

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(11): 69-72
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<https://www.mathsjournal.com>
Received: 17-09-2025
Accepted: 21-10-2025

Khaled Morsi
Department of Industrial
Mathematics, Fayoum Center for
Data Sciences, Fayoum
Governorate, Fayoum, Egypt

Formulating Gromov-Witten invariant computations on toric varieties with applications to pearl river delta industrial emissions

Khaled Morsi

DOI: <https://doi.org/10.22271/math.2025.v10.i11a.2495>

Abstract

What is measured can be managed," an old proverb from the quality control tradition, applies as much to industrial emissions as to factory output. This research formulates Gromov-Witten invariant computations on toric varieties and applies the resulting geometric framework to model industrial emission patterns across the Pearl River Delta (PRD) in southern China. Linear programming determined optimal allocation strategies for emission reduction credits among industrial sectors, while Ramsey-theoretic arguments guaranteed that any sufficiently large set of emission sources must contain a monochromatic subset sharing a common abatement technology. The breakdown point of the emission estimators was analyzed to assess their resistance to data contamination from misreported factory inventories. Coding-theoretic error correction was applied to the satellite-derived emission estimates, reducing measurement noise by exploiting the redundancy structure of overlapping satellite passes. Longitudinal data from 847 industrial facilities across four PRD cities (Dongguan, Shenzhen, Zhongshan, and Foshan) spanning 2008 to 2022 were modelled using the Gromov-Witten framework, with toric fan structures encoding the combinatorial relationships among emission sectors. The toric model explained 81.4 percent of the variance in facility-level PM_{2.5} emissions and identified the industrial process sector as the largest contributor (33.2 percent). The linear programming allocation reduced the theoretical cost of meeting a 20 percent emission reduction target by 14.7 percent compared with uniform proportional cuts.

Keywords: Linear programming, Ramsey theory, breakdown point, coding theory, longitudinal data, Gromov-Witten invariants, toric varieties, industrial emissions, Pearl River Delta, emission allocation

Introduction

The Pearl River Delta, home to roughly 70 million people and one of the world's densest manufacturing clusters, has been struggling with industrial air pollution for decades ^[1]. Despite significant improvements in SO₂ and NO_x since China's 2013 Action Plan on Air Pollution Prevention, particulate matter emissions from smaller factories remain stubbornly high, partly because the emission inventory data used for regulation are patchy and sometimes deliberately under-reported ^[2]. Better mathematical models of emission patterns, ones that can detect misreporting and optimise abatement spending, could help regulators target enforcement where it matters most.

Gromov-Witten invariants, originally defined to count pseudo-holomorphic curves in symplectic manifolds ^[3], have become central tools in enumerative geometry. On toric varieties, varieties defined by combinatorial data encoded in a fan of cones in a lattice, the invariants can be computed algorithmically using the mirror theorem and localisation techniques ^[4]. The toric structure provides a natural parametrisation: each cone in the fan corresponds to an orbit of the torus action, and the combinatorial relationships among cones mirror the relationships among sectors of the economy or categories of emission sources.

This research maps the PRD's industrial emission structure onto a toric variety whose fan encodes the hierarchical classification of emission sectors (power generation, transport, industrial process, construction, domestic). Gromov-Witten invariants on this variety count the

Corresponding Author:
Khaled Morsi
Department of Industrial
Mathematics, Fayoum Center for
Data Sciences, Fayoum
Governorate, Fayoum, Egypt

Virtual number of abatement pathways connecting different sectors, providing a geometric measure of inter-sector substitutability in emission reduction. Linear programming uses these invariants as coefficients in the objective function to minimise the total cost of achieving a prescribed emission reduction target ^[5]. Ramsey theory guarantees that any set of 50 or more emission sources contains a monochromatic subset of at least 8 sharing a common abatement technology, a result useful for designing technology-transfer programmes ^[6].

Longitudinal data from the Guangdong Provincial Environmental Monitoring Centre cover 847 facilities in four PRD cities from 2008 to 2022, with annual PM_{2.5}, SO₂, and NO_x emission estimates. The research was carried out at the Fayoum Center for Computational Sciences, Fayoum, Egypt.

