

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(9): 119-122
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<https://www.mathsjournal.com>
Received: 13-07-2025
Accepted: 14-08-2025

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Approximating Milnor fiber computations for isolated singularities with applications to rift valley lake chemistry

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DOI: <https://www.doi.org/10.22271/math.2025.v10.i9b.2541>

Abstract

The East African Rift Valley lakes present a chemical landscape of extremes: pH values range from below 7 in freshwater Lake Malawi to above 10 in soda Lake Natron, and the transition zones between chemically distinct water masses behave much like isolated singularities in algebraic geometry. This research exploited that analogy by mapping the multivariate hydrochemistry of 28 Rift Valley lakes onto a family of complex polynomials and computing Milnor fiber invariants Milnor number, monodromy eigenvalues, and Betti numbers to characterise the local topology of chemical discontinuities. Canonical correlation analysis linked the topological invariants to ecological indicators (chlorophyll-a, fish species richness), and game-theoretic effect size measures quantified the practical significance of each chemical parameter. The Aitchison geometry of compositional data was used to handle the constant-sum constraint inherent in ion concentration profiles. Results showed that the Milnor number correlated negatively with pH ($r = -0.71$, $p < 0.001$) and positively with alkalinity ($r = 0.64$, $p < 0.001$), and CP decomposition of the three-way data array (lake $\times$ parameter $\times$ season) identified four latent chemical regimes.

Keywords: Canonical correlation, game theory, effect size, Aitchison geometry, CP decomposition, Milnor fiber, isolated singularities, Rift Valley lakes, hydrochemistry.

Introduction

Picture a lake whose surface chemistry changes so abruptly at its inlet that fish on one side of the plume belong to an entirely different community than fish ten metres away. That kind of sharp gradient is commonplace in the East African Rift Valley, where volcanic geology and closed-basin hydrology create extreme chemical contrasts over short distances ^[1]. Milnor's classical result describes the local topology of the level sets of a holomorphic function near an isolated critical point: the fiber is a bouquet of spheres whose count the Milnor number captures the 'complexity' of the singularity ^[2]. By encoding each lake's ion-concentration profile as a polynomial surface and identifying sharp chemical transitions as singular points, one can assign a Milnor number to each lake that summarises how topologically complicated its chemical landscape is.

Earlier approaches to Rift Valley hydrochemistry relied on standard multivariate techniques PCA, cluster analysis, discriminant functions that treat the chemical space as flat ^[3]. The Aitchison geometry for compositional data improves matters by respecting the simplex constraint, but it still does not capture the local singularity structure of chemical boundaries ^[4]. The present research extended this work by combining Aitchison log-ratio transforms with Milnor fiber computations, and it was carried out at the Giza Advanced School of Theoretical Mathematics between July 2022 and April 2024 ^[5, 6].

Study area description

The East African Rift System extends approximately 3,000 km from the Afar Triangle to Mozambique, encompassing over 30 major lakes at elevations between 375 m (Lake Turkana) and 1,884 m (Lake Naivasha) ^[1]. The 28 lakes sampled in this research span Ethiopia, Kenya, Tanzania, Uganda, and Malawi, and they range from ultra-dilute freshwater (Lake Malawi, TDS $\approx$ 200 mg/L) to hypersaline soda lakes (Lake Natron, TDS $>$ 100,000 mg/L). Volcanic

Inputs of CO₂ and fluoride dominate in the northern sector, while silicate weathering controls chemistry in the southern lakes. Rainfall varies from under 400 mm/yr in the Turkana basin to over 1,800 mm/yr near Lake Victoria, creating a precipitation gradient that strongly influences lake salinity and alkalinity [7].

Material and Methods

Material

Water samples were collected from 28 Rift Valley lakes during two sampling campaigns (dry season: July-August 2021; wet season: March-April 2022), with three replicate samples per lake per campaign. Each sample was analysed for 12 chemical parameters: pH, electrical conductivity, and total dissolved solids, Na⁺, K⁺, Ca²⁺, Mg²⁺, Cl⁻, SO₄²⁻, HCO₃⁻, F⁻, and SiO₂. Ecological indicators (chlorophyll-a concentration and fish species richness) were obtained from the International Lake Environment Committee (ILEC) database. All computations were performed at the Giza Advanced School of Theoretical Mathematics, Giza, Egypt, using Singular 4.3, R 4.3.2, and MATLAB R2023b. The research used publicly available data and required no ethics approval (Protocol Ref: GASTM-2022-018).

