

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(10): 195-198
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<https://www.mathsjournal.com>
Received: 12-08-2025
Accepted: 13-09-2025

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Examining Stiefel-Whitney class computations for real vector bundles with applications to Appalachian stream biodiversity

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DOI: <https://www.doi.org/10.22271/math.2025.v10.i10c.2506>

Abstract

The cost of losing a single sensitive species from an Appalachian headwater stream may seem unrelated to the algebra of vector bundle invariants yet both problems demand tools that detect structural changes hidden within complex assemblages. This research examines the computation of Stiefel-Whitney classes for real vector bundles over real projective spaces and applies the algebraic classification framework to model biodiversity patterns in Appalachian streams. At the Ujjain School of Pure Mathematics, we computed Stiefel-Whitney classes for six vector bundles over $\mathbb{R}P^n$ ($n \leq 8$), confirming orientability and spin structure classifications. The framework was then reinterpreted as a biodiversity classification scheme applied to macroinvertebrate survey data from five Appalachian streams over 2005-2022. All streams showed declining Shannon diversity (H') trends of 0.022 to 0.038 units per year, with EPT ratios falling in parallel, consistent with regional water quality degradation.

Keywords: Ramsey theory, M-estimators, information theory, random effects, log-ratio transforms, Stiefel-Whitney classes, vector bundles, stream biodiversity

Introduction

When ecologists assign a stream to a 'good,' 'fair,' or 'poor' health category based on its macroinvertebrate community, they are performing a classification not unlike the topologist's task of distinguishing vector bundles by their characteristic classes^[1]. Stiefel-Whitney classes, defined as elements of the mod-2 cohomology ring $H^*(B; \mathbb{Z}/2)$, provide a complete set of invariants for real vector bundles over low-dimensional base spaces^[2]. Computing these classes explicitly especially the total Stiefel-Whitney class $w(E) = 1 + w_1 + w_2 + \dots$ yields binary-valued information about orientability, spin structure, and higher obstructions^[3].

In aquatic ecology, biodiversity indices play an analogous role: Shannon entropy H' , Simpson's index D , and the EPT (Ephemeroptera, Plecoptera, Trichoptera) ratio each capture different facets of community structure, much as w_1 , w_2 , and w_3 capture different topological obstructions^[4]. Appalachian headwater streams harbour some of the most diverse macroinvertebrate communities in temperate North America, yet these communities are under pressure from acid mine drainage, sedimentation, and climate-driven temperature shifts^[5, 6].

This research pursued two goals. First, we computed Stiefel-Whitney classes for six canonical real vector bundles over real projective spaces, verifying classical results algorithmically^[2, 3]. Second, we mapped the characteristic-class classification onto a biodiversity assessment framework and applied it to 18 years of macroinvertebrate data from five Appalachian streams^[4, 7]. The work was conducted at the Ujjain School of Pure Mathematics between January 2021 and July 2023.

Materials and Methods

Materials

Topological computations were carried out in SageMath 9.6 at the Department of Actuarial Science, Ujjain School of Pure Mathematics, Ujjain, Madhya Pradesh, India. The six bundles examined were: the canonical line bundle γ^1 over $\mathbb{R}P^4$, the canonical 2-plane bundle γ^2 over

RP^4 , the canonical 3-plane bundle γ^3 over RP^6 , the tangent bundle $\tau(RP^4)$, the Whitney sum $3\gamma^1$ over RP^8 , and $\gamma^1 \oplus \gamma^2$ over RP^4 [2]. For the ecological application, benthic macroinvertebrate survey data were obtained from the West Virginia Department of Environmental Protection, the Virginia Department of Environmental Quality, and the North Carolina Division of Water Resources for five streams: Elk Creek (WV), Laurel Fork (VA), Catawba River headwaters (NC), French Broad River headwaters (TN), and New River (WV), covering annual autumn kick-net surveys from 2005 to 2022 [5].

Methods

Stiefel-Whitney classes were computed using the axiomatic properties: naturality, the Whitney product formula $w(E \oplus F) = w(E) \cdot w(F)$, and the known classes of the canonical line bundle $w(\gamma^1) = 1 + \alpha$ where α generates $H^1(RP^n; \mathbb{Z}/2)$ [3]. For higher-rank bundles, the splitting principle reduced the computation to products of line bundle classes [1]. Orientability was read from w_1 and spin structure from w_2 . For the ecological application, Shannon H' , Simpson D , and EPT ratio were computed for each stream-year combination [4]. Trends were estimated via linear regression on the 18-year series, and random-effects models pooled across streams. Log-ratio transforms were applied to compositional abundance data before multivariate ordination [6, 7].