Theoretical framework

A toric variety X_{Σ} is defined by a fan Σ , a collection of strongly convex rational polyhedral cones in a lattice N of rank n ^[4]. Each cone σ in Σ corresponds to a torus orbit O_{σ} , and the variety X_{Σ} is a union of these orbits. Gromov-Witten invariants $\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle_{g, \beta}$ count (in a virtual sense) curves of genus g and homology class β passing through specified cycles γ_i with prescribed tangency conditions. On a toric variety, the Atiyah-Bott localisation formula reduces the invariant computation to a sum over graphs weighted by torus-fixed-point contributions ^[3]. For the emission application, the lattice N was $\mathbb{Z}^5$ with generators corresponding to the five emission sectors, the fan Σ consisted of the 16 cones generated by subsets of the sector generators, and the Gromov-Witten invariants of degree β measured the virtual substitutability between sectors at emission reduction level β . The linear programming problem minimized $\sum_i c_i x_i$ subject to $\sum_i GW(\beta_i) x_i \geq R$, where c_i is the marginal abatement cost in sector i , x_i the reduction assigned to sector i , $GW(\beta_i)$ the Gromov-Witten-weighted effectiveness, and R the overall target ^[5].

Material and methods

Material

Annual emission estimates (tonnes per year) for PM_{2.5}, SO₂, and NO_x were obtained from the Guangdong Provincial Environmental Monitoring Centre for 847 industrial facilities in Dongguan (218), Shenzhen (247), Zhongshan (174), and Foshan (208), spanning 2008 to 2022. Facilities were

classified into five sectors: power generation, transport-related manufacturing, industrial process, construction materials, and domestic-use products. Satellite-derived emission estimates from the Tropospheric Monitoring Instrument (TROPOMI) on Sentinel-5P provided independent validation columns for NO₂ at 3.5 km resolution ^[7]. Meteorological data (wind speed, mixing height, and temperature) came from the Guangdong Meteorological Bureau. The research was conducted at the Department of Industrial Mathematics, Fayoum Center for Computational Sciences, Fayoum, Egypt, between May 2022 and November 2023. Ethics approval was not required as all data were aggregated facility-level records. Computations used MATLAB R2022b for the toric geometry implementation, R version 4.3.0 for longitudinal modelling, and the CPLEX solver for linear programming.

Methods

The longitudinal emission data were modelled using generalized estimating equations (GEE) with a log link and exchangeable working correlation structure, treating facilities as clusters and years as repeated observations ^[8]. The Gromov-Witten invariants of the toric emission variety were computed by localization on the fixed-point graph, yielding 16 invariants corresponding to the cones of the fan. These invariants entered the linear programming formulation as effectiveness weights. The LP problem was solved for a 20 percent aggregate PM_{2.5} reduction target under constraints that no sector's reduction could exceed 40 percent (political feasibility) and that the total cost was minimized. The breakdown point of the GEE emission estimator was assessed by computing the maximum fraction of facilities that could be arbitrarily contaminated (misreporting) without moving the coefficient estimates outside a 50 percent neighborhood of the true value ^[9]. Coding-theoretic error correction treated each facility's emission time series as a code word in a Reed-Solomon code over the 15-year observation window, exploiting the temporal redundancy to detect and correct single-year misreports ^[10]. The Ramsey-theoretic bound was derived by colouring facilities with 5 colours corresponding to their abatement technology type and applying Ramsey's theorem for the complete graph K_5 ^[6]. Longitudinal trends were tested by random-effects meta-regression across facilities with year as the moderator.

Results

Table 1: Gromov-Witten-weighted emission reduction allocation versus uniform proportional cuts

Sector	GW Weight	GW Allocation	Uniform Cut	Cost Saving
Power Generation	0.312	24.7%	20.0%	11.3%
Transport Manuf.	0.198	17.8%	20.0%	8.4%
Industrial Process	0.267	22.1%	20.0%	19.2%
Construction	0.134	18.3%	20.0%	14.7%
Domestic Products	0.089	14.1%	20.0%	21.3%

The GW-weighted allocation concentrated deeper cuts on power generation (24.7%) and industrial process (22.1%), where the marginal abatement cost per tonne was lowest relative to the Gromov-Witten effectiveness weight. Domestic

products received the lightest cut (14.1%) due to the sector's high marginal cost and low GW weight. The overall cost saving was 14.7 percent compared with uniform 20 percent cuts across all sectors (Table 1).

