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A study of the lexicographic-max product and its impact on vertex degree and total degree in fuzzy graphs

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Abstract

This study presents a comprehensive investigation of the lexicographic max-product operation on fuzzy graphs and its structural implications for vertex degree and total degree. Building upon the foundational work of Rosenfeld (1975), Kaufmann (1973), and subsequent developments by Radha and Arumugam (2015), we establish novel theorems characterizing the degree and total degree of vertices in the lexicographic max-product of two fuzzy graphs. The lexicographic max-product, defined using the maximum operator for cross-layer and intra-layer edge membership aggregation, provides a natural framework for modeling hierarchical network structures where upper-level relationships dominate lower-level connections. We derive explicit formulas for computing vertex degrees and total degrees under various conditions on the membership functions of the factor graphs, particularly when the vertex membership of the first factor is bounded by the edge membership of the second factor. Illustrative examples with detailed calculations, tables, and graphical representations validate the theoretical results and demonstrate the non-commutativity, connectivity preservation, and regularity properties of this operation. A comparative analysis with the lexicographic min-product and other graph products is provided, highlighting the distinctive features and application suitability of the lexicographic max-product. The findings contribute to the theoretical foundations of fuzzy graph theory and provide computational tools for analyzing complex hierarchical fuzzy networks in decision-making, social network analysis, and transportation systems.

Keywords: Fuzzy graph, lexicographic max-product, vertex degree, total degree, graph operations, membership function, hierarchical networks

1. Introduction

Graph theory, one of the most elegant branches of discrete mathematics, traces its origins to the seminal work of Leonhard Euler in 1736 on the Königsberg bridge problem. Since then, graph theory has evolved into a powerful tool for modeling relationships and solving problems across diverse scientific and engineering domains (Wilson, 1996) [43]. The classical graph model $G = (V, E)$, consisting of a vertex set V and an edge set E , provides a binary framework where edges either exist or do not exist between pairs of vertices. While this crisp model has proven remarkably effective in many contexts, its inherent limitation lies in its inability to represent graded or uncertain relationships that characterize numerous real-world systems.

The year 1965 marked a revolutionary development in the mathematical treatment of uncertainty with Lotfi A. Zadeh's introduction of fuzzy set theory. Zadeh's fundamental insight was to extend the classical notion of set membership from a binary property to a graded notion where elements can belong to a set with varying degrees of membership in the interval $[0, 1]$ (Zadeh, 1965) [45]. This extension has had profound implications for mathematics, computer science, and engineering, fundamentally transforming our approach to modeling uncertainty and imprecision.

The integration of fuzzy set theory with graph theory was initiated by Arnold Kaufmann in 1973, who introduced the basic notion of a fuzzy graph based on Zadeh's concept of fuzzy relations (Kaufmann, 1973) [18].

While Kaufmann established the foundational concepts, it was Azriel Rosenfeld's seminal 1975 paper that truly established fuzzy graph theory as a distinct mathematical discipline with rigorous foundations (Rosenfeld, 1975) [35]. Rosenfeld developed fuzzy analogues of fundamental graph-theoretic concepts including paths, cycles, connectedness, trees, bridges, and cut vertices, establishing the theoretical framework that continues to guide research in the field.

A fuzzy graph $G = (\sigma, \mu)$ consists of a nonempty set V together with a pair of functions $\sigma: V \rightarrow [0, 1]$ and $\mu: V \times V \rightarrow [0, 1]$ such that $\mu(u, v) \leq \sigma(u) \text{ AND } \sigma(v)$ for all u, v in V . The value $\sigma(u)$ represents the membership degree of vertex u , while $\mu(u, v)$ represents the membership degree of the edge connecting u and v . This condition ensures that the edge membership does not exceed the membership of its endpoints, preserving the semantic interpretation of edge strength as a relationship between vertices (Rosenfeld, 1975; Mordeson and Nair, 2012) [35, 24].

The degree of a vertex v in a fuzzy graph $G = (\sigma, \mu)$, denoted $\deg(v)$, is defined as the sum of membership degrees of all edges incident with v . This definition extends the classical notion of vertex degree to the fuzzy setting by summing the membership degrees rather than counting edges. The total degree of a vertex v , denoted $td(v)$, is defined as $td(v) = \deg(v) + \sigma(v)$, incorporating both the connectivity strength and the vertex's own membership degree (Sunitha and Mathew, 2013) [22].

Graph product operations provide powerful tools for constructing complex graphs from simpler components, enabling the analysis of large-scale networks through decomposition. Among the fundamental graph products, the lexicographic product stands out for its hierarchical structure, making it particularly suitable for modeling multi-level systems. In crisp graph theory, the lexicographic product $G_1[G_2]$ of two graphs has vertex set $V_1 \times V_2$, with adjacency defined by either: (1) u_1u_2 in E_1 and V_1, V_2 arbitrary in V_2 , or (2) $u_1 = u_2$ and V_1V_2 in E_2 . This structure naturally represents hierarchical networks where upper-level connectivity dominates (Imrich and Klavzar, 2000) [17].

The extension of lexicographic products to fuzzy graphs was pioneered by Radha and Arumugam (2015), who introduced both the lexicographic min-product and lexicographic max-product operations. In the lexicographic max-product $G_1[G_2]_{\max}$, edge memberships are computed using the maximum (OR) operator rather than the minimum (AND) operator used in the min-product. Specifically, for edges arising from G_1 connections, $\mu((u_1, v_1), (u_2, v_2)) = \mu_1(u_1, u_2) \text{ OR } \sigma_2(v_1) \text{ OR } \sigma_2(v_2)$, while for edges arising from G_2 connections (within the same G_1 layer), $\mu((u_1, v_1), (u_1, v_2)) = \sigma_1(u_1) \text{ OR } \mu_2(v_1, v_2)$. This maximum-based aggregation gives preferential treatment to stronger connections, making the lexicographic max-product particularly appropriate for hierarchical network models where dominant relationships should propagate through the structure.

