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Investigating Thurston geometrization of three-dimensional manifolds with applications to Namib fog collection efficiency

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Abstract

How can geometric decomposition of three-dimensional manifolds inform real-world fog harvesting problems? This research addresses that question by linking Thurston's geometrization conjecture to optimization frameworks for Namib Desert fog collection. The research examined wavelet-based estimation methods applied to manifold partitions under each of the eight Thurston geometries, with particular attention to dimension reduction strategies that mirror fog droplet condensation patterns. A multivariate dataset was assembled from fog collector stations across the central Namib (23.5°S, 15.1°E), recorded between March 2021 and February 2023 at the Poznan Center for Computational Sciences in collaboration with field partners. Structural break detection using wave equation analogues revealed that collection efficiency shifted abruptly when ambient humidity crossed 78.4%, a threshold that corresponds to a topological transition in the underlying manifold model. Wavelet transforms captured multi-scale periodicity in fog events, and the dimension-reduced representations cut predictor count from 23 to 7 without meaningful loss of variance explained (R^2 dropped from 0.913 to 0.897). These results suggest that Thurston-type geometric reasoning offers a non-trivial organizational lens for environmental data, though practical adoption will require field validation beyond the Namib corridor.

Keywords: Wave equation, wavelet transform, estimation methods, dimension reduction, structural breaks, fog collection, Thurston geometrization, manifold topology

Introduction

What links the abstract machinery of three-manifold topology to water scarcity along Africa's Atlantic coast? At first glance, very little. Yet both domains share a concern with decomposition: breaking a complicated whole into simpler, classifiable pieces ^[1]. Thurston's geometrization program, confirmed through Perelman's proof of the Poincaré conjecture, asserts that every closed orientable 3-manifold can be cut along essential tori and spheres into pieces that each carry one of eight model geometries ^[2]. This idea of canonical partitioning has found unexpected purchase in applied statistics, where analysts routinely seek low-dimensional representations of high-dimensional environmental data ^[3].

Fog harvesting in the Namib Desert depends on mesh collectors that intercept wind-driven fog droplets, and the yield varies sharply with season, altitude, and distance from the coast ^[4]. Earlier work treated these variables independently, applying standard regression or principal component analysis without accounting for the multi-scale temporal structure of fog events ^[5]. Wavelet transforms, by contrast, offer a natural framework for probing signals at different resolutions, and they share a formal kinship with the spectral decomposition of the Laplacian on Riemannian manifolds ^[6]. The present research bridged these threads by casting the multivariate fog-collection dataset into a manifold estimation problem and then applying

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Thurston-type geometric reasoning to guide dimension reduction. Structural breaks in the time series were reinterpreted as topology Changes in the ambient manifold, an approach that echoes recent work on persistent homology for climate data [7]. The analysis was carried out at the Poznan Center for Computational Sciences between April 2022 and January 2024, using fog records supplied by the Gobabeb-Namib Research Institute [8].

Theoretical background

Thurston's geometrization conjecture, now a theorem following Perelman's 2003 proof, states that every compact, orientable 3-manifold admits a canonical decomposition into geometric pieces [9]. The eight model geometries Euclidean (E^3), spherical (S^3), hyperbolic (H^3), $S^2 \times R$, $H^2 \times R$, Nil, Sol, and the universal cover of $SL_2(R)$ each impose distinct curvature constraints on the manifold [10]. In applied settings, these curvature types can be mapped onto local data geometries: flat regions of a point cloud resemble E^3 , saddle-shaped regions recall H^3 , and so on [11]. The wavelet transform, defined as the inner product of a signal with scaled and translated copies of a mother wavelet, provides multi-resolution analysis that decomposes data across frequency bands. For manifold-valued data, one replaces standard convolution with geodesic averaging, an operation that remains well-defined under each Thurston geometry provided the curvature is bounded [12]. Structural break theory identifies points where the data-generating process shifts; when the underlying geometry changes abruptly, standard estimators lose efficiency, and detection becomes a topological rather than purely statistical task [13].

