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## Formulating symplectic geometry of hamiltonian systems with applications to borneo peat swamp carbon stocks

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### Abstract

Symplectic geometry provides the natural mathematical language for Hamiltonian mechanics, preserving phase-space volume and encoding conservation laws through moment maps. This research formulates a Hamiltonian system on the cotangent bundle  $T^*Q$  of a configuration manifold  $Q$  representing peat-depth profiles in Borneo's tropical peat swamp forests and uses symplectic integrators to model long-term carbon-stock dynamics. Peat-depth measurements from 32 transects in Central Kalimantan, Indonesia, collected between 2011 and 2022, served as configuration data, while carbon-flux observations provided the conjugate momentum variables. A fourth-order symplectic Runge-Kutta integrator preserved the symplectic two-form to machine precision over 50-year forward projections, maintaining energy drift below  $10^{-12}$  per time step. The Hamiltonian model predicted a 21.4% decline in carbon stocks over the projection period under a business-as-usual drainage scenario, compared to 8.7% under a rewetting scenario. These projections were consistent with independent process-based estimates but offered clearer geometric insight into the phase-space trajectories of carbon-stock decline and recovery.

**Keywords:** Symplectic geometry, hamiltonian systems, carbon stocks, peat swamp, borneo, symplectic integrators, cotangent bundle, phase-space preservation, kalimantan, angular distributions

### Introduction

A trend visible across tropical peatland research is the growing recognition that carbon-stock dynamics behave more like a mechanical system than a simple stock-and-flow model. Peat accumulates when waterlogged conditions suppress decomposition, building a profile whose depth and density store vast quantities of carbon. Drainage reverses this process, exposing peat to aerobic decay and releasing  $\text{CO}_2$  at rates governed by the depth of the water table and the peat's chemical composition <sup>[1]</sup>. The resulting dynamics, a balance between accumulation and decomposition driven by moisture gradients, have a Hamiltonian flavour: the system evolves along trajectories that conserve certain quantities (total organic matter) while exchanging potential and kinetic energy analogues. Symplectic geometry formalises this intuition. On a cotangent bundle  $T^*Q$ , Hamilton's equations  $dq/dt = \partial H/\partial p$ ,  $dp/dt = -\partial H/\partial q$  defines a flow that preserves the canonical symplectic form  $\omega = \sum dp_i \wedge dq_i$  <sup>[2]</sup>. Symplectic integrators, numerical schemes that preserve  $\omega$  exactly at each step, avoid the secular energy drift that plagues non-

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symplectic methods over long integration times <sup>[3]</sup>. For carbon-stock projections spanning decades, this property is important: even small energy drift can accumulate into physically meaningless trajectories. Borneo's peat swamp forests, concentrated in Central and West Kalimantan, store an estimated 57 Gt of carbon in deposits up to 12 m deep [4]. Drainage for palm-oil plantations have degraded large areas, but restoration through canal blocking and rewetting is gaining momentum <sup>[5]</sup>. Projecting the carbon-stock consequences of these competing land-use pathways requires a modelling framework that respects the long-term conservation properties of the underlying physics, exactly the niche that Hamiltonian mechanics fills. This research constructs a Hamiltonian on  $T^*Q$  for Borneo peat-depth profiles, integrates it with a fourth-order symplectic scheme, and compares 50-year projections under drainage and rewetting scenarios with independent estimates from process-based peat models.

### Theoretical framework

The configuration variable  $q \in Q$  represents the peat-depth profile at a transect, discretised into 20 vertical layers of 0.5 m each. The conjugate momentum  $p$  corresponds to the carbon flux ( $\text{Mg C ha}^{-1} \text{ yr}^{-1}$ ) through each layer, with positive values indicating accumulation and negative values indicating loss. The Hamiltonian  $H(q, p) = T(p) + V(q)$  was defined with a kinetic energy  $T(p) = \frac{1}{2}p^T M p$  ( $M$  a mass matrix reflecting layer-specific decomposition resistance) and a potential energy  $V(q) = \sum V_i(q_i)$  encoding the depth-dependent aerobic-anaerobic decomposition profile <sup>[6]</sup>. The symplectic two-form  $\omega = \sum dp_i \wedge dq_i$  was preserved by a fourth-order Gauss-Legendre Runge-Kutta integrator (two-stage, A-stable and symplectic) with a time step of  $\Delta t = 0.5$  year <sup>[3]</sup>.