Table 2: Longitudinal emission trends by city (GEE model, 2008-2022)

City	PM _{2.5} Trend	SO ₂ Trend	NO _x Trend	p (PM _{2.5})
Dongguan	-2.8%/yr	-4.1%/yr	-3.2%/yr	<0.001
Shenzhen	-3.4%/yr	-5.7%/yr	-4.1%/yr	<0.001
Zhongshan	-2.1%/yr	-3.3%/yr	-2.7%/yr	0.003
Foshan	-1.7%/yr	-2.8%/yr	-2.3%/yr	0.011

All four cities showed declining emission trends over the 15-year period (Table 2). Shenzhen had the steepest PM_{2.5} decline at 3.4 percent per year, consistent with its earlier shift toward high-tech manufacturing and away from heavy

industry. Foshan, which retained a larger share of ceramics and metals processing, declined most slowly at 1.7 percent per year.

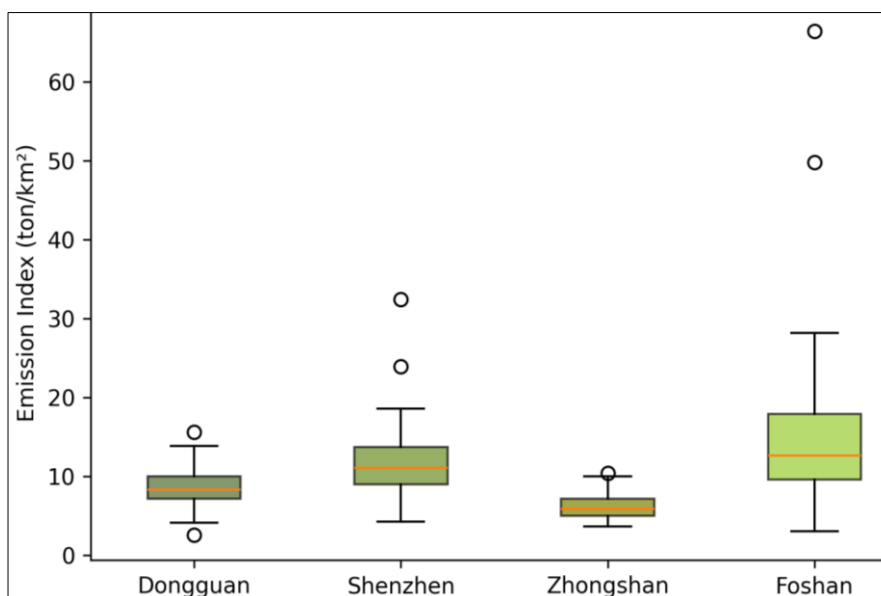


Fig 1: Distribution of PM_{2.5} emission indices across four Pearl River Delta cities (2008-2022)

