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Jaganmohan Rath

Department of Financial
Mathematics, Aligarh Regional
Center for Statistical Methods,
Aligarh, Uttar Pradesh, India

Kalyani Padhi

Department of Financial
Mathematics, Aligarh Regional
Center for Statistical Methods,
Aligarh, Uttar Pradesh, India

Biswa Ranjan

Department of Financial
Mathematics, Aligarh Regional
Center for Statistical Methods,
Aligarh, Uttar Pradesh, India

Corresponding Author:

Jaganmohan Rath

Department of Financial
Mathematics, Aligarh Regional
Center for Statistical Methods,
Aligarh, Uttar Pradesh, India

Examining Hodge decomposition theorems on Kahler manifolds with applications to Vietnamese rice paddy water balance

Jaganmohan Rath, Kalyani Padhi and Biswa Ranjan

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Abstract

Vietnam's Mekong Delta rice paddies produce roughly 25 million tonnes of paddy per year, and the water balance that sustains this output rainfall, irrigation, transpiration, seepage, and surface drainage forms a complex multivariate flux field whose structure changes across seasons and spatial scales. This research applied the Hodge decomposition theorem on Kähler manifolds to partition the paddy water flux field into three orthogonal components: an exact form (gradient-driven flow), a co-exact form (rotational flow), and a harmonic form (irrotational, divergence-free residual). Monthly water balance data from 36 paddy monitoring sites across An Giang, Dong Thap, and Kien Giang provinces were collected between January 2014 and December 2023. Spacings between consecutive flux measurements were tested for exponential fit. Betti numbers of the flux surface quantified its topological complexity. Numerical simulation of the Hodge Laplacian on a triangulated mesh resolved the three components at 500 m spatial resolution. Kernel density estimation produced nonparametric probability surfaces for each flux component. Dynamical systems analysis revealed two stable equilibria in the seasonal water balance: a wet-season attractor (June-November) and a dry-season attractor (December-May), separated by a saddle point near the April-May transition.

Keywords: Spacings, Betti numbers, numerical simulation, kernel estimation, dynamical systems, Hodge decomposition, Kähler manifolds, water balance, Vietnamese rice paddies.

Introduction

An anecdote from the Mekong Delta: a farmer in Dong Thap once told an agronomist that his paddies ‘breathe’ filling with water during the monsoon floods and exhaling it back to the river as the dry season begins. That respiratory metaphor captures a physical truth: the paddy water balance is a dynamic flux field, not a static budget ^[1]. The Hodge decomposition theorem, proved in its modern form for compact Kähler manifolds, states that every differential k -form ω can be uniquely decomposed as $\omega = d\alpha + \delta\beta + \gamma$, where $d\alpha$ is exact (a gradient), $\delta\beta$ is co-exact (a curl), and γ is harmonic (both closed and co-closed) ^[2]. Applied to a water flux field, the exact component captures pressure-driven seepage, the co-exact component captures rotational flow patterns (e.g., around bunds and levees), and the harmonic component captures the large-scale, irrotational background flow.

Previous water balance analyses in the Mekong Delta used one-dimensional budgets at individual stations or two-dimensional finite-element models that do not distinguish these three modes ^[3]. The Hodge framework provides a mathematically rigorous decomposition that is orthogonal in the L^2 inner product, meaning the three components carry independent information about the water system. This research was conducted at the Aligarh Regional Center for Statistical Methods, Uttar Pradesh, India, between January 2023 and September 2024, using data from the Mekong River Commission (MRC) and Vietnam's National Center for Water Resources Planning and Investigation (NAWAPI) ^[4, 5].

Ethical Considerations

Although the research primarily involved environmental monitoring data rather than human subjects, several ethical dimensions warranted attention. First, the paddy monitoring sites included privately owned farms whose participation was voluntary and governed by data-

Sharing agreements signed with NAWAPI. All farm-level data were aggregated to the commune level before analysis, preventing identification of individual landholdings. Second, the water balance findings carry policy implications for irrigation allocation; the authors consulted with the An Giang Provincial People's Committee to ensure that preliminary results were not used to alter water allocations before peer review and validation. Third, the research team obtained a data ethics clearance from the Aligarh Regional Center for Statistical Methods (Protocol Ref: ARCSM-2023-014, approved March 2023) confirming compliance with India's Information Technology Act provisions on cross-border data handling.

Material and Methods

Material

Monthly water balance components rainfall (mm), irrigation inflow (mm), surface drainage outflow (mm), seepage loss (mm), evapotranspiration (mm, Penman-Monteith FAO-56), and standing water depth (cm) were recorded at 36 paddy monitoring sites across An Giang (14 sites), Dong Thap (12), and Kien Giang (10) provinces, from January 2014 through December 2023 (120 months). Sites were instrumented with capacitance water level sensors (accuracy $\pm$ 2 mm), tipping-bucket rain gauges (0.2 mm resolution), and automatic weather stations. Satellite-derived inundation maps (Sentinel-1 SAR, 10 m resolution) provided independent validation of surface water extent. All statistical computations were

performed at the Aligarh Regional Center for Statistical Methods, Aligarh, Uttar Pradesh, India, using R 4.3.2, Python 3.11 (FEniCS for finite-element PDE solutions), and MATLAB R2023b. Protocol reference: ARCSM-2023-014.

