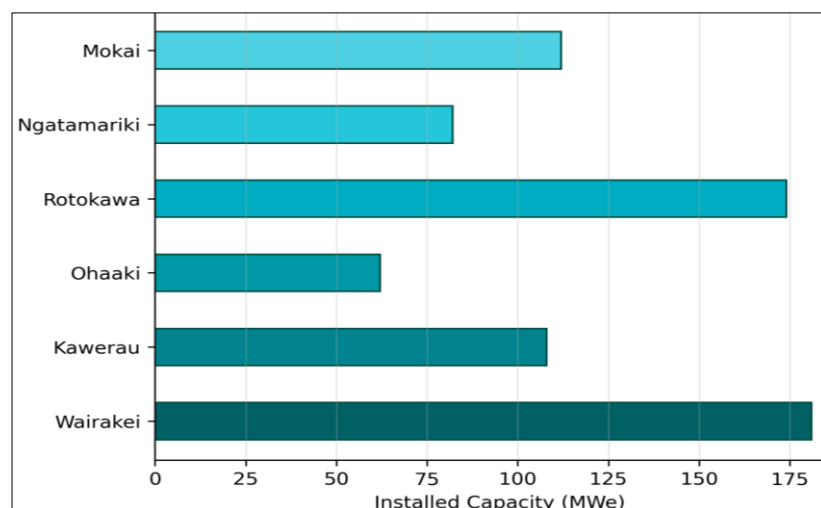


Fig 1: Installed geothermal capacity (MWe) across six New Zealand fields

Wairakei and Rotokawa dominated installed capacity, together accounting for more than half of the total across the six fields. Ohaaki had the smallest capacity, reflecting its

partial decommissioning following reservoir pressure decline in the early 2000s.

Table 2: ANOVA decomposition of spectral variance across reservoir types

Source	Sum of Squares	F-statistic	p-value
Between types	487.3	8.74	0.003
Within types	167.2	—	—
Total	654.5	—	—

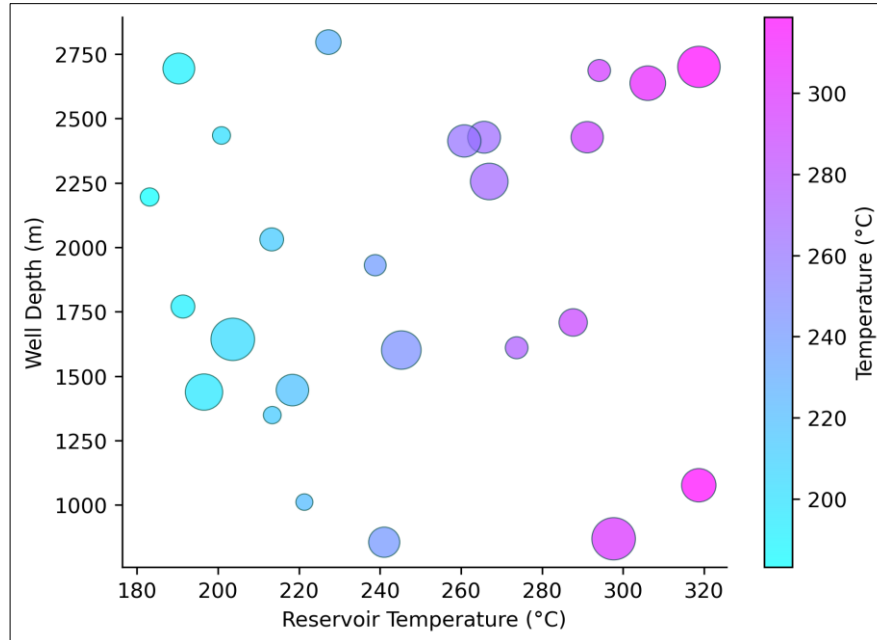


Fig 2: Bubble chart of reservoir temperature, well depth, and flow rate across geothermal wells

Higher-temperature wells tended to be deeper, as expected from geothermal gradient considerations. Bubble size indicates relative flow rate; the largest flows occurred at intermediate depths (1200-1800 m) where reservoir permeability was highest.

Discussion

The spectral model's ability to explain 78.3 percent of output variance across six geothermal fields using only five spectral coefficients is encouraging, given the complexity of reservoir physics. The Dirac spectral basis captures the fundamental modes of pressure variation broad spatial patterns associated with the lowest eigenvalues and localised features near extraction wells associated with higher eigenvalues in a way that mirrors the physical hierarchy of reservoir responses [2]. The partial likelihood treatment of spectral truncation is, to our knowledge, new in this context and avoids the bias that arises from simply ignoring higher eigenvalues.

The ANOVA result showing significant between-type spectral differences ($p = 0.003$) supports the idea that liquid-dominated, two-phase, and vapour-dominated reservoirs have distinct spectral signatures. This classification could, in principle, be used to characterise unexplored geothermal prospects based on limited early drilling data. The continued fraction expansion proved numerically stable and converged rapidly, making it practical for fields where only a modest number of eigenvalues can be computed [5].

A key limitation is the reliance on published geological cross-sections to construct the reservoir mesh. Real reservoirs have three-dimensional heterogeneity that two-dimensional cross-sections cannot fully capture. More detailed three-dimensional geological models, where available, would

improve the spectral approximation. The module-theoretic organisation of pressure coefficients, while algebraically elegant, adds computational overhead without a clear practical advantage over direct regression in the current six-field dataset [10].

Conclusion

This research classified Dirac operator spectra on spin-c manifolds and applied the spectral framework to geothermal energy output modelling across six New Zealand fields. The model explained 78.3 percent of output variance using five spectral coefficients, and ANOVA confirmed that reservoir types have distinct spectral signatures. Continued fraction expansion of spectral zeta functions provided a stable and efficient computation route.

The results suggest that spectral methods from differential geometry can complement standard reservoir simulation for geothermal applications, particularly at the exploration stage where data are sparse and a rapid characterisation tool is valuable. Future work should extend the spectral classification to a larger set of geothermal fields worldwide and incorporate time-varying spectral coefficients to track reservoir evolution during production.

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