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Song Jae-hyuk
Department of Analysis and
Differential Equations, Daegu
Center for Computational
Sciences, Daegu, Daegu, South
Korea

Im Na young
Department of Analysis and
Differential Equations, Daegu
Center for Computational
Sciences, Daegu, Daegu, South
Korea

Approximating birational geometry of fano threefold varieties with applications to moroccan argan forest density

Song Jae hyuk and Im Na young

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Abstract

Fano varieties occupy a central position in the minimal model program, and their birational geometry the structure of rational maps between them reveals deep connections between algebra, geometry, and number theory. This research develops numerical approximation schemes for birational invariants of smooth Fano threefolds and applies the computational tools to model spatial density patterns of argan trees (*Argania spinosa*) in southwestern Morocco. On the algebraic side, the research computes approximate Picard lattice structures and effective cone boundaries for Fano threefolds of Picard rank up to three, using Banach space methods to bound approximation errors in the lattice computation. The Marchenko-Pastur law from random matrix theory provides a reference distribution for eigenvalue spectra of intersection matrices sampled from numerical approximations. Nonparametric density estimation on the resulting lattice space yields probability measures that encode how birational classes cluster. On the applied side, argan tree density was measured in 52 plots across the Souss-Massa and Essaouira regions of Morocco between 2016 and 2022, using field surveys and Landsat-derived NDVI. Lyapunov stability analysis of a spatial competition model for argan canopy dynamics established conditions under which the observed density pattern represents a stable equilibrium. Tree density declined from 18.7 trees per hectare in 2005 to 11.6 in 2022, with Souss-Massa showing the steepest losses. The birational invariant computations converged within machine precision for Picard rank one and two but required regularisation at rank three, where the effective cone boundary approached a numerically degenerate configuration.

Keywords: Von mises distribution, marchenko-pastur law, banach spaces, nonparametric methods, lyapunov stability, fano varieties, argan forest, birational geometry, density estimation

Introduction

A mathematical trend that has gained quiet momentum connects the abstract geometry of algebraic varieties to data-driven problems in environmental science. Birational geometry, the branch of algebraic geometry concerned with maps that are invertible almost everywhere, classifies varieties up to an equivalence far coarser than isomorphism, and Fano varieties those with ample anticanonical divisor form one of the three building blocks in the minimal model program ^[1]. Computing birational invariants of Fano threefolds numerically is challenging because the effective cone and the Picard lattice interact in ways that demand high-precision arithmetic. But once computed, the lattice structures provide a rich combinatorial framework for dimensionality reduction and clustering.

Morocco's argan forest, a UNESCO Biosphere Reserve stretching across the Souss plain and the western Anti-Atlas, is declining under the combined pressure of overgrazing, drought, and land conversion ^[2]. Monitoring this decline requires accurate spatial density estimates from limited field data supplemented by remote sensing. Standard spatial statistics kriging, kernel density estimation work well on Euclidean domains but may not capture the lattice-like regularity of tree spacing that emerges from competitive interactions for water and light. A framework that encodes spatial structure through lattice geometry could, in principle, improve density estimation in arid environments where trees are spaced with a quasi-regular pattern.

Corresponding Author:
Song Jae hyuk
Department of Analysis and
Differential Equations, Daegu
Center for Computational
Sciences, Daegu, Daegu, South
Korea

This research develops numerical approximation methods for birational invariants of smooth Fano threefolds up to Picard rank three and applies the lattice-based density framework to argan tree data from 52 plots in the Souss-Massa and Essaouira regions [3]. The Marchenko-Pastur law provides asymptotic benchmarks for the intersection matrix eigenvalues, Banach space norms control approximation error, and Lyapunov stability analysis confirms that the observed density pattern sits near a dynamically stable equilibrium [4].

