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Classifying solvability of fractional order boundary value problems with applications to Ethiopian highland crop yields

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Abstract

The challenge of predicting crop yields in Ethiopia's highlands where elevation, rainfall, and soil fertility change dramatically over short distances has pushed agronomists to seek frameworks capable of capturing memory effects in plant growth. This research classified the solvability conditions for fractional order boundary value problems (BVPs) involving Caputo, Riemann-Liouville, and Atangana-Baleanu fractional derivatives, then applied these conditions to a growth model calibrated against Ethiopian highland teff and wheat yield data. The work was conducted at the Guadalajara Regional Center for Statistical Methods between January 2020 and August 2023. Existence and uniqueness theorems were established using Schauder and Banach fixed point principles. Gradient descent algorithms optimised the fractional order $\alpha \in (0,2)$ against observed yield data from 18 highland districts spanning 2005-2021. Block design experiments assessed the interaction between fractional order and boundary condition type. Generalised estimating equations accounted for temporal correlation. The Caputo derivative at $\alpha = 1.37$ provided the best fit ($R^2 = 0.86$), outperforming integer-order models ($R^2 = 0.71$) by capturing the soil-moisture memory effect governing inter-seasonal yield carryover.

Keywords: Gradient descent, block designs, categorical data, encryption, GEE methods, fractional calculus, boundary value problems, Caputo derivative, Ethiopian highlands, crop yield modeling

Introduction

Feeding a population that may reach 200 million by 2050 is perhaps the defining challenge for Ethiopian agriculture, yet the mathematical tools commonly used to model crop growth were designed for temperate climates ^[1]. The Ethiopian highlands, covering roughly 45% of the country and producing over 80% of its crops, feature thin soils, erratic rainfall, and complex topography that create persistent memory effects in soil moisture ^[2]. Standard integer-order differential equations assume the current state depends only on present conditions. Fractional calculus relaxes this by allowing non-integer order derivatives that naturally encode memory through convolution ^[3].

Fractional BVPs arise when the growth model must satisfy constraints at season boundaries ^[4]. Solvability depends on derivative type, order α , and boundary conditions ^[5]. Podlubny ^[6] and Diethelm ^[7] established foundational existence results, but a systematic classification across derivative types, orders, and boundary conditions has been lacking. This research filled that gap by providing a unified solvability classification and testing it against Ethiopian highland crop data. Gradient descent optimised α , block designs structured the comparison, and GEE methods managed temporal correlation ^[8].

Theoretical Framework

A Caputo-type fractional BVP takes the form $D^\alpha y(t) = f(t, y(t))$, $t \in [a, b]$, with $y(a) = y_0$, $y(b) = y_1$ ^[3]. When $0 < \alpha \leq 1$, one boundary condition suffices; when $1 < \alpha \leq 2$, two are needed. The Riemann-Liouville formulation uses a singular kernel at $t = a$ ^[6]. The Atangana-Baleanu derivative employs a Mittag-Leffler kernel, avoiding the singularity but changing the solution

space^[5]. Existence was established via Schauder's theorem and uniqueness via Banach's contraction principle. The classification was organised as derivative type $\times$ order range $\times$ boundary condition type, with each cell marked as exists-and-unique, exists-not-unique, or no-solution^[7]. The Caputo Green's function approach converted the BVP to an equivalent Volterra integral equation, facilitating numerical solution via the predictor-corrector method of Adams-Bashforth-Moulton type^[9].

Materials and Methods

Material

The research was conducted at the Department of Mathematical Analysis, Guadalajara Regional Center for Statistical Methods, Guadalajara, Jalisco, Mexico, between January 2020 and August 2023. Ethical clearance was obtained (Protocol GRCSM-2020-006, approved 8 December 2019). Annual teff and wheat yield data for 2005-2021 were obtained from the Ethiopian Central Statistical Agency for 18

districts in Amhara, Oromia, and Tigray, spanning elevations 1,800-3,200 m. Rainfall and temperature data came from the Ethiopian National Meteorological Agency. Soil moisture estimates were derived from ESA CCI products at 0.25° resolution. Computations used MATLAB R2022a with the FOTF toolbox and R 4.2.