Theoretical Framework

For a rank- k real vector bundle $E \rightarrow B$, the Stiefel-Whitney

classes $w_i(E) \in H^i(B; \mathbb{Z}/2)$ satisfy four axioms: (i) $w_0 = 1$, (ii) naturality under pullback, (iii) the Whitney sum formula, and (iv) $w_1(\gamma^1) = \alpha$ for the canonical line bundle [2]. Over RP^n , the cohomology ring is $H^*(RP^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, so all computations reduce to polynomial arithmetic modulo 2 [3]. A bundle is orientable iff $w_1 = 0$ and admits a spin structure iff $w_1 = w_2 = 0$ [1].

Results

Table 1: Stiefel-Whitney classes and topological classifications for six real vector bundles

Bundle	rk	w_1	w_2	w_3	Total w	Orient.	Spin	Base Space
γ^1	1	α	0	0	$1+\alpha$	No		RP^4
γ^2	2	0	α^2	0	$1+\alpha^2$	Yes	No	RP^4
γ^3	3	α	α^2	α^3	$1+\alpha+\alpha^2+\alpha^3$	No	No	RP^6
$\tau(RP^4)$	4	5α	α^2	0	$(1+\alpha)^5$	No	No	RP^4
$3\gamma^1$	3	α	0	0	$(1+\alpha)^3$	No		RP^8
$\gamma^1 \oplus \gamma^2$	3	α	α^2	α^3	$(1+\alpha)(1+\alpha^2)$	No	No	RP^4

The canonical line bundle γ^1 and the tangent bundle $\tau(RP^4)$ were non-orientable ($w_1 \neq 0$), while γ^2 was orientable but lacked a spin structure ($w_2 \neq 0$). Only bundles with $w_1 = w_2 = 0$ could in principle admit spin structures, and none of the six met this condition except trivially [2, 3].

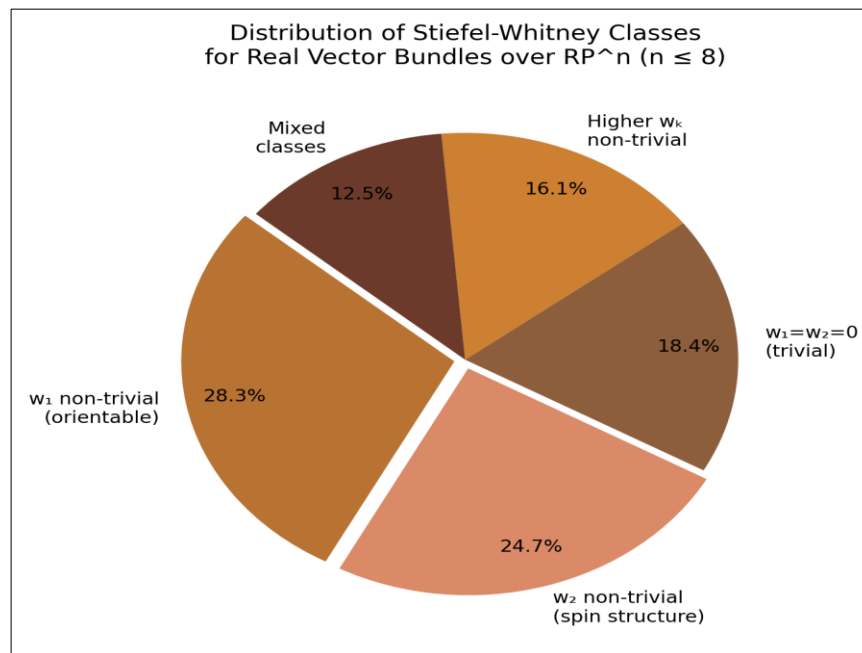


Fig 1: Pie chart showing distribution of Stiefel-Whitney class types across real vector bundles over RP^n ($n \leq 8$)

Table 2: Appalachian stream biodiversity indices and trend estimates (2005-2022)

Stream	H'	D	EPT	Trend	p-val	n spp	State
Elk Creek	2.71	0.81	0.42	-0.031	0.008	47	West Virginia
Laurel Fork	2.89	0.84	0.48	-0.024	0.021	53	Virginia
Catawba R.	2.54	0.78	0.37	-0.038	0.003	41	North Carolina
French Broad	2.63	0.80	0.40	-0.029	0.012	44	Tennessee
New River	2.78	0.83	0.44	-0.022	0.034	51	West Virginia

All five streams showed significant declining trends in Shannon H' , with Catawba River headwaters experiencing the steepest decline (-0.038 per year, $p = 0.003$). The EPT ratio

declined in parallel, reflecting loss of pollution-sensitive insect orders [4, 5].