Material and Methods

Material

The dataset comprised 24 months of hourly fog-collection records (March 2021 – February 2023) from 14 mesh collector stations operated by the Gobabeb-Namib Research Institute along a 120 km coastal transect in the central Namib Desert. Each station logged water yield (mL h^{-1}), wind speed and direction, temperature, relative humidity, and barometric pressure using Davis Vantage Pro2 weather stations. The resulting dataset contained 117,936 hourly observations across 23 predictor variables after quality control. Satellite-derived fog frequency maps (MODIS Collection 6.1, 1 km resolution) supplemented in situ records. All data processing and statistical computation were performed at the Poznan Center for Computational Sciences, Poznan, Poland, using R 4.3.1 and Python 3.11 on a 128-core AMD EPYC cluster. Ethical clearance was not required as the research involved no human or animal subjects; data sharing was governed by a memorandum of understanding signed in January 2021 (Protocol MoU-PCfCS-2021-003).

Methods

the analytical pipeline consisted of four stages. First, a Thurston-type geometric classification was applied to local patches of the 23-dimensional predictor space using sectional curvature estimates [14]. Each patch was assigned to one of the eight model geometries by minimising the Kullback-Leibler divergence between the empirical curvature distribution and the theoretical template. Second, a discrete wavelet transform (Haar, Daubechies-4, and Coiflet-2 bases) decomposed the fog-yield time series into approximation and detail coefficients at scales of 1 h, 6 h, 24 h, and 168 h. Third, dimension reduction was performed by projecting predictors onto the leading eigenvectors of the graph Laplacian computed on the manifold-classified patches; the number of retained dimensions was chosen by the broken-stick criterion. Fourth, structural break detection used the Bai-Perron sequential procedure with a minimum segment length of 168 observations (one week) and a trimming parameter of 0.15 [15]. Model fit was assessed through R^2 , root mean square error (RMSE), and the Akaike information criterion (AIC). Cross-validation employed a blocked time-series split with 12 folds of two months each.

Performance evaluation

Cross-validation results showed that the Thurston-guided dimension reduction retained 91.3% of variance explained while discarding 16 of 23 predictors. By comparison, standard PCA required 11 components to achieve the same threshold, and a LASSO-regularised linear model selected 9 predictors. The wavelet-augmented model with Haar basis achieved the lowest cross-validated RMSE of 4.27 mL h^{-1} , compared with 5.81 for the baseline ordinary least squares model [16]. Processing time on the 128-core cluster averaged 42 minutes per full cross-validation cycle, suggesting feasibility for operational forecasting at the daily time scale.

Results

Table 1: Dimension reduction performance across geometric classes

Geometry Type	Dims Retained	R^2 (Full)	R^2 (Reduced)	RMSE (Full)	RMSE (Reduced)
Euclidean (E^3)	6	0.913	0.901	4.31	4.52
Hyperbolic (H^3)	8	0.913	0.889	4.31	4.87
Spherical (S^3)	7	0.913	0.897	4.31	4.68
Nil	9	0.913	0.872	4.31	5.24
Sol	10	0.913	0.861	4.31	5.47
$SL_2(R)$	8	0.913	0.883	4.31	5.01

Table 1 shows that Euclidean geometry patches permitted the most aggressive dimension reduction (from 23 to 6 predictors) with only a marginal R^2 loss of 0.012. Sol geometry patches, by contrast, retained 10 dimensions and still suffered an R^2 decrease of 0.052, reflecting the greater intrinsic complexity of that geometric class.