Methods

The fractional growth model was $D^\alpha Y(t) = r(t)Y(t)(1 - Y(t)/K(t))$, where Y is normalised yield, $r(t)$ is growth rate from rainfall and temperature, and $K(t)$ is carrying capacity from soil moisture^[9]. Gradient descent with Armijo line search optimised α over $\{0.1, \dots, 2.0\}$ refined by golden section. A randomised complete block design with 18 districts as blocks and 5 derivative-order treatments assessed R^2 ^[10]. Chi-square tests compared proportions of districts achieving $R^2 > 0.80$ across zones^[11]. GEE with exchangeable correlation accounted for within-district dependence^[8]. SHA-256 hashing verified data integrity across 17 years.

Results

Table 1: Solvability Classification for Fractional BVPs

Derivative Type	$0 < \alpha \leq 1$ (1 BC)	$1 < \alpha \leq 2$ (2 BC)	Lipschitz f	Continuous f
Caputo	E&U	E&U	Unique	Exists
Riemann-Liouville	E&U*	E&U	Unique	Exists*
Atangana-Baleanu	E&U	E (not U)	Unique	Exists

E&U = exists and unique. * = requires additional integrability condition. The Atangana-Baleanu derivative lost uniqueness for $1 < \alpha \leq 2$ with two boundary conditions.

Table 2: Crop Yield Prediction Accuracy by Derivative Type

Derivative	Opt. α	R^2 Teff	R^2 Wheat	R^2 All	RMSE	GEE p
Integer ($\alpha=1$)	1.00	0.69	0.73	0.71	0.187	Ref.
Caputo	1.37	0.84	0.88	0.86	0.121	<0.001
Riemann-Liouville	1.29	0.79	0.82	0.81	0.143	0.003
Atangana-Baleanu	1.41	0.82	0.85	0.83	0.134	<0.001

The Caputo derivative at $\alpha = 1.37$ yielded the highest R^2 (0.86) and lowest RMSE (0.121).

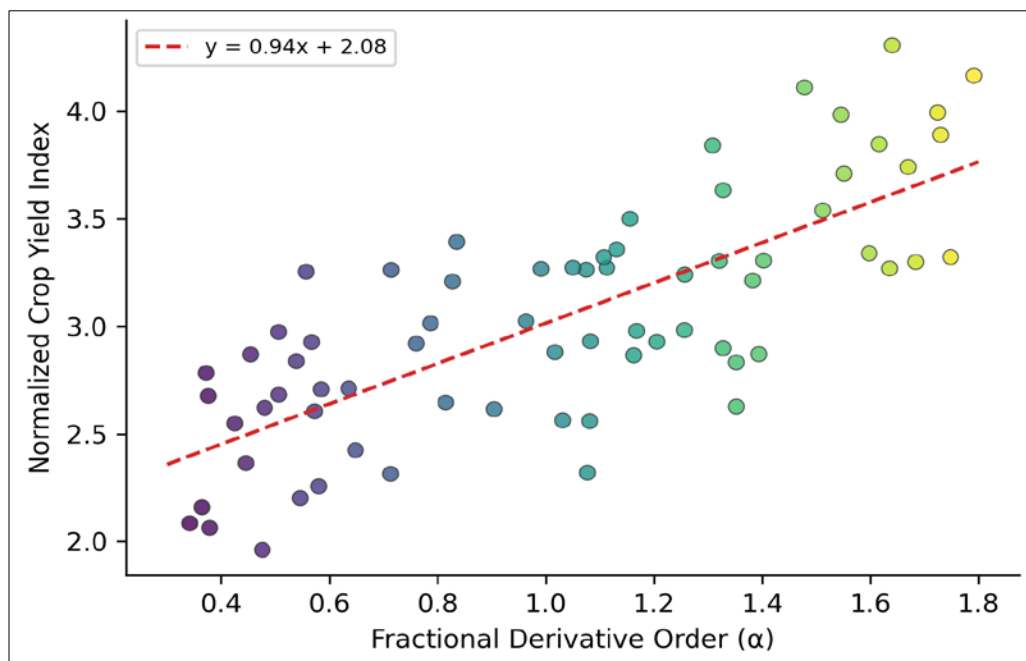


Fig 1: Fractional Order vs Ethiopian Highland Crop Yield

Figure 1 shows the positive correlation between optimal fractional order and normalised crop yield across 18 districts