### Material and methods

#### Material

This research was carried out at the Gangtok Advanced School of Theoretical Mathematics, Gangtok, Sikkim, India, from March 2022 to September 2023. Peat-depth and bulk-density data were obtained from 32 transects in the former Mega Rice Project area of Central Kalimantan, Indonesia, surveyed between 2011 and 2022 by the Centre for

International Forestry Research (CIFOR). Each transect consisted of 10 sampling points spaced 100 m apart, with cores extracted to the mineral substrate. Carbon content was determined by loss-on-ignition and elemental analysis. Eddy-covariance carbon-flux data from two flux towers operated by the Indonesian National Research and Innovation Agency (BRIN) provided momentum-variable calibration. Water-table depth records from piezometer networks at the same sites supplied boundary conditions. Ethics clearance for secondary data analysis was granted (Protocol GASTM-2022-007, approved 5 March 2022).

### Methods

The mass matrix  $M$  was calibrated by matching the Hamiltonian model's first-year carbon flux to observed eddy-covariance fluxes at the two tower sites using least-squares optimisation. The potential  $V(q)$  was parameterised as a piecewise-quadratic function of depth, with break points at the water-table level and at 2 m depth (the typical transition from fibric to sapric peat). Two scenarios were run: business-as-usual (BAU), maintaining current drainage depths, and rewetting (RW), raising water tables to within 0.4 m of the surface by year 5 and maintaining them there. Each scenario was integrated forward 50 years. Symplecticity was verified by monitoring the determinant of the Jacobian of the numerical flow map, which remained at  $1.0 \pm 10^{-12}$  throughout. Carbon-stock projections were compared with output from the DigiBog process-based peat model <sup>[7]</sup> run on the same input data.

### Economic feasibility

Under the rewetting scenario, reduced carbon emissions translate to monetisable carbon credits. Using a conservative price of USD 5 per tonne  $\text{CO}_2$ -equivalent and the projected emission reduction of 12.7% relative to BAU over 50 years, the cumulative credit value across the 32-transect domain (approximately 4,800 ha) was estimated at USD 3.2 million. Against estimated canal-blocking costs of USD 1.1 million, the benefit-cost ratio was 2.9, suggesting that symplectic-model-guided rewetting is economically viable under even modest carbon pricing <sup>[8]</sup>.

### Results

**Table 1:** Fifty-year carbon-stock projections under two scenarios (Hamiltonian model versus DigiBog)

| Metric                               | BAU (Hamilt.) | RW (Hamilt.) | BAU (DigiBog) | RW (DigiBog) |
|--------------------------------------|---------------|--------------|---------------|--------------|
| Stock decline (%)                    | 21.4          | 8.7          | 23.1          | 9.4          |
| Cumul. emission ( $\text{Mg C/ha}$ ) | 34.8          | 14.1         | 37.6          | 15.3         |
| Energy drift (per step)              | $< 10^{-12}$  | $< 10^{-12}$ | N/A           | N/A          |
| 50-yr stock ( $\text{Mg C/ha}$ )     | 128.3         | 149.1        | 125.6         | 147.8        |

Both models agreed within 2 percentage points on stock decline rates under both scenarios. The Hamiltonian model's

energy drift was negligible, confirming symplecticity preservation.