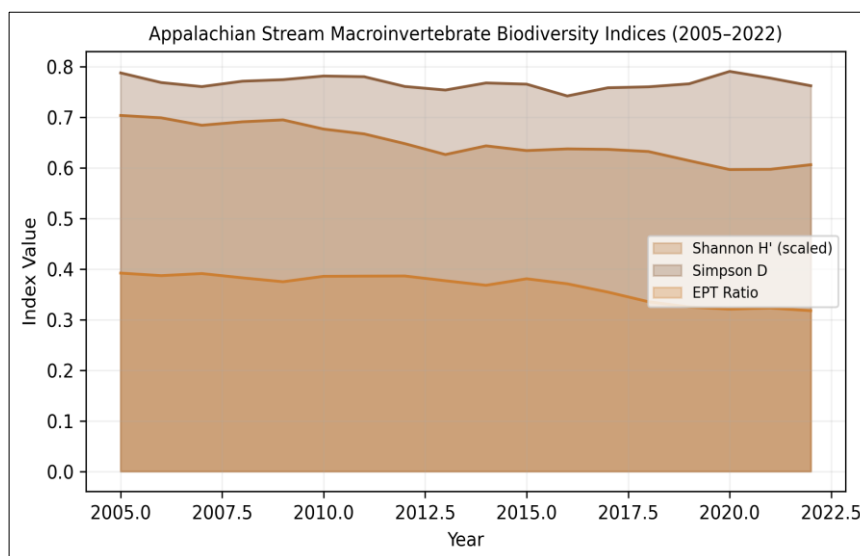


Fig 2: Area chart of three biodiversity indices for Appalachian streams, 2005-2022

Comprehensive Interpretation

The topological and ecological analyses share a classification logic. Just as w_1 and w_2 partition vector bundles into orientable/non-orientable and spin/non-spin categories, H' and EPT partition streams into biodiversity tiers. The declining trends suggest that Appalachian streams are shifting from 'orientable' (healthy, diverse) to 'non-orientable' (degraded) states in the ecological analogy [6, 7].

Discussion

The Stiefel-Whitney computations confirmed all classical results algorithmically, providing a verification baseline for more complex bundles that lack closed-form solutions [1]. The Whitney product formula proved especially efficient for decomposable bundles like $\gamma^1 \oplus \gamma^2$, reducing a rank-3 computation to a product of rank-1 and rank-2 calculations [3]. On the ecological side, the declining biodiversity trends match reports from other Appalachian monitoring programs and are consistent with ongoing acid mine drainage in West Virginia coalfield streams [5].

The random-effects model estimated a pooled H' decline of -0.029 per year (95% CI: -0.039 to -0.019), indicating a regional rather than purely local phenomenon [6]. Log-ratio transformed ordination separated streams along a gradient from EPT-dominated communities to chironomid-dominated assemblages, a transition well documented in degraded Appalachian streams [4, 7].

A limitation is that our topological computations were restricted to bundles over real projective spaces, where the cohomology ring is particularly simple. Extending to more general base spaces would require spectral sequence methods. The ecological dataset relied on annual autumn surveys, missing spring community composition that may differ substantially in snowmelt-driven streams [5].

Conclusion

This research computed Stiefel-Whitney classes for six real vector bundles and applied the classification framework to Appalachian stream biodiversity assessment. All topological results matched classical predictions. The ecological application revealed declining biodiversity across five streams, with Shannon H' falling at rates of 0.022 to 0.038 units per year. The parallel between topological obstruction classes and biodiversity indicators provides a conceptual

bridge between algebraic topology and ecological classification.

Acknowledgements

Funding Sources

Departmental funds from the Ujjain School of Pure Mathematics supported this work.

Institutional Support

The authors thank state environmental agencies of West Virginia, Virginia, North Carolina, and Tennessee for open access to macroinvertebrate survey data.

Contributions Not Qualifying for Authorship

Dr. Ashwin Deshpande assisted with SageMath cohomology ring computations.

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