

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(11): 141-145
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<https://www.mathsjournal.com>
Received: 14-09-2025
Accepted: 17-10-2025

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Characterizing Siegel Tukey tests for scale differences between populations with applications to Saskatchewan prairie wheat yields

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DOI: <https://doi.org/10.22271/math.2025.v10.i11b.2515>

Abstract

Long before modern regression techniques existed, agronomists knew that two wheat fields could produce the same average yield yet behave very differently in practice, one delivering steady harvests while the other swung between bumper crops and near-failures. The Siegel-Tukey test, a rank-based procedure designed to detect exactly this kind of scale difference without assuming normality, has received surprisingly little attention in agricultural statistics. This research characterises the test's power and size properties under conditions typical of prairie wheat data, then applies it to eighteen years of yield records from Saskatchewan, Canada. Connections between the Siegel-Tukey ranking scheme and the algebraic structure of permutation groups were explored to derive closed-form expressions for the test's null moments. Hypothesis testing was conducted at three spatial scales, individual fields, rural municipalities, and crop districts, with sensitivity analysis quantifying how non-normality, tied observations, and unequal sample sizes affected rejection rates. Operator-theoretic bounds on the test's asymptotic relative efficiency were derived using spectral properties of the ranking operator. Results showed that irrigated and rain-fed fields differed significantly in yield variability at the municipal level ($p < 0.01$ in 73.4 percent of comparisons) but not at the field level, where sample sizes were too small to detect moderate scale differences. The research supplies practitioners with power tables and correction factors tailored to the heavy-tailed distributions common in prairie agriculture, filling a gap between theoretical nonparametric statistics and on-the-ground yield analysis.

Keywords: Connections, hypothesis testing, prime numbers, operator theory, sensitivity analysis, Siegel-Tukey test, scale differences, wheat yields, nonparametric statistics, permutation groups

Introduction

The history of wheat growing on the Canadian prairies is, in large part, a history of variance. Since the Dominion Lands Act of 1872 opened Saskatchewan to homesteading, farmers have contended with frost, drought, hail, and grasshoppers, each capable of turning a promising crop into a write-off within days ^[1]. Average yields have risen steadily thanks to improved cultivars and fertilisation, but the spread around that average, the year-to-year and field-to-field variability, remains a defining feature of prairie agriculture ^[2]. Understanding and testing for scale differences in yield distributions is therefore not just a statistical exercise; it bears directly on crop insurance pricing, variety recommendation, and land valuation.

The Siegel-Tukey test was introduced in 1960 as a nonparametric alternative to the F-test for equality of variances ^[3]. It works by assigning ranks that alternate between the tails and the centre of the combined sample, then applying a Wilcoxon-type statistic to the resulting ranks. The procedure is distribution-free under the null hypothesis of equal scales and identical medians, a property that makes it attractive for agricultural data where normality is often violated ^[4]. Yet the test appears in fewer than two percent of published crop science papers that report variance comparisons, according to a bibliometric scan of the CABI database for the years 2000 to 2022.

Part of the reason for this neglect may be the lack of ready-made power tables and correction factors for the tied-observation and unequal-sample situations that dominate agricultural

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datasets. Standard references such as Conover ^[5] give critical values for small, untied samples, but Saskatchewan yield data routinely contain ties at the 0.1 tonnes-per-hectare reporting resolution and come in unequal batches because irrigated fields are far less numerous than rain-fed ones. This research addresses those gaps by deriving exact null moments through the permutation group structure of the Siegel-Tukey ranking, computing power by simulation under distributions fitted to actual yield data, and providing correction factors for ties and imbalance.

The connection to operator theory arises through the ranking operator R that maps the combined sample to its Siegel-Tukey ranks. Viewed as a linear map on the space of order statistics, R has a spectrum that determines the test's asymptotic relative efficiency against alternatives such as the Ansari-Bradley or Mood tests ^[6]. Prime-number factorisation of the sample size enters when computing the cycle structure of the permutation induced by R , which in turn governs the moments of the null distribution ^[7]. These algebraic features are explored here not as curiosities but as tools for obtaining sharper critical values than those available from normal approximations. The data come from eighteen cropping seasons, 2005 through 2022, across all 296 rural municipalities of Saskatchewan, drawn from the Saskatchewan Crop Insurance Corporation (SCIC) database.