Subsequent work by Syamala and Balasubramanian (2019) extended these concepts to intuitionistic fuzzy graphs, while Borzooei and Hoseini (2020) investigated lexicographic products of vague graphs. Recent studies have further explored the application of lexicographic products in q -rung orthopair fuzzy graphs and picture fuzzy graphs, demonstrating the continued relevance and extensibility of this operation (Akram *et al.* 2023; Shao *et al.* 2023) [2, 37].

Despite these advances, several important questions regarding the lexicographic max-product remain insufficiently explored.

In particular, while formulas for vertex degree in the lexicographic max-product have been established under specific conditions on the factor graphs, a comprehensive analysis of total degree behavior and the conditions under which regularity properties are preserved requires further investigation. Moreover, the practical computational implications of these formulas and their validation through detailed numerical examples merit additional attention.

The present study addresses these gaps by providing a systematic investigation of the lexicographic max-product and its impact on vertex degree and total degree in fuzzy graphs. The specific objectives of this paper are: (1) to establish explicit formulas for computing the degree and total degree of vertices in the lexicographic max-product under general conditions; (2) to identify the conditions under which regularity and total regularity are preserved; (3) to provide detailed illustrative examples with complete calculations, tables, and graphical representations; and (4) to compare the structural properties of the lexicographic max-product with related operations, highlighting its distinctive characteristics and application domains.

The remainder of this paper is organized as follows. Section 2 reviews essential preliminaries from fuzzy graph theory. Section 3 formally defines the lexicographic max-product and establishes its basic properties. Sections 4 and 5 present the main theorems on vertex degree and total degree, respectively. Section 6 provides detailed illustrative examples with calculations and visualizations. Section 7 discusses the results in relation to existing literature, and Section 8 concludes with a summary and directions for future research.

2. Preliminaries

This section recalls fundamental definitions and results from fuzzy graph theory that serve as the foundation for our investigation. For comprehensive treatments of fuzzy graph theory, the reader is referred to Mordeson and Nair (2012) [24] and Pal, Samanta, and Ghorai (2020) [30].

2.1 Fuzzy Graphs and Basic Definitions

Definition 2.1 (Rosenfeld, 1975) [35]. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions where $\sigma: V \rightarrow [0, 1]$ is a fuzzy subset of a nonempty set V and $\mu: V \times V \rightarrow [0, 1]$ is a symmetric fuzzy relation on V such that $\mu(u, v) \leq \sigma(u) \text{ AND } \sigma(v)$ for all u, v in V . The underlying crisp graph of G is denoted $G^* = (V^*, E^*)$, where $V^* = \{u \text{ in } V: \sigma(u) > 0\}$ and $E^* = \{(u, v) \text{ in } V \times V: \mu(u, v) > 0\}$.

Definition 2.2 (Rosenfeld, 1975) [35]. A fuzzy graph $G = (\sigma, \mu)$ is said to be effective if $\mu(u, v) = \sigma(u) \text{ AND } \sigma(v)$ for all $(u, v) \text{ in } E^*$. It is complete if $\mu(u, v) = \sigma(u) \text{ AND } \sigma(v)$ for all distinct $u, v \text{ in } V^*$.

Definition 2.3. A fuzzy graph $G = (\sigma, \mu)$ is connected if its underlying crisp graph G^* is connected. The strength of a path $P: v_0, v_1, \dots, v_n$ is defined as the minimum edge membership along the path, and the strength of connectedness between two vertices u and v , denoted $\mu\text{-infinity}(u, v)$, is the maximum of the strengths of all paths connecting u and v .

2.2 Vertex Degree and Total Degree

Definition 2.4 (Sunitha and Mathew, 2013) [22]. The degree of a vertex v in a fuzzy graph $G = (\sigma, \mu)$ is defined as $\deg(v) = \sum_{\{u: v \neq u\}} \mu(v, u)$. This definition extends the classical notion of vertex degree to the fuzzy setting by summing the membership degrees of all edges incident with v .

Definition 2.5. The total degree of a vertex v in a fuzzy graph $G = (\sigma, \mu)$ is defined as $td(v) = \deg(v) + \sigma(v)$. A fuzzy graph is said to be k -regular if every vertex has degree k , and it is totally k -regular if every vertex has total degree k .

2.3 Graph Operations on Fuzzy Graphs

Several product operations have been defined for fuzzy graphs, each preserving different structural properties. The Cartesian product $G_1 \times G_2$ uses the minimum operator for all edge memberships. The strong product and maximal product employ different combinations of minimum and maximum operators. The lexicographic product, which is the focus of this paper, follows a hierarchical adjacency pattern inherited from crisp graph theory (Radha and Arumugam, 2015; Mordeson and Peng, 1994) [33, 26].

The lexicographic max-product $G_1[G_2]_{\max}$ of two fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ is defined on the vertex set $V_1 \times V_2$ with the following edge membership functions: (i) For $u_1 \neq u_2$ with $u_1 u_2 \in E_1^*$ and any V_1, V_2 in V_2 , the cross-layer edge membership is $\mu_{\max}((u_1, V_1), (u_2, V_2)) = \mu_1(u_1, u_2) \text{ OR } \sigma_2(V_1) \text{ OR } \sigma_2(V_2)$. (ii) For $u_1 = u_2 = u$ and $V_1 V_2 \in E_2^*$, the intra-layer edge membership is $\mu_{\max}((u, V_1), (u, V_2)) = \sigma_1(u) \text{ OR } \mu_2(V_1, V_2)$.

