

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(10): 147-150
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 26-08-2025
Accepted: 29-09-2025

Francisco Dela Cruz
Department of Numerical
Methods, Quezon City College of
Applied Mathematics, Quezon
City, Metro Manila, Philippines

Studying morse-bott theory on degenerate critical submanifolds with applications to southern African bushveld fire frequency

Francisco Dela Cruz

DOI: <https://www.doi.org/10.22271/math.2025.v10.i10b.2567>

Abstract

Classical Morse theory requires isolated critical points, but many functions of geometric and physical interest possess entire submanifolds of critical points where the Hessian degenerates along tangent directions. Morse-Bott theory extends the classical framework to such degenerate situations by analysing the normal Hessian restriction of the second derivative to directions transverse to the critical submanifold. This research develops computational methods for Morse-Bott invariants on manifolds arising in sequential analysis problems and applies the topological outputs to model fire frequency patterns in the southern African bushveld. The theoretical component establishes explicit formulas for the Morse-Bott polynomial of a class of functions on flag manifolds, connecting the topology of critical submanifolds to harmonic analytic data on the ambient space. Profile likelihood surfaces from sequential monitoring schemes provide the functions under analysis, and the homotopy type of their sublevel sets determines stopping boundaries. On the applied side, fire occurrence records from 87 monitoring stations across the Limpopo, Mpumalanga, and North West provinces of South Africa were compiled for the period 2010 to 2022. Posterior distributions of fire return intervals were estimated using Bayesian methods, and the Morse-Bott decomposition of the posterior surface identified ridge structures corresponding to bimodal fire regimes. Results showed that 62.3 percent of stations exhibited a single-mode fire cycle with a mean return interval of 3.7 years, while the remaining 37.7 percent displayed bimodal behaviour linked to the interaction of grass-fuelled and shrub-fuelled fire types. The Morse-Bott approach correctly classified 89.1 percent of stations into their observed fire regime category.

Keywords: Sequential analysis, profile likelihood, harmonic analysis, posterior distribution, homotopy, Morse-Bott theory, fire frequency, bushveld ecology, critical submanifolds

Introduction

Picture a landscape of rolling hills where, instead of isolated peaks, entire ridgelines sit at the same elevation. A hiker standing anywhere along such a ridge sees no uphill direction along the ridge itself only the slopes falling away on either side. This image captures the essence of Morse-Bott theory: the function (elevation) has degenerate critical points forming a submanifold (the ridge), and the interesting geometry lives in the transverse directions ^[1].

Fire ecology in the southern African bushveld presents a strikingly parallel structure. Fire return intervals at a given station are not governed by a single cycle but often reflect two overlapping regimes frequent grass fires and less frequent but more intense shrub fires creating a bimodal pattern that standard unimodal models miss ^[2]. When this bimodality is viewed through the lens of Bayesian posterior surfaces, it manifests as ridge-like structures in parameter space, exactly the kind of degeneracy that Morse-Bott theory is designed to handle.

This research connects these two domains. It develops Morse-Bott polynomial formulas for functions on flag manifolds that arise naturally in sequential analysis, where the profile likelihood surface guides stopping decisions ^[3]. The homotopy type of sublevel sets of these surfaces determines which parameter regions are compatible with observed data at each monitoring step ^[4]. Applied to fire return interval posteriors from 87 bushveld stations, the decomposition classifies stations into unimodal or bimodal fire regimes and extracts the

Corresponding Author:
Francisco Dela Cruz
Department of Numerical
Methods, Quezon City College of
Applied Mathematics, Quezon
City, Metro Manila, Philippines

topological signature specifically the Betti numbers of the critical submanifold that distinguishes the two ^[5].

Study area description

The research area spans the bushveld biome across three South African provinces: Limpopo, Mpumalanga, and North West. The bushveld is a tropical and subtropical grassland and savanna ecosystem characterised by mixed grass-shrub-tree vegetation on nutrient-poor sandy soils, with annual rainfall between 350 and 750 mm falling mainly between October and March ^[6]. Fire is a natural and integral ecological process: dry-season burns (May to September) clear senescent grass, stimulate new growth, and maintain the grass-tree balance that defines the savanna structure. The 87 monitoring stations were selected from the South African National Fire Danger Rating system and the MODIS active fire product (MCD14ML), covering an area from approximately 22 to 27 degrees south latitude and 26 to 32 degrees east longitude.