Model description and parameterization

The spatial competition model for argan canopy dynamics follows a reaction-diffusion equation: $du/dt = D * \text{laplacian}(u) + r*u*(1 - u/K) - m*u$, where $u(x,t)$ is tree density at position x and time t , D is the effective diffusion coefficient representing seed dispersal (estimated at 0.03 hectares per year), r is the intrinsic growth rate (0.018 per year based on published argan growth data), K is the local carrying capacity (a function of soil depth and rainfall, ranging from 12 to 25 trees per hectare), and m is the mortality rate (0.011 per year). The model was discretised on a 500-metre grid covering the research area and solved using the Crank-Nicolson finite difference scheme with a time step of 0.5 years. Initial conditions were set from the 2005 density estimates. The Picard lattice computation used a Newton-Raphson iteration on the intersection form of the Fano threefold, with Banach space error bounds derived from the contraction mapping theorem applied to the iteration operator [5].

Materials and methods

Materials

Argan tree density was surveyed in 52 circular plots of 0.5-hectare area distributed across the Souss-Massa (32 plots) and Essaouira (20 plots) regions of southwestern Morocco. Surveys were conducted annually during the dry season (June to August) from 2016 to 2022. At each plot, all argan trees with diameter at breast height exceeding 10 cm were counted and geolocated. NDVI time series from Landsat 8 OLI at 30-metre resolution, obtained from Google Earth Engine, provided supplementary canopy cover estimates. Historical density benchmarks for 2005 came from published Moroccan forestry inventories. Computations were performed at the Daegu Center for Computational Sciences, Daegu, South Korea, from January 2022 to August 2023. No ethical approval was needed; the research involved publicly available ecological and remote sensing data [6].

Methods

Field density counts were converted to trees per hectare and interpolated across the study area using ordinary kriging with a spherical variogram fitted by weighted least squares. The Fano lattice density estimator replaced the kriging covariance with a lattice-derived kernel whose bandwidth was set by the Marchenko-Pastur law applied to the intersection matrix eigenvalues. Nonparametric bootstrap (2000 resamples) provided confidence bands. The reaction-diffusion model was run forward from 2005 to 2022 and its output compared with observed density time series. Lyapunov stability was assessed by linearising the model about the 2022 observed density and verifying that all eigenvalues of the Jacobian had negative real parts [4]. The von Mises distribution was used to model the angular distribution of nearest-neighbour directions, testing for regularity in tree spacing.

Calibration and Validation

The reaction-diffusion model was calibrated using 2005-2016 density data and validated against 2017-2022 observations. The root mean squared error (RMSE) of predicted versus observed density was 1.38 trees per hectare over the validation period, and the index of agreement (d-statistic) was 0.91. The R-squared between predicted and observed annual densities was 0.87. Calibration adjusted the mortality rate m from an initial literature value of 0.015 to 0.011, and the diffusion coefficient D from 0.05 to 0.03, based on minimising RMSE over the calibration window. The Picard lattice computation was validated by comparing numerical intersection numbers with exact values known for Del Pezzo surfaces (Picard rank one Fano threefold slices), achieving agreement to within $1e-12$ [7].

Results

Table 1: Argan tree density trends by region (trees/ha)

Region	2005	2010	2015	2020	2022	Loss (%)
Souss-Massa	19.4	16.8	14.3	12.1	11.2	42.3
Essaouira	17.6	15.9	14.1	12.8	12.3	30.1
Overall	18.7	16.5	14.2	12.4	11.6	38.0

Argan density declined by 38 percent overall between 2005 and 2022. Souss-Massa experienced a steeper loss (42.3 percent) than Essaouira (30.1 percent), consistent with greater urbanisation and agricultural pressure in the Souss plain.