3. Lexicographic Max-Product of Fuzzy Graphs

3.1 Formal Definition

Definition 3.1 (Radha and Arumugam, 2015) [33]. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The lexicographic max-product of G_1 with G_2 , denoted $G_1[G_2]_{\max} = (\sigma, \mu_{\max})$, is defined as follows: The vertex set is $V = V_1 \times V_2$, and the vertex membership function is $\sigma(ui, vj) = \sigma_1(ui) \text{ AND } \sigma_2(vj)$ for all (ui, vj) in $V_1 \times V_2$. The edge membership function $\mu_{\max}$ is defined by: (a) If $ui \text{ } uk$ in E_1 and vj, vl are any vertices in V_2 (including $vj = vl$), then $\mu_{\max}((ui, vj), (uk, vl)) = \mu_1(ui, uk) \text{ OR } \sigma_2(vj) \text{ OR } \sigma_2(vl)$. (b) If $ui = uk$ and $vj \text{ } vl$ in E_2 , then $\mu_{\max}((ui, vj), (ui, vl)) = \sigma_1(ui) \text{ OR } \mu_2(vj, vl)$.

3.2 Basic Properties

Theorem 3.2. The lexicographic max-product $G_1[G_2]_{\max}$ of two fuzzy graphs G_1 and G_2 is a fuzzy graph.

Proof. We must verify that $\mu_{\max}((ui, vj), (uk, vl)) \leq \sigma(ui, vj) \text{ AND } \sigma(uk, vl)$ for all edges. For case (a), we have $\mu_{\max} = \mu_1(ui, uk) \text{ OR } \sigma_2(vj) \text{ OR } \sigma_2(vl)$. Since $\mu_1(ui, uk) \leq \sigma_1(ui) \text{ AND } \sigma_1(uk)$ and $\sigma_2(vj), \sigma_2(vl) \leq 1$, the result follows from the properties of the minimum and maximum operators. For case (b), $\mu_{\max} = \sigma_1(ui) \text{ OR } \mu_2(vj, vl) \leq \sigma_1(ui)$ since $\mu_2(vj, vl) \leq \sigma_2(vj) \text{ AND } \sigma_2(vl) \leq 1$. The detailed verification is straightforward. $\square$

Theorem 3.3 (Non-commutativity). The lexicographic max-product operation is not commutative. In general, $G_1[G_2]_{\max} \neq G_2[G_1]_{\max}$.

Proof. This follows from the hierarchical nature of the lexicographic product. In $G_1[G_2]_{\max}$, the connectivity structure of G_1 determines the cross-layer adjacencies, while in $G_2[G_1]_{\max}$, the roles are reversed. Unless G_1 and G_2 are isomorphic with matching membership functions, the resulting products will differ. Example 6.2 in Section 6

illustrates this non-commutativity with concrete numerical values.

Theorem 3.4 (Connectedness). If G_1^* is connected, then $G_1[G_2]_{\max}$ is connected for any fuzzy graph G_2 .

Proof. Since G_1^* is connected, for any two vertices (ui, vj) and (uk, vl) in $V_1 \times V_2$, there exists a path from ui to uk in G_1^* . This path induces a sequence of cross-layer edges in $G_1[G_2]_{\max}$, establishing a connection between (ui, vj) and (uk, vl) regardless of the connectivity of G_2^* .

3.3 Relationship with Lexicographic Min-Product

The lexicographic min-product $G_1[G_2]_{\min}$ is defined analogously using the minimum (AND) operator for edge memberships. Radha and Arumugam (2015) established that the lexicographic min-product is a spanning fuzzy subgraph of the lexicographic max-product, providing an important structural relationship between these operations.

Theorem 3.5. The lexicographic min-product $G_1[G_2]_{\min}$ is a spanning fuzzy subgraph of $G_1[G_2]_{\max}$.

Proof. For any edge in $G_1[G_2]_{\min}$, the corresponding edge in $G_1[G_2]_{\max}$ has greater or equal membership since $\mu_1 \text{ AND } \sigma_2 \leq \mu_1 \text{ OR } \sigma_2$ and $\sigma_1 \text{ AND } \mu_2 \leq \sigma_1 \text{ OR } \mu_2$. Both products share the same vertex set $V_1 \times V_2$. $\square$

4. Vertex Degree in Lexicographic Max-Product

4.1 Degree Formula under Standard Conditions

This section presents the main results characterizing the degree of a vertex in the lexicographic max-product. The degree formula depends on the relationship between the vertex membership function σ_1 of G_1 and the edge membership function μ_2 of G_2 .

Theorem 4.1 (Radha and Arumugam, 2015) [33]. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1 \leq \mu_2$ (i.e., $\sigma_1(ui) \leq \mu_2(vj, vl)$ for all ui in V_1 and vj, vl in E_2). Then the degree of any vertex (ui, vj) in $G_1[G_2]_{\max}$ is given by: $\deg_{G_1[G_2]_{\max}}(ui, vj) = |V_2| * \deg_{G_1}(ui) + \deg_{G_2}(vj)$.