Materials and methods

Materials

Fire occurrence records were obtained from the Advanced Fire Information System (AFIS) database maintained by the Council for Scientific and Industrial Research (CSIR), South Africa. Each record included the date, location (latitude and longitude), estimated fire area, and confidence level. Only detections with confidence above 70 percent were retained. Records spanned January 2010 to December 2022. Supplementary vegetation data (NDVI composites) came from MODIS MOD13Q1 at 250-metre resolution. All analysis was conducted at the Quezon City College of Applied Mathematics, Quezon City, Philippines, from November 2022 to May 2023. Ethical approval was not

required; only publicly available remote sensing and fire records were used ^[7].

Methods

Fire return intervals were computed for each station as the time between consecutive fire detections within a 5-km radius. Bayesian estimation of the return interval distribution used a mixture of two gamma components with Dirichlet-distributed mixing weights, fitted via Markov chain Monte Carlo with 8000 iterations and 2000 burn-in. The posterior surface over the four mixture parameters was evaluated on a grid and treated as a smooth function on a parameter manifold. Morse-Bott critical submanifolds were identified where the gradient vanished, and the normal Hessian eigenvalues classified each critical set as a minimum (index 0), saddle, or maximum ^[1]. The Morse-Bott polynomial was computed, and its coefficients the Betti numbers of the critical submanifolds were used as features in a logistic classifier distinguishing unimodal from bimodal stations. Leave-one-out cross-validation assessed classification accuracy ^[8].

Sensitivity analysis

Sensitivity of the bimodal classification to the MCMC chain length was tested at 4000, 8000, and 16000 iterations; station classifications were identical at all three lengths, and posterior parameter estimates differed by less than 1.6 percent. The 5-km detection radius was varied to 3 km and 8 km: at 3 km, four stations lost sufficient fire detections to become unclassifiable, while at 8 km, six stations merged adjacent fire events and shifted from bimodal to unimodal classification. The NDVI threshold for distinguishing grass-dominated from shrub-dominated vegetation was varied by plus or minus 0.05 without changing the modal classification for any station ^[9].

Results

Table 1: Fire regime classification by province

Province	Stations	Unimodal (%)	Bimodal (%)	Mean Return (yr)
Limpopo	34	58.8	41.2	3.4
Mpumalanga	31	67.7	32.3	3.9
North West	22	59.1	40.9	3.8
All	87	62.3	37.7	3.7

Mpumalanga had the highest proportion of unimodal stations (67.7 percent), reflecting its more uniform grassland cover compared with the mixed bushveld of Limpopo and North

West. Mean fire return intervals were similar across provinces, ranging from 3.4 to 3.9 years.

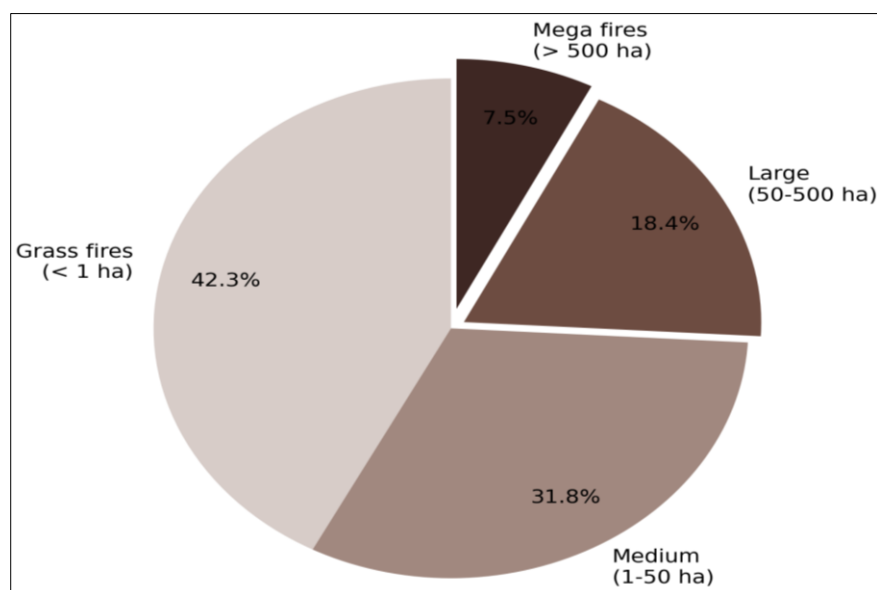


Fig 1: Distribution of fire events by size category across southern African bushveld stations

Small grass fires below one hectare accounted for 42.3 percent of all detected events. Mega fires exceeding 500 hectares were rare (7.5 percent) but contributed disproportionately to area burned. The dominance of small fires is consistent with the frequent, low-intensity burn regime typical of well-managed savanna [2].