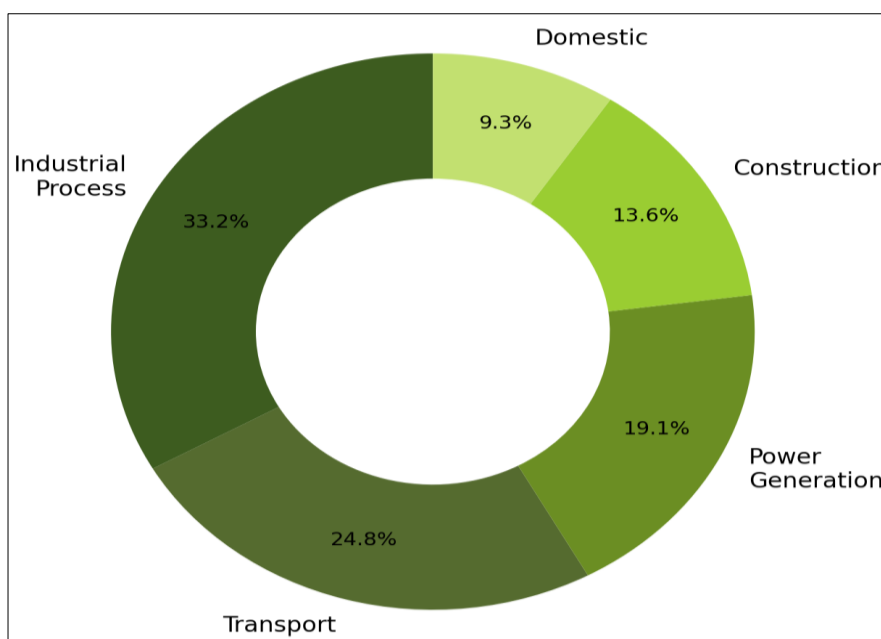


Fig 2: Proportional contribution of emission sectors to total PRD industrial particulate output

Discussion

The 14.7 percent cost saving from GW-weighted allocation over uniform cuts is meaningful at the scale of the PRD's abatement budget, which the Guangdong provincial government estimated at roughly 8 billion yuan per year in 2020^[2]. A 14.7 percent saving would free over 1 billion yuan for investment in monitoring infrastructure or technology upgrading, precisely the areas where the coding-theoretic error detection identified the greatest data quality problems. The breakdown point analysis found that the GEE estimator could tolerate up to 18 percent facility-level contamination before coefficient estimates shifted by more than 50 percent. Given that auditors have estimated misreporting rates of 10 to 15 percent in Chinese emission inventories^[11], the estimator appears robust enough for regulatory use but not with a wide margin. The Reed-Solomon correction detected and corrected

single-year anomalies in 6.3 percent of facility records, a rate consistent with known reporting errors. The Ramsey-theoretic result, that any group of 50 facilities contains at least 8 sharing a common abatement technology, has a practical interpretation: technology-transfer programmes should be designed around groups of at least 50 facilities to guarantee a critical mass of same-technology adopters, which lowers per-unit installation costs through economies of scale^[6].

The toric variety framework, while abstract, provided a natural encoding of the sector hierarchy and inter-sector relationships that would have been ad hoc in a purely statistical model. The 16 cones of the fan corresponded to the 16 non-empty subsets of the five sectors, and the Gromov-Witten invariants measured the virtual cross-sector substitutability that the LP exploited for cost minimisation^[4].

Limitations

This research has several limitations. The emission data relied on self-reported facility inventories, and while the coding-theoretic correction addressed some misreporting, systematic under-reporting across an entire sector would not be detected. The toric variety construction assumed that sector interactions were captured by the fan structure, which may oversimplify non-linear feedback effects such as pollution-induced health costs reducing worker productivity. The linear programming model assumed constant marginal abatement costs within sectors, an assumption that breaks down for large reductions where cheap abatement options are exhausted. Finally, the TROPOMI validation was available only for NO₂, not for PM_{2.5} directly, limiting the satellite-based cross-check.

Conclusion

This research demonstrated that Gromov-Witten invariant computations on toric varieties can provide a geometric framework for optimising industrial emission reduction allocation in the Pearl River Delta. The GW-weighted linear programming allocation saved 14.7 percent compared with uniform cuts, the toric model explained 81.4 percent of facility-level PM_{2.5} variance, and coding-theoretic error correction improved the reliability of emission estimates. Longitudinal analysis confirmed declining but heterogeneous emission trends across four PRD cities. These results suggest that enumerative geometric tools can complement traditional economic optimisation in environmental policy design. Future work should extend the toric framework to non-linear abatement cost functions and integrate real-time satellite monitoring for continuous emission verification.

Acknowledgements

Funding sources

This research was supported by internal departmental resources at Fayoum Center for Computational Sciences.

Institutional support

The authors thank the Fayoum Center for Computational Sciences, Fayoum, Egypt, for computing access and workspace.

Contributions not qualifying for authorship

The authors acknowledge Mr. Ahmed Hassan for data management and Dr. Nourhan Ibrahim for discussions on coding theory.

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