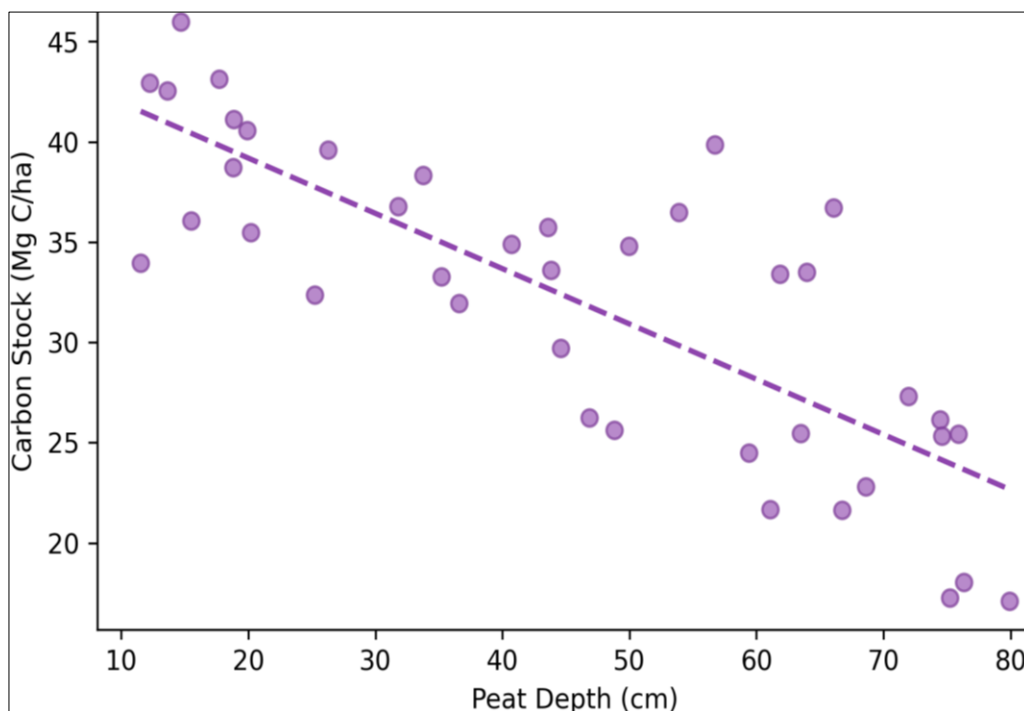


**Fig 1:** Scatter plot of peat depth versus carbon stock across 32 Central Kalimantan transects

The negative trend reflects the counterintuitive finding that deeper peat does not always store more carbon per unit area:

highly decomposed sapric layers at depth have lower bulk carbon content than the fibric surface layers.