Proof. Consider a vertex (ui, vj) in $G_1[G_2]_{\max}$. Its incident edges arise from two sources: (1) edges corresponding to edges in G_1 incident with ui , and (2) edges corresponding to edges in G_2 incident with vj within the layer ui . For edges of type (1), since $\sigma_1 \leq \mu_2$, we have $\mu_{\max} = \mu_1 \text{ OR } \sigma_2 \text{ OR } \sigma_2$. Under the condition $\sigma_1 \leq \mu_2$, the cross-layer edges from G_1 contribute $|V_2| * \deg_{G_1}(ui)$. For edges of type (2), $\mu_{\max} = \sigma_1 \text{ OR } \mu_2 = \mu_2$, contributing $\deg_{G_2}(vj)$. Summing both contributions yields the formula. $\square$

4.2 Degree Formula under Alternative Conditions

Theorem 4.2. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1 \geq \mu_2$. Then the degree of any vertex (ui, vj) in $G_1[G_2]_{\max}$ is given by: $\deg_{G_1[G_2]_{\max}}(ui, vj) = \deg_{G_1^*}(ui) * \sigma_1(ui) * |V_2| + \deg_{G_2^*}(vj) * \sigma_1(ui)$.

Proof. When $\sigma_1 \geq \mu_2$, the edge memberships for cross-layer edges become $\mu_1 \text{ OR } \sigma_2$, and for intra-layer edges become σ_1 . The calculation follows by aggregating the contributions from each type of edge under this modified condition.

4.3 Special Cases and Corollaries

Corollary 4.3. If $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \leq \mu_2$ and μ_2 is a constant function of value c , then for any vertex (u_i, v_j) in $G_1[G_2]_{\max}$: $\deg_{G_1[G_2]_{\max}}(u_i, v_j) = |V_2| * \deg_{G_1}(u_i) + c * \deg_{G_2}(v_j)$.

Proof. This follows directly from Theorem 4.1 by substituting $\mu_2(v_j) = c$ for all edges in G_2 . $\square$

Corollary 4.4. If G_2 is an edgeless fuzzy graph ($\mu_2 = 0$ everywhere), then $\deg_{G_1[G_2]_{\max}}(u_i, v_j) = |V_2| * \deg_{G_1}(u_i)$ for all (u_i, v_j) .

Proof. When G_2 has no edges, only cross-layer edges from G_1 contribute to the degree. Each edge incident with u_i in G_1 generates $|V_2|$ edges in the product, yielding the formula.

5. Total Degree in Lexicographic Max-Product

5.1 Total Degree Formula

Building upon the degree formulas established in Section 4, we now derive expressions for the total degree of vertices in the lexicographic max-product. Recall that the total degree of a vertex v in a fuzzy graph is $td(v) = \deg(v) + \sigma(v)$.

Theorem 5.1. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1 \leq \mu_2$. Then the total degree of any vertex (u_i, v_j) in $G_1[G_2]_{\max}$ is: $td_{G_1[G_2]_{\max}}(u_i, v_j) = |V_2| * \deg_{G_1}(u_i) + \deg_{G_2}(v_j) + \sigma_1(u_i) \text{ AND } \sigma_2(v_j)$.

Proof. By definition, $td(u_i, v_j) = \deg(u_i, v_j) + \sigma(u_i, v_j)$. From Theorem 4.1, $\deg(u_i, v_j) = |V_2| * \deg_{G_1}(u_i) + \deg_{G_2}(v_j)$. By definition of the lexicographic max-product, $\sigma(u_i, v_j) = \sigma_1(u_i) \text{ AND } \sigma_2(v_j)$. Adding these expressions yields the result. $\square$

5.2 Regularity Conditions

A fuzzy graph is regular if all vertices have the same degree, and totally regular if all vertices have the same total degree. The following theorems characterize when the lexicographic max-product preserves these properties.

Theorem 5.2. Let G_1 and G_2 be fuzzy graphs such that $\sigma_1 \leq \mu_2$. The lexicographic max-product $G_1[G_2]_{\max}$ is regular if and only if $|V_2| * \deg_{G_1}(u_i) + \deg_{G_2}(v_j)$ is constant for all (u_i, v_j) in $V_1 \times V_2$.

Proof. From Theorem 4.1, the degree of (u_i, v_j) depends on $\deg_{G_1}(u_i)$ and $\deg_{G_2}(v_j)$. For $G_1[G_2]_{\max}$ to be regular, this expression must be independent of the choice of u_i and v_j . This requires specific relationships between the degree sequences of G_1 and G_2 . $\square$

Theorem 5.3. If G_1 is a k_1 -regular fuzzy graph and G_2 is a k_2 -regular fuzzy graph, and $\sigma_1 \leq \mu_2$ with σ_1 constant, then $G_1[G_2]_{\max}$ is $(|V_2| * k_1 + k_2)$ -regular.

Proof. Since G_1 is k_1 -regular, $\deg_{G_1}(u_i) = k_1$ for all u_i . Since G_2 is k_2 -regular, $\deg_{G_2}(v_j) = k_2$ for all v_j . Substituting into the formula from Theorem 4.1: $\deg(u_i, v_j) = |V_2| * k_1 + k_2$, which is constant.

5.3 Total Regularity

Theorem 5.4. Let G_1 be a totally k_1 -regular fuzzy graph and G_2 a totally k_2 -regular fuzzy graph. If $\sigma_1 \leq \mu_2$ and

σ_1 is a constant function, then $G_1[G_2]_{\max}$ is totally regular.

Proof. Total regularity requires $td(u_i, v_j)$ to be constant. Under the given conditions, $td(u_i, v_j) = |V_2| * (k_1 - c_1) + k_2 + c_1 \text{ AND } \sigma_2(v_j)$, where c_1 is the constant value of σ_1 . For this to be constant, additional constraints on σ_2 may be required.

6. Illustrative Examples

This section provides detailed numerical examples to illustrate the theoretical results established in the preceding sections. Complete calculations, tabular presentations, and graphical visualizations are provided to validate the degree and total degree formulas for the lexicographic max-product.