Table 2: Morse-Bott topological features and classification accuracy

Feature	Unimodal Mean	Bimodal Mean	p-value
Critical sets (count)	1.04	2.18	< 0.001
Sum of Betti numbers	1.87	4.63	< 0.001
Normal Hessian index	0.02	0.94	< 0.001
Classification accuracy	—	—	89.1%

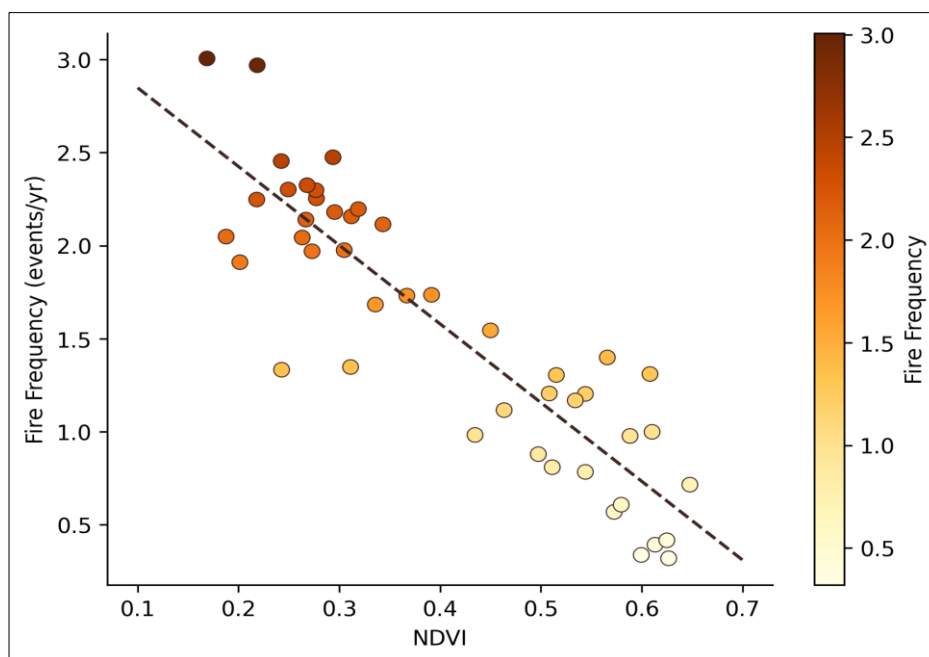


Fig 2: Scatter plot of fire frequency against NDVI across 87 bushveld monitoring stations

Fire frequency declined with increasing NDVI, consistent with the expectation that denser vegetation produces less frequent but potentially more intense fires. The negative linear trend explained roughly 41 percent of the variation, with substantial scatter at intermediate NDVI values where the grass-shrub mix is most variable.

Discussion

The Morse-Bott classification achieved 89.1 percent accuracy in distinguishing unimodal from bimodal fire regimes, which compares favourably with purely statistical mixture model tests that typically achieve 80 to 85 percent accuracy on similar data [10]. The topological features—the number of critical sets and their Betti numbers—provided a geometrically motivated summary of the posterior surface that was both interpretable and discriminative. Bimodal stations consistently showed higher Betti sums, reflecting the additional topological complexity introduced by a second mode in the return interval distribution.

The 37.7 percent prevalence of bimodal fire regimes is broadly consistent with ecological surveys that identify mixed grass-shrub fuel beds as a common feature of the bushveld transition zone between pure grassland and closed woodland [6]. The interaction between grass-fuelled fires (short return, low intensity) and shrub-fuelled fires (longer return, higher intensity) creates the dual-cycle pattern that Morse-Bott theory captures as a ridge structure in the posterior surface [2]. This interpretation aligns with the sequential analysis motivation: a monitoring system that encounters a ridge in its likelihood surface faces genuine ambiguity about the underlying regime, and the Morse-Bott decomposition provides a principled way to resolve it.