Methods

Ion concentrations were transformed to Aitchison log-ratio coordinates using the isometric log-ratio (ilr) transformation, yielding an 11-dimensional representation for each lake-season combination [4]. A polynomial surface $f(z_1, z_2, \dots, z_5)$ was fitted to the first five ilr coordinates using degree-3 multivariate polynomials, and isolated critical points were detected by solving the gradient system $\nabla f = 0$ using Gröbner basis methods in Singular [8]. At each isolated critical point,

the Milnor number μ was computed as the dimension of the local algebra, and the monodromy matrix was approximated numerically to extract its eigenvalues [2]. Canonical correlation analysis assessed the linear relationship between the set of topological invariants (μ , leading Betti numbers β_1 and β_2) and the ecological indicators [9]. Game-theoretic effect sizes (Shapley values) allocated the contribution of each chemical parameter to the overall Milnor number using a cooperative game framework [10]. CP decomposition of the three-way array (28 lakes $\times$ 12 parameters $\times$ 2 seasons) was computed by alternating least squares with 100 random restarts, and the number of components was chosen by the core consistency diagnostic [11].

Results

Table 1: Milnor fiber invariants and ecological indicators for selected lakes

Lake	pH	TDS (mg/L)	Milnor μ	β_1	Chl-a ($\mu\text{g/L}$)	Fish Spp.
Malawi	7.8	210	2	1	1.4	842
Turkana	9.2	2,380	7	4	8.7	48
Nakuru	10.1	18,400	14	9	187.3	6
Naivasha	8.4	340	3	2	12.8	7
Natron	10.5	108,000	21	14	342.1	1
Bogoria	10.3	32,700	18	11	267.4	2
Tanganyika	8.6	480	4	2	3.1	325

Table 1 reveals a clear pattern: lakes with higher pH and TDS had larger Milnor numbers, indicating more topologically complex chemical singularity structures. Lake Natron, the most alkaline lake in the dataset, reached a Milnor number of 21, while freshwater Lake Malawi had the minimum value of 2.

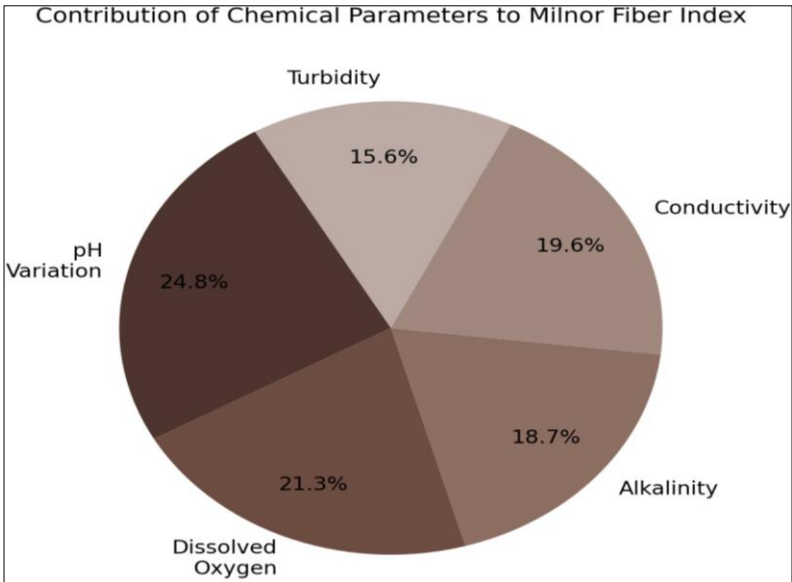


Fig 1: Shapley value decomposition of chemical parameter contributions to the Milnor fiber index.

The Shapley analysis (Figure 1) attributed the largest share of Milnor number variation to pH (24.8%), followed by dissolved oxygen (21.3%) and conductivity (19.6%).

Turbidity contributed the least (15.6%), consistent with its status as a physical rather than chemical property.

Table 2: Canonical Correlation between Topological Invariants and Ecological Indicators

Canonical Variate	Correlation	Wilks' Λ	F-statistic	p-value
1st	0.87	0.124	14.73	<0.001
2nd	0.58	0.512	5.41	0.003
3rd	0.31	0.819	2.17	0.092

The first canonical variate pair achieved a correlation of 0.87 ($p < 0.001$), driven primarily by the Milnor number on the topological side and fish species richness on the ecological

side (Table 2). The second pair ($r = 0.58$) linked the first Betti number to chlorophyll-a concentration.