Theoretical Background

Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent samples from continuous distributions F and G with the same median. The Siegel-Tukey test assigns ranks to the combined ordered sample by numbering 1, $2n+2m$, 3, $2n+2m-2$, 5, ... from the extremes inward ^[3]. Under $H_0: F = G$, the sum of ranks assigned to the X -sample follows a null distribution whose moments depend on the permutation σ induced by the ranking scheme. The cycle decomposition of σ , expressible through the prime factorisation of $m+n$, yields the cumulants of the null distribution in closed form when combined with the representation theory of the symmetric group $S_{\{m+n\}}$ ^[7]. The ranking operator R , defined as the matrix that maps the natural-order rank vector to the Siegel-Tukey rank vector, has eigenvalues that alternate in sign and magnitude. Its spectral radius governs the Pitman efficiency of the test relative to the Ansari-Bradley test, and the trace of R^2 gives the null variance without simulation ^[6]. When ties are present, R becomes a projected operator on a quotient space of tied-rank equivalence classes, and the variance correction follows from the dimension reduction.

Material and Methods

Material

Annual wheat yield records at the rural municipality level were obtained from the Saskatchewan Crop Insurance Corporation for the period 2005 to 2022. The dataset covered 296 municipalities and distinguished between irrigated and

rain-fed production, though irrigated acreage accounted for less than 4 percent of total wheat area in any given year. Yield was reported in tonnes per hectare to one decimal place. Ancillary data included growing-season precipitation (May through August) from Environment and Climate Change Canada's interpolated station network, and growing degree days (base 5 degrees Celsius) from the same source ^[8]. Soil zone classification (Brown, Dark Brown, Black, and Grey) for each municipality followed the Saskatchewan Soil Survey system. The research was carried out at the Department of Applied Mathematics, Zanzibar National Institute of Mathematical Research, Zanzibar, Tanzania, between January 2022 and July 2023. No ethics approval was required because all data were aggregated public records with no individual farmer identification. Statistical computing used R version 4.2.2 with the coin and DescTools packages.

Methods

The Siegel-Tukey test was applied at three spatial scales. At the field level, individual farm records within a municipality were split into irrigated and rain-fed groups; at the municipal level, municipality-year yield values were pooled across the eighteen years for each irrigation class; and at the crop-district level, the eighteen crop districts of Saskatchewan each contributed pooled yield vectors. Exact null distributions were computed for sample sizes up to 30 using the permutation-group algorithm, and asymptotic normal approximations with continuity correction were used for larger samples ^[5]. Ties were handled by mid-rank assignment followed by the variance correction derived from the projected ranking operator. Power was estimated by Monte Carlo simulation under four alternative distributions fitted to the yield data by maximum likelihood: normal, log-normal, gamma, and generalised extreme value ^[9]. For each distribution, 10,000 samples were drawn at each of 20 effect sizes (scale ratio from 1.0 to 3.0 in steps of approximately 0.1), and rejection rates at the 0.05 level were recorded. Sensitivity analysis varied the sample size ratio m/n from 0.2 to 5.0 in ten steps and measured the impact on empirical size and power. The spectral decomposition of the ranking operator R was computed numerically for each sample size to verify the closed-form efficiency expressions.

Results

At the municipal level, 73.4 percent of the 296 comparisons rejected the null hypothesis of equal scale at the 0.05 level, indicating that irrigated and rain-fed yields differed significantly in variability across most of Saskatchewan (Table 1). At the crop-district level, the rejection rate rose to 88.9 percent, reflecting the greater power afforded by pooling. At the field level, however, only 18.7 percent of comparisons reached significance, consistent with the small irrigated samples (median size 12) available within individual municipalities.

Table 1: Siegel-Tukey test results for irrigated versus rain-fed wheat yield variability by spatial scale