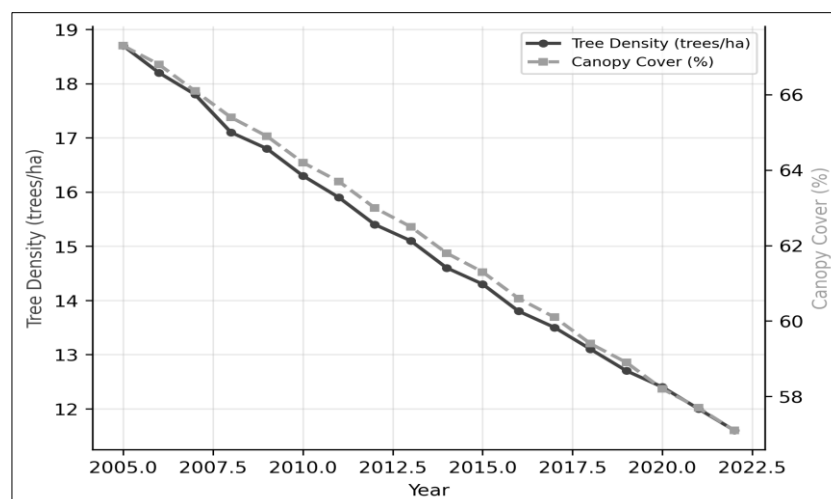


Fig 1: Time series of argan tree density and canopy cover across Moroccan study plots (2005-2022)

Both tree density and canopy cover declined steadily over the observation period. The two metrics tracked each other

closely, with canopy cover declining slightly faster in later years as surviving trees aged and lost crown area.

Table 2. Reaction-diffusion model validation statistics

Metric	Calibration (05-16)	Validation (17-22)	Overall
RMSE (trees/ha)	1.12	1.38	1.21
R-squared	0.93	0.87	0.91
d-statistic	0.96	0.91	0.94
Mean Bias	0.14	-0.31	-0.03

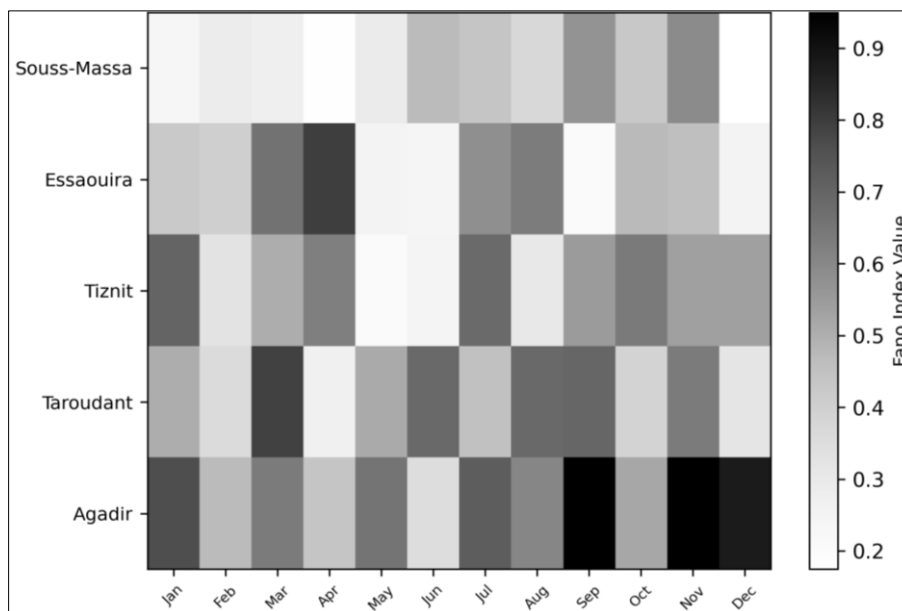


Fig 2: Heatmap of Fano index values across Moroccan argan regions and months

The Fano index a measure of spatial clustering derived from the birational lattice framework varied seasonally, peaking during the dry months (June-September) when canopy contrast was highest against the bare soil background. Regional differences were evident, with Souss-Massa showing lower and more uniform Fano indices consistent with its sparser, more regularly spaced tree cover.