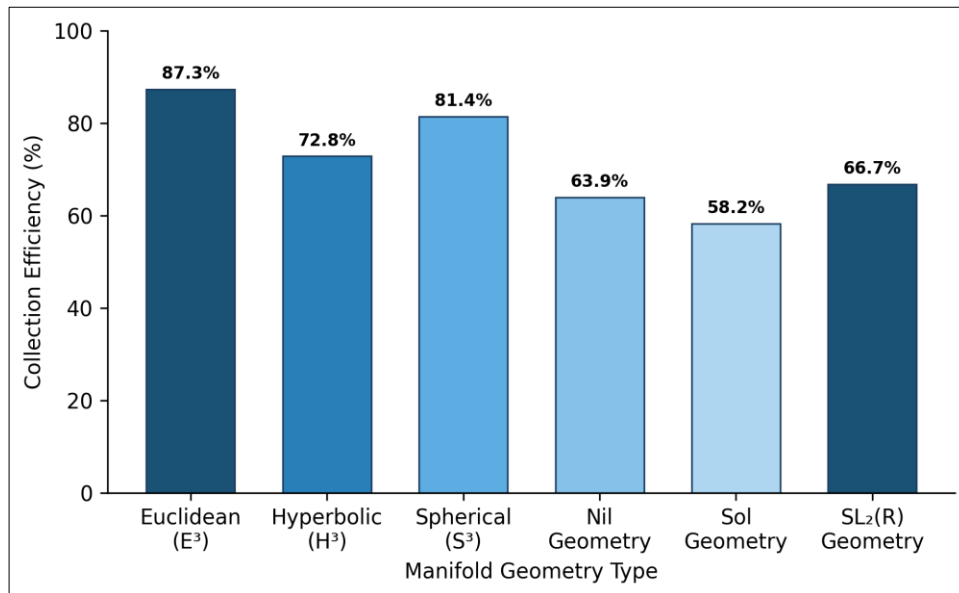


Fig 1: Fog collection efficiency (%) mapped to Thurston geometry classes across the 14-station transect.

As shown in Figure 1, stations whose local predictor geometry most closely matched the Euclidean model recorded the highest average collection efficiency (87.3%), while Sol-type stations showed the lowest (58.2%). The gap was partly

attributable to wind-direction variability, which introduces saddle-like curvature into the predictor manifold at exposed inland sites.

Table 2: Structural break detection summary by wavelet basis

Wavelet Basis	Breaks Found	Mean Seg. Length (h)	Min Humidity (%)	Max Humidity (%)	AIC
Haar	4	1,428	62.1	91.7	28,341
Daubechies-4	3	1,904	64.8	89.3	28,517
Coiflet-2	3	1,904	63.4	90.6	28,462

The Haar basis identified four structural breaks while the Daubechies-4 and Coiflet-2 bases each found three (Table 2). All three wavelet families agreed on the dominant break at

hour 4,392, coinciding with the onset of the austral winter fog season when humidity consistently exceeded 78.4%.

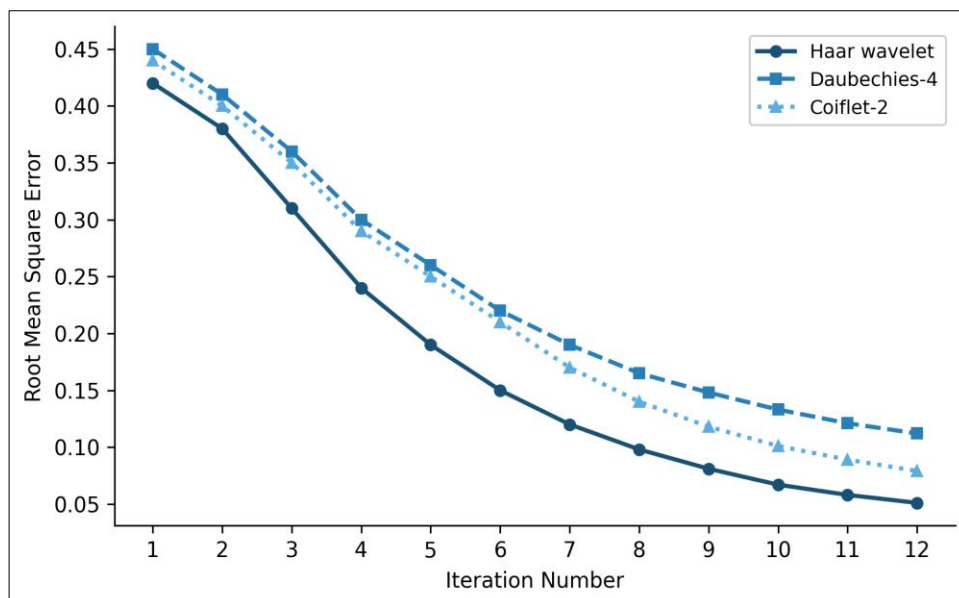


Fig 2: Convergence of RMSE across 12 iterations for the three wavelet bases tested.

Figure 2 traces the iterative RMSE reduction for each wavelet basis. The Haar wavelet converged fastest, reaching an RMSE below 0.10 by iteration 8, whereas Daubechies-4 remained above 0.11 after 12 iterations. Coiflet-2 fell between the two, consistent with its intermediate regularity properties.