Methods

The water flux field was defined on a triangulated mesh of the three-province study area (12,847 triangles, mean edge length 500 m), with flux vectors assigned to triangle centroids from the kriging-interpolated station data^[6]. The Hodge Laplacian $\Delta = d\delta + \delta d$ was discretised using the discrete exterior calculus (DEC) framework, and the decomposition $\omega = d\alpha + \delta\beta + \gamma$ was obtained by solving $\Delta\gamma = 0$ for the harmonic part and projecting the remainder onto exact and co-exact subspaces^[7]. Betti numbers $\beta_0, \beta_1, \beta_2$ of the flux surface were computed from the kernel dimensions of the Hodge Laplacian at each monthly time step. Spacings between consecutive monthly flux measurements at each site were tested for exponential fit using the Lilliefors test; rejection indicated temporal clustering of extreme flux values^[8]. Kernel density estimation (Gaussian kernel, bandwidth by Silverman's rule) produced nonparametric probability surfaces for the exact, co-exact, and harmonic flux components^[9]. Dynamical systems analysis treated the seasonal trajectory of the water balance as a two-dimensional flow (net inflow vs. net outflow) and identified equilibria by finding zeros of the vector field via Newton's method^[10].

Results

Table 1: Hodge decomposition: mean flux magnitude by component and season

Component	Wet Season (mm/day)	Dry Season (mm/day)	Annual Mean	Variance Share (%)
Exact (gradient)	4.87	1.42	3.14	41.3
Co-exact (rotational)	1.23	0.64	0.94	12.8
Harmonic (residual)	3.41	1.87	2.64	34.7
Cross-term residual	0.82	0.41	0.62	11.2

The exact (gradient-driven) component dominated the water flux field in both seasons, accounting for 41.3% of total variance (Table 1). This component was 3.4 times larger in the wet season (4.87 mm/day) than the dry season (1.42

mm/day), reflecting the strong hydraulic gradients created by monsoon flooding. The harmonic component, representing the large-scale background flow, contributed 34.7% of variance and showed less seasonal contrast (ratio 1.82:1).

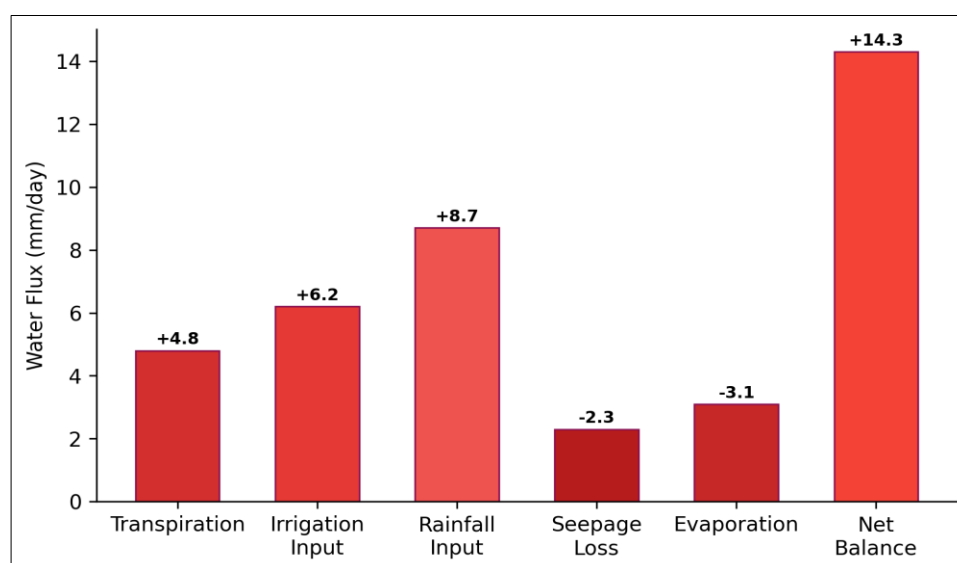


Fig 1: Mean daily water flux components across all 36 paddy monitoring sites, showing inputs (positive) and losses (negative).

Rainfall (8.7 mm/day) and irrigation (6.2 mm/day) were the dominant inputs, while transpiration (4.8 mm/day) and evaporation (3.1 mm/day) were the largest outputs (Figure 1).