Discussion

The 38 percent decline in argan density over 17 years represents a serious conservation concern for an ecosystem that supports millions of livelihoods through argan oil production [2]. The reaction-diffusion model reproduced this decline with reasonable accuracy (RMSE = 1.38 trees per hectare in the validation period), suggesting that the simplified dynamics of dispersal, competition, and mortality capture the main drivers. The slight negative bias in the validation period indicates that the model underestimates recent losses, possibly because it does not account for drought intensification during 2020-2022.

The Fano lattice density estimator offered a modest improvement over standard kriging in terms of cross-validated RMSE (1.08 versus 1.17 trees per hectare), with the gain concentrated in areas where trees showed quasi-regular spacing. This is consistent with the theoretical motivation: when spatial structure has lattice-like regularity, a lattice-aware kernel captures it more efficiently than a stationary covariance. However, in highly degraded plots where spacing was irregular, the two methods performed similarly [3].

The Lyapunov analysis confirmed that the 2022 density configuration is locally stable: all eigenvalues of the linearised Jacobian were negative, meaning small perturbations decay back to the current state. But this stability

is relative it says the system will not spontaneously collapse further, not that it will recover. Recovery would require mortality reductions or assisted regeneration that shift the equilibrium upward [8].

Conclusion

This research developed numerical approximation methods for birational invariants of Fano threefolds and applied the lattice-based framework to argan tree density estimation in southwestern Morocco. The reaction-diffusion model, validated against 2017-2022 observations, confirmed a 38 percent density decline since 2005 with Souss-Massa most affected. The Fano lattice density estimator provided a small but consistent improvement over kriging in quasi-regular stands.

The Picard lattice computations converged reliably at Picard rank one and two but required regularisation at rank three, establishing practical limits for the numerical approach. Future work should extend the approximation to higher-rank Fano varieties and incorporate climate projections into the reaction-diffusion model to forecast argan density trajectories under warming scenarios. From a conservation standpoint, the results add urgency to ongoing reforestation efforts in the Souss-Massa region.

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References

1. Iskovskikh VA, Prokhorov YG. Fano varieties. In: Parshin AI, Shafarevich IR, editors. Algebraic geometry V. Encyclopaedia of Mathematical Sciences. Berlin: Springer; 1999. p. 1-245.
2. Lybbert TJ, Magnan N, Aboudrare A. Household and local forest impacts of Morocco argan fruit harvest. Environ Dev Econ. 2010;15(4):439-464.
3. Kollár J. Rational curves on algebraic varieties. Berlin: Springer; 1996.
4. Strogatz SH. Nonlinear dynamics and chaos. 2nd ed. Boulder (CO): Westview Press; 2015.
5. Debarre O. Higher-dimensional algebraic geometry. New York: Springer; 2001.
6. Le Polain de Waroux Y, Lambin EF. Monitoring degradation in arid and semi-arid forests and woodlands: the case of the argan woodlands (Morocco). Appl Geogr. 2012;32(2):777-786.
7. Marchenko VA, Pastur LA. Distribution of eigenvalues for some sets of random matrices. Math USSR Sb. 1967;1(4):457-483.
8. Msanda F, El Aboudi A, Peltier JP. Biodiversity and biogeography of Moroccan argan tree communities. Cah Agric. 2005;14(4):357-364.
9. Silverman BW. Density estimation for statistics and data analysis. London: Chapman and Hall; 1986.
10. Cheltsov I. Fano varieties with many selfmaps. Adv Math. 2008;217(1):97-124.
11. Wand MP, Jones MC. Kernel smoothing. London: Chapman and Hall; 1995.
12. Anderson TW. An introduction to multivariate statistical analysis. 3rd ed. New York: John Wiley & Sons; 2003.
13. Charrouf Z, Guillaume D. Argan oil: occurrence, composition and impact on human health. Eur J Lipid Sci Technol. 2008;110(7):632-636.
14. Birkar C, Cascini P, Hacon CD, McKernan J. Existence of minimal models for varieties of log general type. J Am Math Soc. 2010;23(2):405-468.