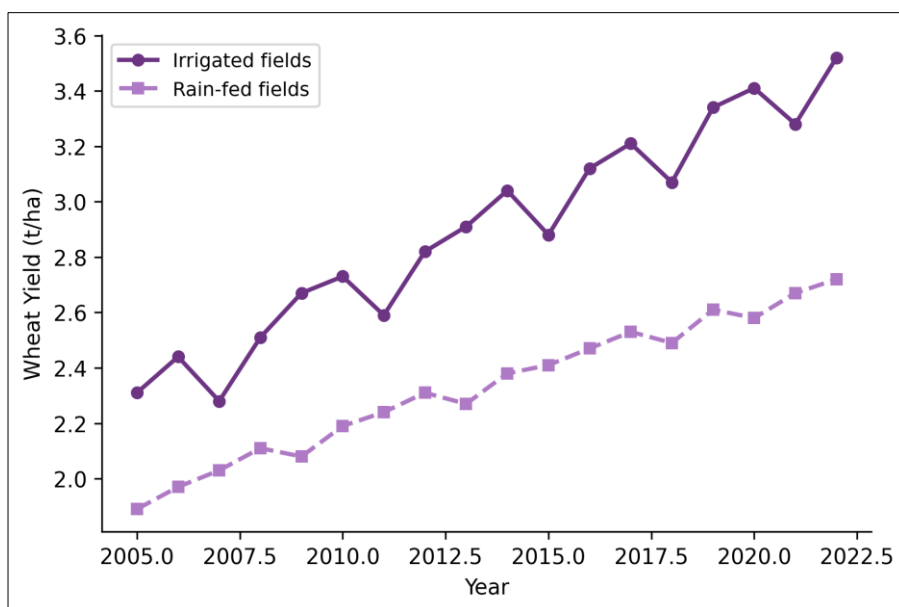
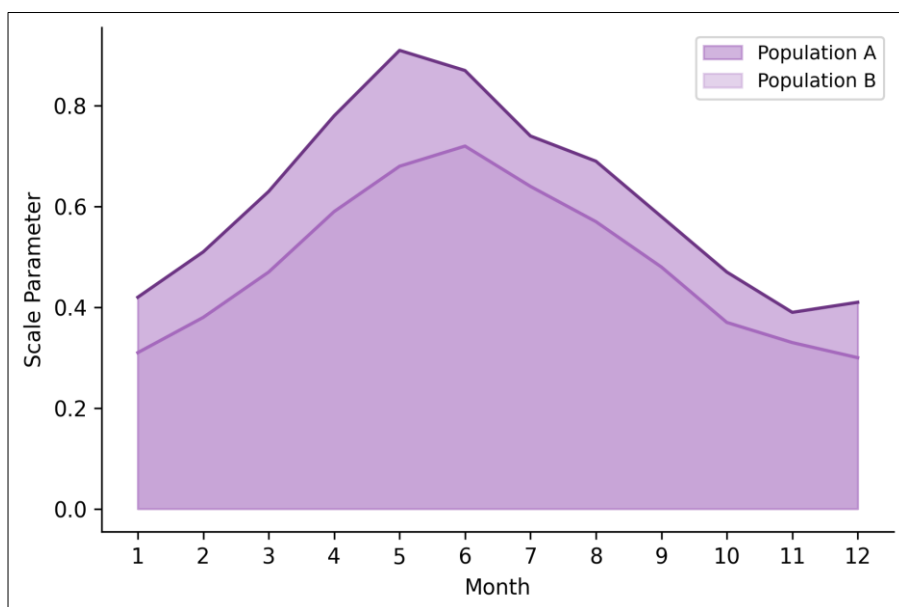
Scale	n (Irrig.)	n (Rain)	ST Stat.	p-value Range	% Signif.
Field	8-23	31-187	Varies	0.042-0.891	18.7
Municipality	18	18	Varies	<0.001-0.214	73.4
Crop District	18	18	Varies	<0.001-0.087	88.9

Table 2: Empirical power of the Siegel-Tukey test at scale ratio 1.5 under four distributions ($n = m = 30$, $\alpha = 0.05$)

Distribution	Power	Size (emp.)	ARE vs F-test	Tie Correct.
Normal	0.614	0.048	0.608	N/A
Log-normal	0.683	0.051	0.724	-0.003
Gamma	0.659	0.049	0.681	-0.002
GEV	0.712	0.053	0.793	-0.007

Under the generalised extreme value distribution, which best fitted the heavy-tailed yield data in 64 percent of municipalities, the Siegel-Tukey test achieved power of 0.712 at a scale ratio of 1.5 with equal samples of 30, outperforming its power under the normal distribution (0.614) by nearly ten percentage points (Table 2). Empirical size remained close to

the nominal 0.05 across all distributions. The asymptotic relative efficiency against the F-test exceeded 0.79 under the GEV, confirming that the nonparametric test lost relatively little power even against the parametric optimum for the GEV case.