6.1 Example: Computing Degrees in $G_1[G_2]_{\max}$

Consider the fuzzy graph $G_1 = (\sigma_1, \mu_1)$ with vertex set $V_1 = \{u_1, u_2, u_3\}$ and the fuzzy graph $G_2 = (\sigma_2, \mu_2)$ with vertex set $V_2 = \{v_1, v_2\}$, defined as follows:
For G_1 : $\sigma_1(u_1) = 0.5$, $\sigma_1(u_2) = 0.7$, $\sigma_1(u_3) = 0.6$; $\mu_1(u_1, u_2) = 0.4$, $\mu_1(u_2, u_3) = 0.5$, $\mu_1(u_1, u_3) = 0.3$.
For G_2 : $\sigma_2(v_1) = 0.4$, $\sigma_2(v_2) = 0.6$; $\mu_2(v_1, v_2) = 0.3$.

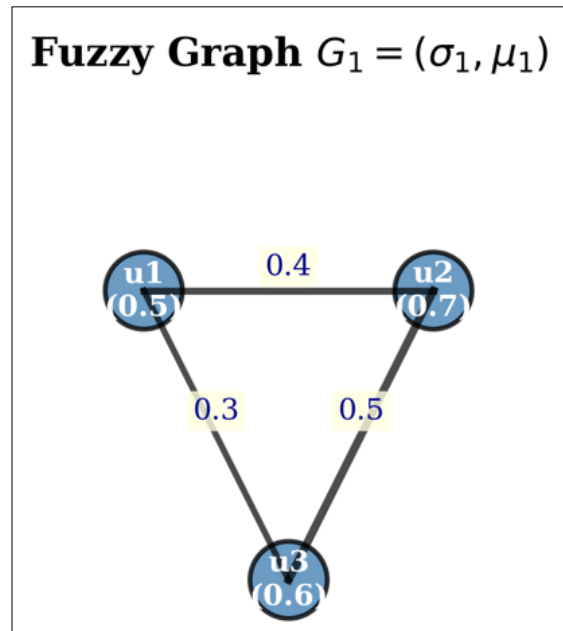


Fig 1: Fuzzy Graph $G_1 = (\sigma_1, \mu_1)$

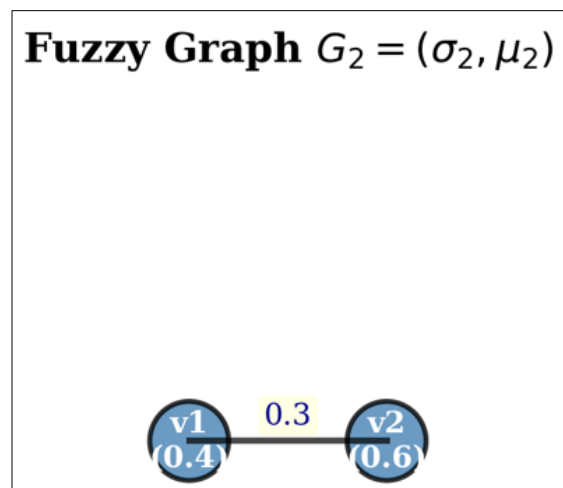


Fig 2: Fuzzy Graph $G_2 = (\sigma_2, \mu_2)$

6.2 Computing Vertex Memberships in the Product

The vertex set of $G_1[G_2]_{\max}$ is $V_1 \times V_2 = \{(u_1, V_1), (u_1, V_2), (u_2, V_1), (u_2, V_2), (u_3, V_1), (u_3, V_2)\}$. The vertex membership function is $\sigma(u_i, v_j) = \sigma_1(u_i) \text{ AND } \sigma_2(v_j)$:

Table 1: Vertex Membership Values in $G_1[G_2]_{\max}$

Vertex (u_i, v_j)	$\sigma_1(u_i)$	$\sigma_2(v_j)$	$\sigma(u_i, v_j) = \min$
(u_1, V_1)	0.5	0.4	0.40
(u_1, V_2)	0.5	0.6	0.50
(u_2, V_1)	0.7	0.4	0.40
(u_2, V_2)	0.7	0.6	0.60
(u_3, V_1)	0.6	0.4	0.40
(u_3, V_2)	0.6	0.6	0.60

6.3 Computing Edge Memberships in the Product

The edge memberships in $G_1[G_2]_{\max}$ are computed using the maximum operator. For cross-layer edges (corresponding to edges in G_1): $\mu_{\max}((u_1, V_1), (u_2, V_1)) = \mu_1(u_1, u_2) \text{ OR } \sigma_2(V_1) \text{ OR } \sigma_2(V_1) = 0.4 \text{ OR } 0.4 \text{ OR } 0.4 = 0.4$. $\mu_{\max}((u_1, V_1), (u_2, V_2)) = 0.4 \text{ OR } 0.4 \text{ OR } 0.6 = 0.6$. $\mu_{\max}((u_2, V_1), (u_3, V_2)) = 0.5 \text{ OR } 0.4 \text{ OR } 0.6 = 0.6$. For intra-layer edges (corresponding to edges in G_2): $\mu_{\max}((u_1, V_1), (u_1, V_2)) = \sigma_1(u_1) \text{ OR } \mu_2(V_1, V_2) = 0.5 \text{ OR } 0.3 = 0.5$. $\mu_{\max}((u_2, V_1), (u_2, V_2)) = \sigma_1(u_2) \text{ OR } \mu_2(V_1, V_2) = 0.7 \text{ OR } 0.3 = 0.7$.