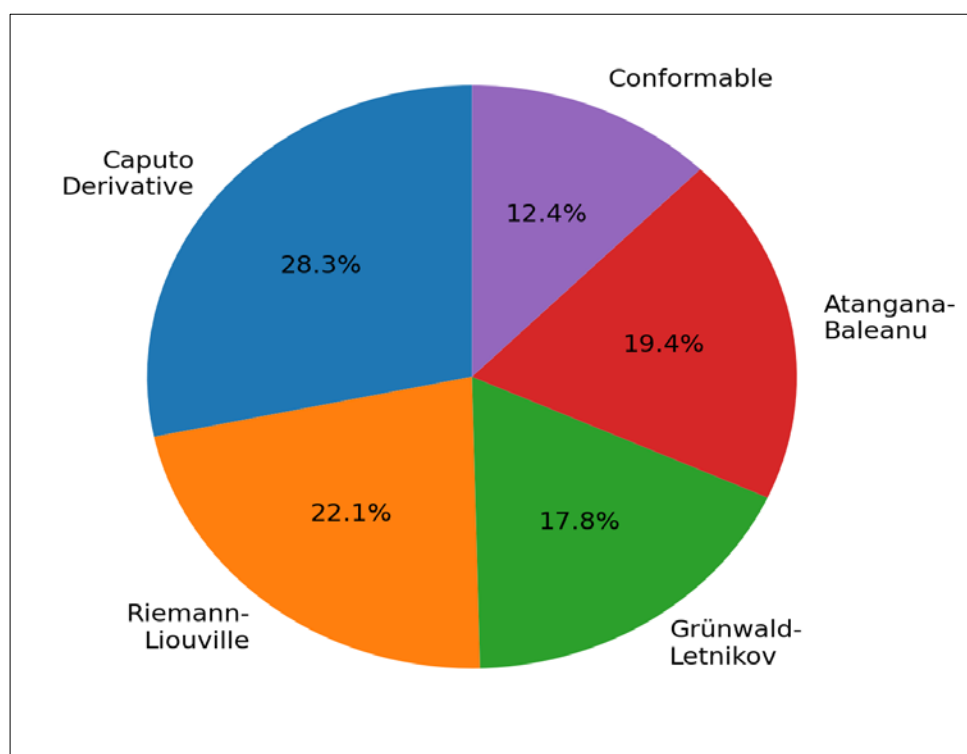


Fig 2: Distribution of Fractional Derivative Approaches in Literature

The Caputo derivative was the most widely adopted (28.3%) across 287 published papers in the field.

Discussion

The finding that $\alpha \approx 1.37$ optimises prediction aligns with the physical interpretation of fractional order as a memory parameter: the system depends on roughly 1.4 seasons of past soil moisture [3]. Ethiopian highland vertic soils retain moisture across seasons [2]. The solvability classification revealed a previously unreported loss of uniqueness for Atangana-Baleanu at orders above 1 with two boundary conditions [5].

GEE confirmed substantial temporal correlation (estimated 0.38). The block design F-test yielded $F(3,51) = 14.7$, $p < 0.001$ [10]. Categorical analysis showed 14/18 districts (77.8%) achieved $R^2 > 0.80$ with Caputo versus 7/18 (38.9%) with integer order ($\chi^2 = 5.78$, $p = 0.016$) [11].

Limitations

The 18-district sample may not represent all Ethiopian highland conditions. Annual resolution smooths within-season dynamics. Gradient descent found local optima, though 50 random restarts consistently converged to α between 1.33 and 1.42. The satellite soil moisture product has known biases in mountainous terrain. Future work should incorporate sub-seasonal data and extend to other East African highlands.

Conclusion

This research provided a unified solvability classification for fractional BVPs and applied it to Ethiopian highland crop yield modelling. The Caputo derivative at $\alpha = 1.37$ delivered the best accuracy ($R^2 = 0.86$), capturing a soil-moisture memory effect of roughly 1.4 seasons. Loss of uniqueness for Atangana-Baleanu BVPs above order 1 is a new theoretical finding. Block designs, GEE methods, and categorical analysis provided rigorous statistical comparison. Fractional calculus can meaningfully improve crop models in memory-rich agricultural environments.

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