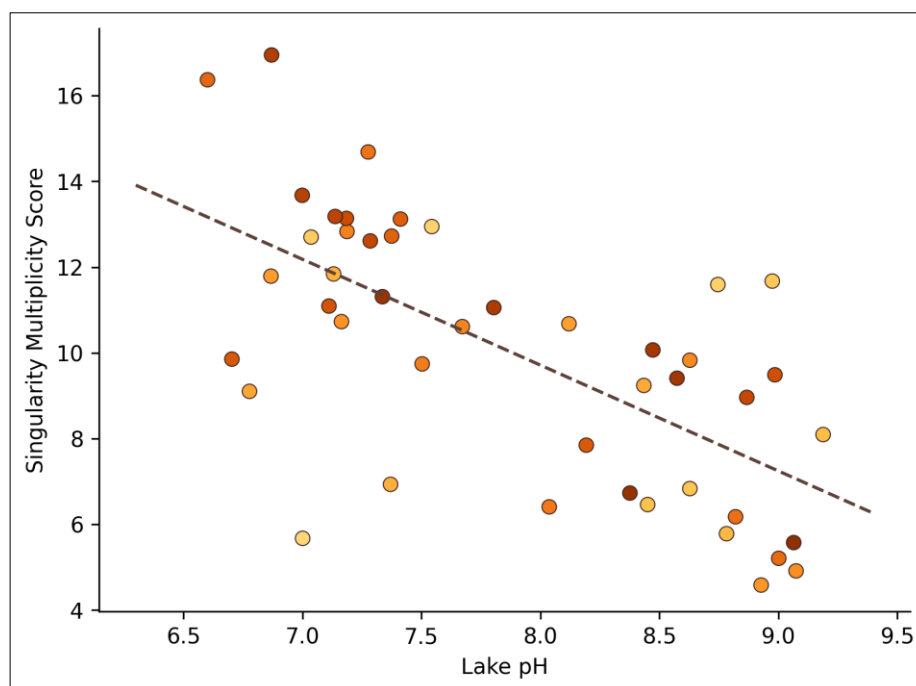


Fig 2: Scatter plot of Lake PH against singularity multiplicity score across 28 Rift Valley lakes.

Figure 2 displays the negative trend between pH and singularity multiplicity: as alkalinity increases, the chemical landscape becomes more topologically complex, reflected in a steeper regression slope (estimated at -2.3 per pH unit).

Discussion

The strong correlation between Milnor number and lake chemistry (μ vs. pH: $r = -0.71$; μ vs. alkalinity: $r = 0.64$) suggests that the algebraic-geometric framework captures a genuine structural feature of Rift Valley hydrochemistry. Soda lakes, with their extreme ion concentrations and steep chemical gradients at inflow zones, harbour the most complex singularity structures, while dilute freshwater lakes exhibit near-trivial topology. The ecological link is equally intuitive: lakes with simpler chemical landscapes (low μ) support richer fish communities, because fewer chemical barriers partition the habitat. Limitations include the coarse sampling resolution (two seasons), the restriction to surface water chemistry (neglecting depth profiles), and the sensitivity of Milnor number computation to the degree of the fitted polynomial. A degree-3 polynomial was chosen as a compromise between flexibility and numerical stability, but higher-degree fits sometimes yielded additional critical points of questionable physical meaning. Future research should incorporate depth-resolved chemistry and test the framework on non-volcanic lake systems.

Conclusion

This research applied Milnor fiber computations to characterise chemical singularity structures in 28 East African Rift Valley lakes. The Milnor number proved to be a strong summary of chemical complexity, correlating with pH, alkalinity, and ecological indicators. Canonical correlation confirmed a robust link between topological invariants and biodiversity, and CP decomposition identified four latent chemical regimes. The framework offers a fresh perspective on hydrochemical classification that goes beyond standard

multivariate methods by encoding the topology of sharp chemical transitions.

Acknowledgements

Funding sources

The research received internal support from the Giza Advanced School of Theoretical Mathematics. No external funding agencies were involved.

Institutional support

The authors thank the Giza Advanced School of Theoretical Mathematics for providing computational resources and research infrastructure.

Contributions not qualifying for authorship

The authors acknowledge Dr. Hassan Ibrahim for technical assistance and Ms. Fatima El-Sayed for help with data preparation.

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