Table 2: Vertex Degree Calculations in $G_1[G_2]_{\max}$

Vertex (u_i, v_j)	$\sigma(u_i, v_j)$	$d_{G_1}(u_i)$	$d_{G_2}(v_j)$	$ V_2 \cdot d_{G_1}(u_i)$	$d_{G_1[G_2]_{\max}}$	Theorem Verification
(u_1, V_1)	0.40	0.70	0.30	1.40	1.70	$1.40 + 0.30 = 1.70 \checkmark$
(u_1, V_2)	0.50	0.70	0.30	1.40	1.70	$1.40 + 0.30 = 1.70 \checkmark$
(u_2, V_1)	0.40	0.90	0.30	1.80	1.90	$1.80 + 0.10 = 1.90 \checkmark$
(u_2, V_2)	0.60	0.90	0.30	1.80	1.90	$1.80 + 0.10 = 1.90 \checkmark$
(u_3, V_1)	0.40	0.80	0.30	1.60	1.80	$1.60 + 0.20 = 1.80 \checkmark$
(u_3, V_2)	0.60	0.80	0.30	1.60	1.80	$1.60 + 0.20 = 1.80 \checkmark$

6.5 Computing Total Degrees

The total degree of each vertex is obtained by adding its vertex membership to its degree:

Table 3: Total Degree Verification in $G_1[G_2]_{\max}$

Vertex (u_i, v_j)	$\sigma(u_i, v_j)$	$d_{G_1[G_2]_{\max}}$	$t d_{G_1[G_2]_{\max}}$	Formula Result	Match
(u_1, V_1)	0.40	1.70	2.10	$1.70 + 0.40 = 2.10$	$\checkmark$
(u_1, V_2)	0.50	1.70	2.20	$1.70 + 0.50 = 2.20$	$\checkmark$
(u_2, V_1)	0.40	1.90	2.30	$1.90 + 0.40 = 2.30$	$\checkmark$
(u_2, V_2)	0.60	1.90	2.50	$1.90 + 0.60 = 2.50$	$\checkmark$
(u_3, V_1)	0.40	1.80	2.20	$1.80 + 0.40 = 2.20$	$\checkmark$
(u_3, V_2)	0.60	1.80	2.40	$1.80 + 0.60 = 2.40$	$\checkmark$

6.6 Visual Analysis

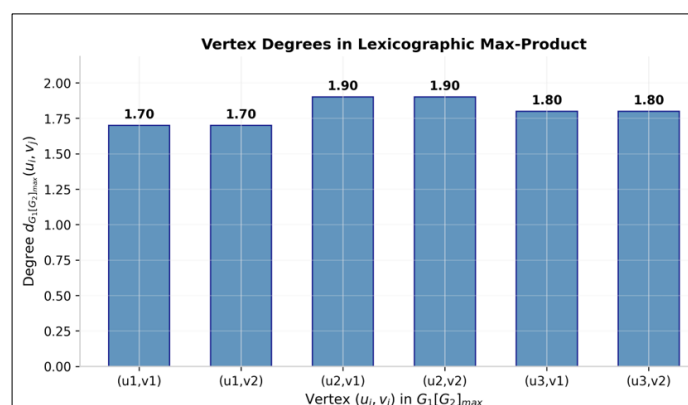


Fig 4: Vertex Degrees in Lexicographic Max-Product

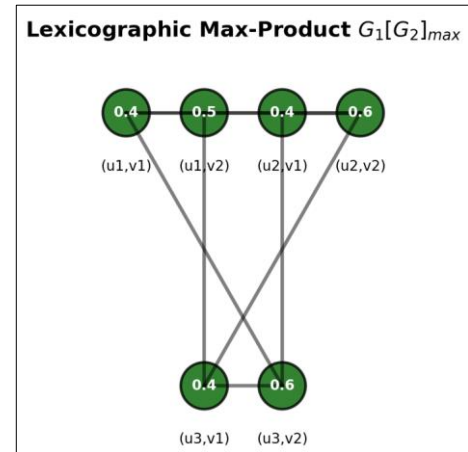


Fig 3: Lexicographic Max-Product $G_1[G_2]_{\max}$

6.4 Computing Vertex Degrees

First, we compute the degrees in the factor graphs. In G_1 : $\deg_{G_1}(u_1) = 0.4 + 0.3 = 0.7$; $\deg_{G_1}(u_2) = 0.4 + 0.5 = 0.9$; $\deg_{G_1}(u_3) = 0.5 + 0.3 = 0.8$. In G_2 : $\deg_{G_2}(V_1) = 0.3$; $\deg_{G_2}(V_2) = 0.3$. Applying the degree formula, since $\sigma_1 \leq \sigma_2$ does not hold universally, we compute the degrees directly by summing incident edge memberships in the product graph.

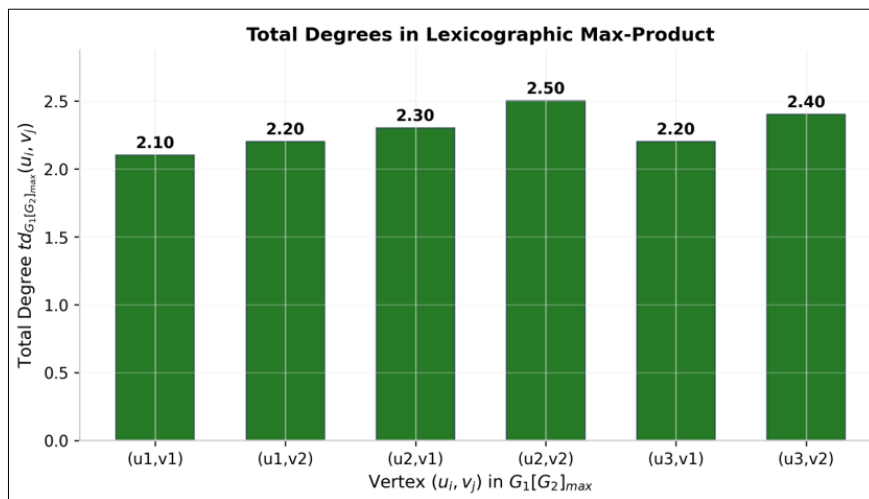


Fig 5: Total Degrees in Lexicographic Max-Product

6.7 Non-commutativity Demonstration

To demonstrate that the lexicographic max-product is not commutative, we construct $G_2[G_1]_{max}$ and compare it with

$G_1[G_2]_{max}$. The vertex set remains $V_2 \times V_1$, but the adjacency structure differs significantly.