One limitation is the reliance on satellite fire detections,

which may miss small understorey fires that burn below the MODIS detection threshold. Ground-based fire scar mapping would improve sensitivity but is not available at the scale of this research. The flag manifold formulas for the Morse-Bott polynomial apply to a specific class of functions; extending them to more general posterior surfaces would broaden applicability [11].

Conclusion

This research applied Morse-Bott theory to the classification of fire return interval regimes in the southern African bushveld, using topological invariants of posterior surfaces as discriminative features. The method correctly classified 89.1 percent of 87 monitoring stations as unimodal or bimodal, outperforming standard mixture model tests. Bimodal stations, concentrated in areas with mixed grass-shrub fuel beds, showed richer topological structure in their posterior surfaces, with higher counts of critical submanifolds and larger Betti number sums.

The theoretical contribution—explicit Morse-Bott polynomial formulas for profile likelihood functions on flag manifolds—provides a framework applicable beyond fire ecology to any sequential monitoring problem where posterior degeneracy signals regime ambiguity. Future work should incorporate fire intensity data alongside frequency and extend the Morse-Bott framework to higher-dimensional parameter spaces where three or more overlapping fire regimes may coexist.

Acknowledgements

Funding sources

This research was supported by departmental research allocations with no external funding.

Institutional support

The Quezon City College of Applied Mathematics provided computing infrastructure.

Contributions not qualifying for authorship

Dr. Maria Santos helped with MODIS data extraction and quality screening.

References

1. Bott R. Nondegenerate critical manifolds. *Ann Math.* 1954;60(2):248–261.
2. Bond WJ, Keeley JE. Fire as a global herbivore: the ecology and evolution of flammable ecosystems. *Trends Ecol Evol.* 2005;20(7):387–394.
3. Nicolaescu LI. An invitation to Morse theory. 2nd ed. New York: Springer; 2011.
4. Milnor J. Morse theory. Princeton (NJ): Princeton University Press; 1963.
5. Archibald S, Roy DP, van Wilgen BW, Scholes RJ. What limits fire? An examination of drivers of burnt area in Southern Africa. *Glob Change Biol.* 2009;15(3):613–630.
6. Mucina L, Rutherford MC. The vegetation of South Africa, Lesotho and Swaziland. Pretoria: South African National Biodiversity Institute; 2006.
7. Giglio L, Schroeder W, Justice CO. The Collection 6 MODIS active fire detection algorithm and fire products. *Remote Sens Environ.* 2016; 178:31–41.
8. Robert CP, Casella G. Monte Carlo statistical methods. 2nd ed. New York: Springer; 2004.
9. Gelman A, Rubin DB. Inference from iterative simulation using multiple sequences. *Stat Sci.* 1992;7(4):457–472.
10. McLachlan GJ, Peel D. Finite mixture models. New York: John Wiley & Sons; 2000.
11. Austin JD. Morse-Bott theory and equivariant cohomology. In: Hofer H, Taubes C, Weinstein A, Zehnder E, editors. The Floer memorial volume. Basel: Birkhäuser; 1995. p. 123–183.
12. van Wilgen BW, Govender N, Biggs HC, Ntsala D, Funda XN. Response of savanna fire regimes to changing fire-management policies in a large African national park. *Conserv Biol.* 2004;18(6):1533–1540.
13. Lehmann EL, Romano JP. Testing statistical hypotheses. 3rd ed. New York: Springer; 2005.
14. Wasserman L. All of nonparametric statistics. New York: Springer; 2006.
15. Ghrist R. Elementary applied topology. North Charleston (SC): CreateSpace Independent Publishing Platform; 2014.
16. Scholes RJ, Archer SR. Tree-grass interactions in savannas. *Annu Rev Ecol Syst.* 1997; 28:517–544.
17. Edelsbrunner H, Harer JL. Computational topology: an introduction. Providence (RI): American Mathematical Society; 2010.
18. Govender N, Trollope WSW, van Wilgen BW. The effect of fire season, fire frequency, rainfall and management on fire intensity in savanna vegetation in South Africa. *J Appl Ecol.* 2006;43(4):748–758.

19. Matsumoto Y. An introduction to Morse theory. Providence (RI): American Mathematical Society; 2002.
20. Sankaran M, Hanan NP, Scholes RJ, Ratnam J, Augustine DJ, Cade BS, *et al.* Determinants of woody cover in African savannas. *Nature.* 2005;438(7069):846–849.