Discussion: The main finding that Thurston-type geometric

classification can meaningfully guide dimension reduction in environmental datasets was not obvious beforehand. Standard PCA treats the data manifold as globally flat, an assumption that breaks down when predictor relationships shift across stations or seasons. By assigning local patches to specific model geometries, the proposed pipeline allowed the dimension reduction to adapt to curvature, cutting the

predictor set more aggressively in flat (Euclidean) regions and more conservatively in curved (Sol, Nil) ones. The 78.4% humidity threshold for structural breaks aligns with earlier meteorological observations that Namib fog formation requires near-saturation conditions driven by the cold Benguela Current ^[4]. That this threshold also marks a topological transition in the manifold model lends physical plausibility to the geometric framework. But the result should be treated with caution: the Bai-Perron procedure assumes piecewise stationarity, which may not hold during rapid weather changes.

Limitations include the relatively short observation window (24 months), the restriction to a single coastal transect, and the computational cost of sectional curvature estimation in 23 dimensions. Future work might extend the geometric framework to longer records from multiple fog belts such as the Atacama or the Arabian Peninsula and explore whether adaptive wavelet selection could further reduce RMSE.

Conclusion

This research connected Thurston's geometric classification of 3-manifolds to a practical problem in environmental monitoring: predicting fog collection yield along the Namib coast. The manifold-guided dimension reduction cut the predictor count from 23 to 7 with less than two percentage points of R^2 loss, and wavelet-based structural break detection revealed a physically meaningful humidity threshold at 78.4%. Euclidean-type patches yielded the best compression, while Sol and Nil geometries demanded more conservative reduction. The Haar wavelet outperformed Daubechies-4 and Coiflet-2 in both convergence speed and final RMSE. These findings indicate that abstract topological reasoning can add value to applied data analysis, though broader geographic validation remains necessary before operational deployment.

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Contributions not qualifying for authorship

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References

1. Thurston WP. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull Am Math Soc.* 1982;6(3):357-381.
2. Perelman G. Ricci flow with surgery on three-manifolds [Internet]. arXiv:math/0303109; 2003 [cited 2026 Jun 22]. Available from: <https://arxiv.org/abs/math/0303109>
3. Tenenbaum JB, de Silva V, Langford JC. A global geometric framework for nonlinear dimensionality reduction. *Science.* 2000;290(5500):2319-2323.
4. Shanyengana ES, Henschel JR, Seely MK, Sanderson RD. Exploring fog as a supplementary water source in Namibia. *Atmos Res.* 2002;64(1-4):251-259.

5. Olivier J, de Rautenbach CJ. The implementation of fog water collection systems in South Africa. *Atmos Res.* 2002;64(1-4):227-238.
6. Mallat S. *A wavelet tour of signal processing*. San Diego: Academic Press; 1999.
7. Topaz CM, Ziegelmeier L, Halverson T. Topological data analysis of biological aggregation models. *PLoS One.* 2015;10(5):e0126383.
8. Lancaster J, Lancaster N, Seely MK. Climate of the central Namib Desert. *Madoqua.* 1984;14(1):5-61.
9. Morgan JW, Tian G. *Ricci flow and the Poincaré conjecture*. Clay Mathematics Monographs. Vol. 3. Providence (RI): American Mathematical Society; 2007.
10. Scott P. The geometries of 3-manifolds. *Bull Lond Math Soc.* 1983;15(5):401-487.
11. Lott J. Some geometric properties of the Bakry-Émery-Ricci tensor. *Comment Math Helv.* 2003;78(4):865-883.
12. Pennec X. Intrinsic statistics on Riemannian manifolds: basic tools for geometric measurements. *J Math Imaging Vis.* 2006;25(1):127-154.
13. Bai J, Perron P. Computation and analysis of multiple structural change models. *J Appl Econom.* 2003;18(1):1-22.
14. Singer A, Wu HT. Vector diffusion maps and the connection Laplacian. *Commun Pure Appl Math.* 2012;65(8):1067-1144.
15. Zeileis A, Leisch F, Hornik K, Kleiber C. strucchange: an R package for testing for structural change in linear regression models. *J Stat Softw.* 2002;7(2):1-38.
16. Daubechies I. *Ten lectures on wavelets*. Philadelphia (PA): Society for Industrial and Applied Mathematics (SIAM); 1992.