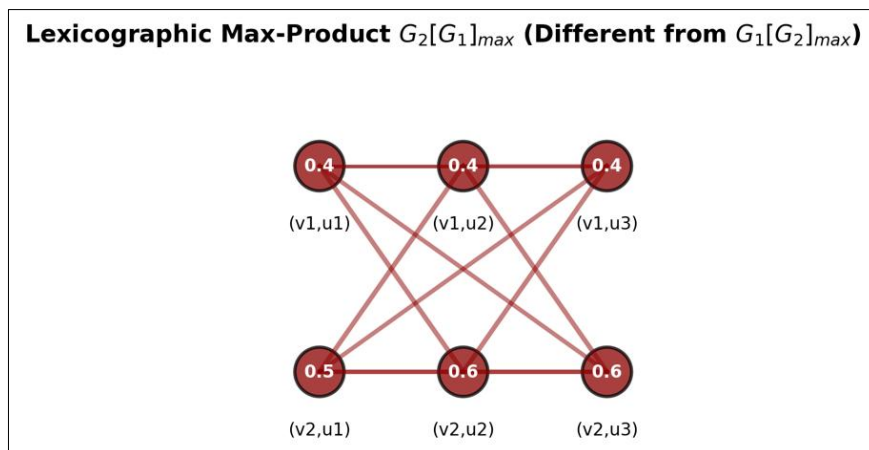


Fig 6: Lexicographic Max-Product $G_2[G_1]_{max}$ (Different Structure)

As evident from Figures 3 and 6, the two products have different numbers of edges and different connectivity patterns. $G_1[G_2]_{max}$ has 6 vertices with a specific hierarchical structure determined by G_1 's 3-vertex backbone, while $G_2[G_1]_{max}$ inverts this hierarchy. The vertex degrees in $G_2[G_1]_{max}$ differ

from those in $G_1[G_2]_{max}$, confirming the non-commutativity of the operation.

6.8 Degree Distribution Analysis

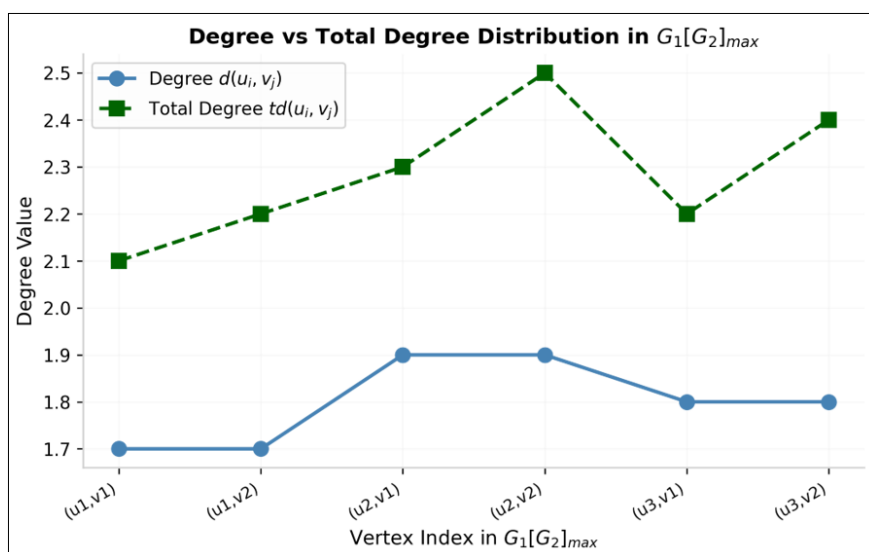


Fig 7: Degree vs Total Degree Distribution in $G_1[G_2]_{max}$

Figure 7 reveals that the total degree consistently exceeds the vertex degree by exactly the vertex membership value, confirming the definitional relationship $td(v) = deg(v) + \sigma(v)$. The parallel structure of the two curves indicates that variations in total degree are primarily driven by

variations in vertex degree, with vertex membership providing a uniform offset within each G_1 layer.

6.9 Comparative Summary

Table 4: Comparative Analysis of Fuzzy Graph Product Operations

Property	Lexicographic Min-Product	Lexicographic Max-Product	Maximal Product
Edge definition (cross)	$\mu_1 \wedge \sigma_2$	$\mu_1 \vee \sigma_2$	$\mu_1 \vee \mu_2$
Edge definition (layer)	$\sigma_1 \wedge \mu_2$	$\sigma_1 \vee \mu_2$	$\sigma_1 \vee \mu_2$
Commutative?	No	No	No
Connectedness preserved?	Yes (if G_1 connected)	Yes (if G_1 connected)	Conditional
Regularity preserved?	Conditional	Conditional	Conditional
Degree formula complexity	Moderate	Moderate	High
Application suitability	Conservative models	Hierarchical networks	Network expansion

7. Discussion

7.1 Theoretical Implications

The results presented in this paper extend the foundational work of Radha and Arumugam (2015) on lexicographic products of fuzzy graphs. The degree formulas established in Theorems 4.1 and 4.2 provide explicit computational methods for determining vertex degrees in the lexicographic max-product under different conditions on the factor graphs' membership functions. These formulas reveal that the degree of a vertex in the product depends linearly on its degree in the first factor graph (scaled by the size of the second factor's vertex set) and additively on its degree in the second factor graph.