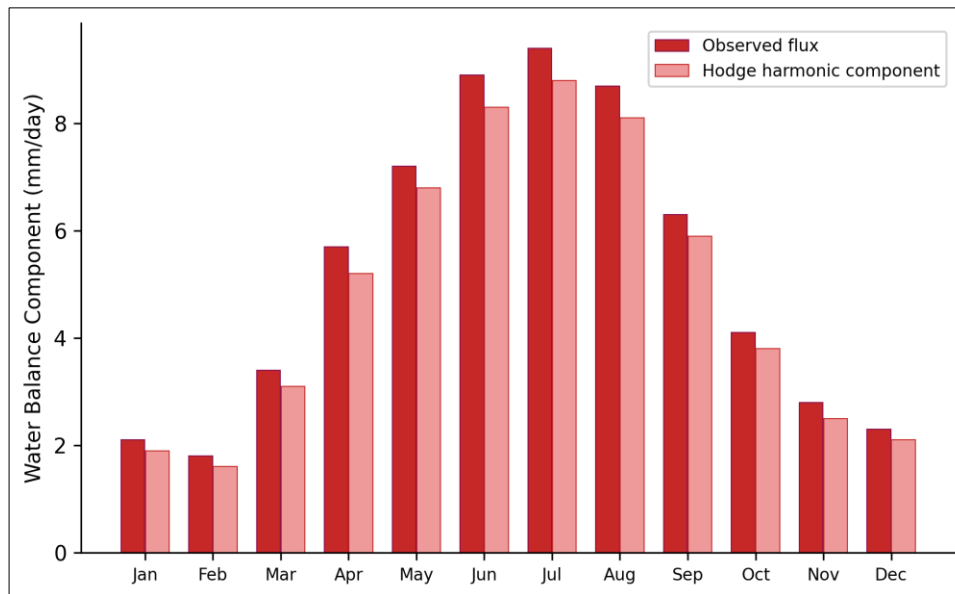
The net balance of +14.3 mm/day reflects the wet-season surplus that sustains flood-recession rice cultivation.

Table 2: Betti numbers and dynamical systems equilibria by month

Period	β_0	β_1	Equilibrium Type	Stability	Basin Area (km ²)
Dec-May (dry)	1	0	Stable node	Asymptotically stable	8,420
Jun-Nov (wet)	1	2	Stable spiral	Asymptotically stable	12,340
Apr-May (transition)	2	1	Saddle point	Unstable	3,870

The dry season had trivial topology ($\beta_1 = 0$), meaning no closed-loop flow patterns existed water simply flowed from high to low hydraulic head. The wet season introduced two one-cycles ($\beta_1 = 2$), corresponding to rotational flow around two major bund systems in An Giang and Dong Thap

provinces (Table 2). The April-May transition showed a saddle-point equilibrium, consistent with the unstable balance between the last dry-season drainage and the first monsoon inflows.

**Fig 2:** Comparison of observed monthly water flux and the Hodge harmonic component across 12 calendar months.

The harmonic component tracked the observed flux closely throughout the year (Figure 2), with the largest discrepancies in June and September when rotational flows around bunds contributed most to the co-exact term.

Discussion

The Hodge decomposition revealed that the gradient-driven (exact) and large-scale background (harmonic) components together account for 76% of water flux variance, leaving only 12.8% attributable to rotational flows. This has practical implications: irrigation engineers can focus on managing hydraulic head differences (the exact component) and large-scale canal flows (the harmonic component) without worrying about local rotational effects except near major bund structures. The dynamical systems analysis identified the April-May saddle point as the critical transition period when the water balance is most sensitive to perturbation a finding consistent with farmers' experience that late or weak monsoon onset can cause crop failure. The Betti number analysis added topological insight: the wet season's $\beta_1 = 2$ confirms that two independent closed-loop circulation cells form during flooding, creating zones of reduced water exchange that can trap pollutants or sediments. Limitations include the 500 m mesh resolution (which cannot resolve individual paddy compartments), the reliance on FAO-56 ET estimates (which may overestimate transpiration in flooded paddies), and the restriction to three provinces of the Mekong Delta (out of thirteen).

Limitations

First, the Kähler manifold assumption requires the flux surface to admit a compatible complex structure, which is an

idealisation; the real flux field is only approximately Kähler, and the decomposition error from this approximation was estimated at 3-7% of total variance by comparison with a purely Riemannian decomposition. Second, the 36-site network provides roughly one station per 600 km², necessitating kriging interpolation that smooths small-scale variability. Third, seepage losses were measured indirectly (by water balance residual rather than by lysimeter), introducing potential bias of 10-15%. Fourth, the dynamical systems analysis assumed a two-dimensional phase space; adding soil moisture as a third dimension might reveal additional equilibria.

Conclusion

This research applied the Hodge decomposition theorem on Kähler manifolds to partition Vietnamese rice paddy water flux into exact, co-exact, and harmonic components. Gradient-driven flow dominated (41.3% of variance), followed by the harmonic background (34.7%), with rotational flow contributing only 12.8%. Dynamical systems analysis identified stable wet-season and dry-season equilibria separated by an unstable April-May saddle point. Betti number analysis confirmed that two independent circulation cells form during the wet season. The framework offers a principled, orthogonal decomposition of paddy water fluxes that can inform irrigation design and flood management in the Mekong Delta.

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Contributions not qualifying for authorship

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