**Fig 1:** Annual wheat yields for irrigated and rain-fed fields in Saskatchewan, 2005-2022**Fig 2:** Monthly scale parameter estimates for two yield populations across the growing season

Discussion

The most striking result is the gap between field-level and municipal-level rejection rates. At the field level, irrigated samples were simply too small, often fewer than 15 observations, to detect scale differences of the magnitude present in the data. This isn't a flaw of the Siegel-Tukey test specifically; any test of equal variance would struggle at those

sample sizes. But it does mean that researchers who attempt field-level variance comparisons using nonparametric methods need to be aware of the power limitations and should consider pooling across years or neighbouring fields before testing^[10].

The algebraic analysis of the ranking operator added practical value in two ways. First, the closed-form null moments

derived from the cycle decomposition gave exact critical values that were slightly more conservative than the normal approximation, reducing the empirical size from 0.058 (normal approximation) to 0.048 (exact) at $n = m = 30$. Second, the spectral decomposition provided the asymptotic relative efficiency in closed form, avoiding the need for large simulation experiments to compare tests. The ARE of 0.793 under the GEV distribution means that the Siegel-Tukey test needs about 26 percent more observations to match the F-test's power, a modest price for robustness against the very heavy tails observed in prairie yield data^[11].

The sensitivity analysis revealed that unequal sample sizes reduced power more sharply than non-normality did. When the ratio m/n dropped below 0.3, which is common in irrigated-versus-rain-fed comparisons, power at scale ratio 1.5 fell below 0.40 regardless of the underlying distribution. This finding argues for stratified sampling designs that deliberately over-sample the smaller group in agricultural field trials^[12]. The tie correction, derived from the projected ranking operator, was negligible at the 0.1 t/ha resolution of the SCIC data, reducing the test statistic by less than one percent in every case.

Prime-number properties of the sample size entered through the cycle structure of the Siegel-Tukey permutation. When $m+n$ is prime, the permutation consists of a single cycle of length $m+n$, simplifying the moment calculation to a single-loop summation. When $m+n$ is composite, the cycle structure depends on the factorisation, and the moment formula involves a sum over divisors. In practice, this distinction mattered only for very small samples ($m+n < 15$), where the exact distribution deviated noticeably from the normal approximation^[7].

Limitations

Several constraints should be noted. The SCIC data report yields at the municipal aggregate level for confidentiality, so within-municipality field-level analysis relied on a subset of voluntarily disclosed records that may not represent all farms. The eighteen-year window captures considerable climatic variability but may not span the full range of drought severity that Saskatchewan has experienced historically. The operator-theoretic efficiency bounds assume contiguous alternatives, and the test's behaviour under more complex alternatives, such as scale differences confined to one tail, was not examined. Finally, the Siegel-Tukey test assumes equal medians; when medians differ, the test can reject for location differences rather than scale differences, a confound that was addressed here by median-centering each sample before testing but that could introduce bias if the median estimates themselves are poor.

Conclusion

This research characterised the Siegel-Tukey test's behaviour under conditions relevant to prairie wheat yield analysis and found it to be a reliable, if sometimes underpowered, tool for detecting scale differences between irrigated and rain-fed populations. At the municipal level, where sample sizes were adequate, 73.4 percent of Saskatchewan municipalities showed significantly greater yield variability in rain-fed than in irrigated fields. The algebraic analysis of the ranking operator yielded closed-form null moments and efficiency bounds that sharpen the test's practical use. Power tables and tie-correction factors tailored to heavy-tailed agricultural distributions were provided. These results fill a gap between the theoretical nonparametric literature and the practical needs

of crop scientists and insurance analysts working with prairie yield data. Future research should extend the analysis to other prairie crops, including canola and barley, and explore adaptive ranking schemes that adjust the Siegel-Tukey rank assignment to the estimated tail behaviour of the data.

Acknowledgements

Funding Sources

This research was conducted without external funding. Departmental resources covered all computational costs.

Institutional Support

The authors thank the Zanzibar National Institute of Mathematical Research, Zanzibar, Tanzania, for providing workspace and computing resources.

Contributions Not Qualifying for Authorship

The authors acknowledge Mr. Hamisi Abdalla for assistance with data cleaning and Dr. Grace Mosha for comments on an early draft.

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