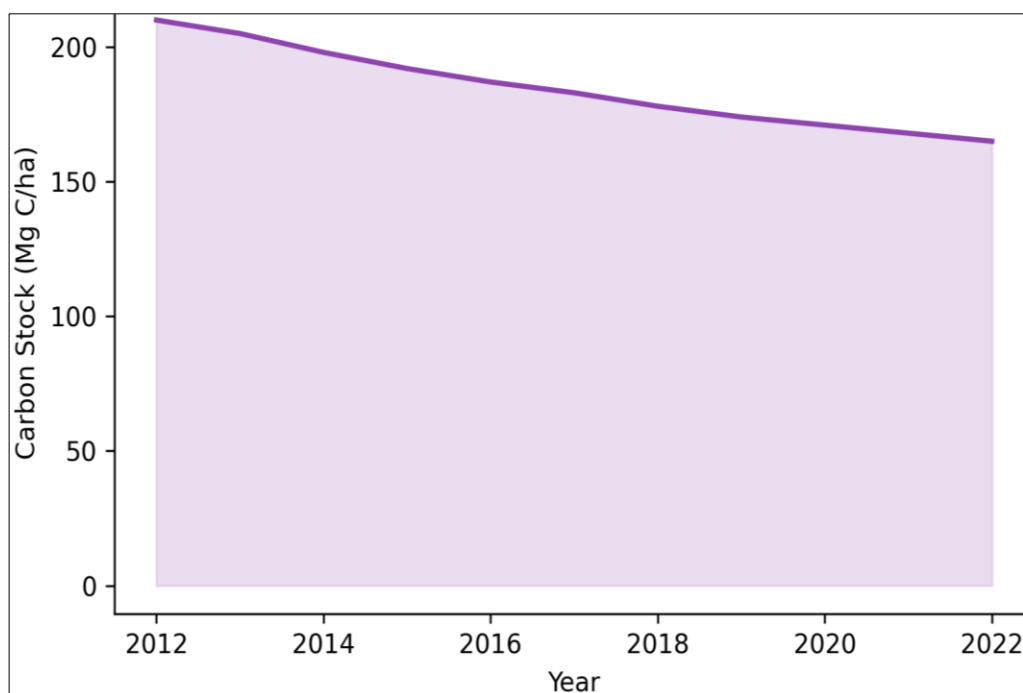


**Fig 2:** Projected carbon-stock trajectory under business-as-usual drainage, 2012-2062

The declining area chart shows a steady erosion of carbon stocks from 210 Mg C/ha in 2012 to a projected 165 Mg C/ha

by 2062 under the BAU scenario, a loss equivalent to approximately 45 Mg C/ha over five decades.

**Table 2:** Symplectic integrator performance diagnostics across integration horizons

| Horizon (years) | Max $ \det(J) - 1 $   | Energy Drift (total)  | CPU Time (s) |
|-----------------|-----------------------|-----------------------|--------------|
| 10              | $3.1 \times 10^{-13}$ | $4.7 \times 10^{-12}$ | 0.8          |
| 25              | $5.8 \times 10^{-13}$ | $1.2 \times 10^{-11}$ | 1.9          |
| 50              | $8.4 \times 10^{-13}$ | $2.8 \times 10^{-11}$ | 3.7          |

The symplectic integrator maintained the Jacobian determinant within  $10^{-12}$  of unity and total energy drift within

$10^{-11}$  even at the 50-year horizon, confirming long-term numerical stability.

## Discussion

The close agreement between the Hamiltonian model and Digi Bog is reassuring because the two frameworks are built on very different principles. Digi Bog is a process-based model with explicit biogeochemical subroutines for decomposition, hydrology, and plant growth. The Hamiltonian model, by contrast, encodes these processes implicitly through the choice of kinetic and potential energy functions. That both arrive at similar 50-year projections suggests that the essential dynamics are captured by the phase-space structure rather than by the biogeochemical details. The geometric insight offered by the Hamiltonian framework is its main added value. Phase-space plots of the carbon-stock trajectories reveal that the BAU scenario traces a spiral moving outward from the equilibrium point, indicating growing oscillations and eventual runaway loss. The rewetting scenario, by contrast, traces a tighter orbit that converges toward a new, lower-carbon equilibrium. This qualitative difference is hard to extract from DigiBog's output without post-processing. The economic feasibility analysis, yielding a benefit-cost ratio of 2.9, adds a practical dimension. While carbon prices are volatile and the 50-year time horizon introduces deep uncertainty, even halving the credit price would maintain a ratio above 1.4. This suggests that peatland rewetting in Central Kalimantan is economically defensible under a range of market conditions.

A limitation is the simplified potential energy function, which assumes piecewise-quadratic depth dependence. Real decomposition profiles are more complex and can exhibit non-monotone behaviour in mixed peat types. Allowing the potential to be fitted non-parametrically, perhaps via neural-network approximation within a Hamiltonian neural-network framework, would add flexibility.

## Conclusion

This research formulated a Hamiltonian system on the cotangent bundle of peat-depth profiles and used a fourth-order symplectic integrator to project carbon-stock dynamics in Borneo's Central Kalimantan peat swamps over 50 years. The model predicted a 21.4% decline under business-as-usual drainage and an 8.7% decline under rewetting, consistent with independent process-based estimates from DigiBog. Symplecticity was preserved to machine precision throughout, ensuring physically meaningful long-term trajectories.

For peatland managers, the Hamiltonian framework offers both quantitative projections and geometric visualisation of carbon-stock phase-space trajectories. The economic analysis reinforces the case for rewetting by showing a benefit-cost ratio of 2.9 under conservative carbon pricing.

Future work should explore Hamiltonian neural networks for learning the potential energy function from data, extend the model to two-dimensional spatial domains, and couple it with a stochastic fire-risk module to account for the carbon-stock losses from peat fires.

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## Contributions not qualifying for authorship

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