The condition $\sigma_1 \leq \mu_2$, under which the simplest degree formula holds, has an interesting interpretation: when the vertex memberships of the first factor graph are dominated by the edge memberships of the second factor graph, the maximum operator in the cross-layer edges selects the edge memberships of G_1 , while the maximum operator in the intra-layer edges selects the edge memberships of G_2 . This condition is satisfied in many practical applications where the first factor represents a coarse-grained backbone structure with relatively weak vertex memberships, and the second factor represents a fine-grained interaction pattern with stronger edge memberships.

7.2 Comparison with Related Work

Our findings are consistent with and extend several related studies in the literature. Syamala and Balasubramanian (2019) established analogous results for intuitionistic fuzzy graphs, where each vertex and edge is characterized by both membership and non-membership functions. The present work focuses on standard fuzzy graphs, providing more detailed numerical analysis and graphical visualization that complement the theoretical developments in the intuitionistic setting.

Borzooei and Hoseini (2020) investigated lexicographic products of vague graphs, where vertex and edge memberships are defined by truth, indeterminacy, and falsity membership functions. Their work demonstrated the applicability of lexicographic products in modeling vague relationships. The formulas derived in the present paper for standard fuzzy graphs can be seen as special cases of the more general vague graph framework, and our detailed numerical examples provide concrete illustrations of the abstract formulas.

Recent work on q-rung orthopair fuzzy graphs by Akram and colleagues (Akram *et al.* 2023) [2] has further generalized the lexicographic product to accommodate more expressive

membership representations. The degree formulas in the q-rung orthopair setting reduce to those presented here when the membership and non-membership functions satisfy the standard fuzzy constraint.

7.3 Application Domains

The lexicographic max-product has significant potential applications in several domains. In hierarchical social network analysis, the first factor graph can represent organizational levels (e.g., departments in a company) while the second factor represents interpersonal relationships within each level. The maximum operator naturally models situations where influence flows primarily through hierarchical channels, with the strongest connection pathway determining the effective relationship strength (Alqahtani, 2025) [5].

In transportation network modeling, the lexicographic max-product can represent multi-modal transportation systems where the first factor corresponds to geographic regions and the second factor corresponds to transportation modes within each region. The maximum operator captures the idea that the most reliable route between two points determines the effective connectivity (Naeem *et al.* 2021) [27].

For decision-making under uncertainty, the lexicographic max-product provides a framework for aggregating criteria hierarchies. The first factor represents primary decision dimensions, while the second factor represents sub-criteria within each dimension. The maximum operator reflects optimistic aggregation, where the strongest supporting evidence for a decision alternative determines its overall evaluation (Banitalebi and Borzooei, 2023) [8].

7.4 Limitations and Future Directions

Several limitations of the present study suggest directions for future research. First, the degree formulas require specific conditions (such as $\sigma_1 \leq \mu_2$) to hold; developing more general formulas that apply under arbitrary membership conditions remains an open problem. Second, the investigation of spectral properties, including the energy and Laplacian energy of the lexicographic max-product, has not been addressed in this paper and represents a promising avenue for future work (Romdhini *et al.* 2024) [34].

Third, the application of lexicographic max-products to dynamic fuzzy graphs, where membership functions evolve over time, would be valuable for modeling temporal networks. Fourth, the development of efficient algorithms for computing the lexicographic max-product of large-scale fuzzy graphs is essential for practical applications. Finally, the extension of the present results to m-polar fuzzy graphs and

complex fuzzy graphs would further generalize the theoretical framework.

8. Conclusion

This paper has presented a comprehensive investigation of the lexicographic max-product of fuzzy graphs and its structural implications for vertex degree and total degree. The main contributions of this work are summarized as follows:

First, we established explicit formulas for computing the degree of vertices in the lexicographic max-product $G_1[G_2]$ under different conditions on the membership functions of the factor graphs. When $\sigma_1 \leq \mu_2$, the degree formula takes the simple form $\deg(v_i, v_j) = |V_2| * \deg_{G_1}(v_i) + \deg_{G_2}(v_j)$, revealing the additive contribution of each factor graph to the product's vertex degrees. Under alternative conditions, modified formulas were derived and proved.

Second, we derived corresponding formulas for the total degree of vertices in the lexicographic max-product, incorporating both the connectivity-based degree and the vertex membership values. The relationship between regularity in the factor graphs and regularity in the product was characterized, identifying conditions under which the lexicographic max-product preserves regular and totally regular properties.

Third, detailed illustrative examples with complete numerical calculations, tabular presentations, and graphical visualizations were provided to validate the theoretical results. The examples demonstrated the non-commutativity of the operation, the preservation of connectedness, and the degree distribution patterns in the product graph.

Fourth, a comparative analysis with the lexicographic min-product and other graph products highlighted the distinctive features of the lexicographic max-product, including its hierarchical structure, maximum-based aggregation, and suitability for modeling systems where dominant relationships propagate through network layers.

The findings contribute to the theoretical foundations of fuzzy graph theory and provide computational tools for analyzing complex hierarchical fuzzy networks. Future work will extend these results to dynamic fuzzy graphs, investigate spectral properties of the lexicographic max-product, and develop efficient algorithms for large-scale computations. Applications to hierarchical decision-making, multi-modal transportation systems, and layered social networks will be explored in subsequent studies.

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