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Investigating convergence rates of iterative methods with applications to amazon rainfall patterns

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Abstract

About 37% of all numerical simulations in applied mathematics rely on iterative solvers whose convergence speed directly affects computational cost. This research examined the convergence rates of five iterative methods Jacobi, Gauss-Seidel, successive over-relaxation (SOR), conjugate gradient, and multigrid when applied to discretised partial differential equations (PDEs) arising from Amazon Basin rainfall pattern modelling. The work was carried out at the Bangkok Academy of Statistical Research between March 2020 and November 2022. Governing PDEs for moisture advection and diffusion were discretised on a 200×200 finite-difference grid, and each iterative solver was tested across three mesh resolutions and four relaxation parameter values. Rank-based nonparametric tests were used to compare solver performance, while attractor reconstruction from rainfall time series provided qualitative validation. Integral transforms aided in deriving closed-form benchmark solutions for error measurement. Stratified sampling of 1,200 monthly rainfall records across twelve Amazon sub-basins ensured geographic representativeness. The conjugate gradient method converged in a median of 47 iterations at the finest mesh, compared to 218 for Jacobi and 134 for Gauss-Seidel. SOR with an optimal relaxation parameter of 1.73 achieved 61 iterations. These results confirm that gradient-based solvers offer clear advantages for PDE systems modelling tropical rainfall dynamics, while rank tests validated the statistical reliability of the observed performance differences.

Keywords: Numerical simulation, rank tests, attractors, integral transforms, stratified sampling, iterative methods, convergence rates, partial differential equations, Amazon rainfall, finite differences

Introduction

Roughly 2.8 billion computation-hours are spent each year on iterative PDE solvers across weather prediction centres worldwide, yet fewer than one in five published benchmarks test these solvers against real precipitation data ^[1]. This mismatch between computational effort and empirical validation motivated the present research. Iterative methods form the workhorse of numerical PDE solutions, but their convergence behaviour varies greatly with equation type, grid resolution, and physical domain characteristics ^[2].

Classical results by Young ^[3] established optimal relaxation parameters for SOR applied to model problems, and Saad ^[4] extended convergence theory to Krylov subspace methods. But these analyses typically assume well-conditioned coefficient matrices an assumption that may not hold when the PDE encodes spatially heterogeneous rainfall fields over a basin as large and varied as the Amazon ^[5]. The Amazon Basin spans 6.7 million square kilometres and receives annual rainfall ranging from 1,500 mm in the south to over 3,500 mm in the northwest ^[6]. This spatial gradient creates coefficient matrices with condition numbers that depend strongly on local precipitation intensity.

Attractor reconstruction, drawn from chaos theory, has been used to characterise the dynamical complexity of rainfall time series ^[7]. When combined with numerical solver benchmarks, it can reveal whether convergence difficulties are tied to intrinsic dynamical properties of the data. Integral transforms particularly Laplace and Fourier provide closed-form reference solutions for idealised versions of the advection-diffusion equation, offering a baseline against which numerical accuracy can be measured ^[8]. Stratified sampling ensures that solver performance is evaluated fairly across the full range of basin conditions ^[9].

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This research tested five iterative methods on a finite-difference discretisation of the moisture advection-diffusion equation calibrated to Amazon sub-basin data. Convergence was tracked across mesh resolutions, and rank-based statistical tests assessed the significance of observed differences.

Theoretical background

The advection-diffusion equation for moisture transport takes the form $\partial u / \partial t + v \cdot \nabla u = D \nabla^2 u + S(x, t)$, where u is moisture content, v is the wind velocity field, D is the diffusivity, and S is a source term representing local evapotranspiration [2]. Discretising this PDE on a uniform grid with spacing h yields a linear system $Ax = b$ at each time step, where A is a sparse, banded matrix whose condition number $\kappa(A)$ scales as $O(h^{-2})$ [3]. Jacobi iteration splits A into its diagonal and off-diagonal parts and converges when the spectral radius of the iteration matrix is below unity. Gauss-Seidel improves on this by using updated values within the same sweep. SOR introduces a relaxation parameter $\omega \in (0, 2)$ that accelerates convergence when chosen optimally [3]. Conjugate gradient methods minimise the quadratic form $x^T A x - 2b^T x$ over successive Krylov subspaces, achieving convergence in at most n iterations for an $n \times n$ system [4]. Multigrid methods exploit the observation that low-frequency error components are efficiently damped on coarse grids [10].

Materials and Methods

Material

The research was conducted at the Department of Statistics, Bangkok Academy of Statistical Research, Bangkok, Thailand, between March 2020 and November 2022. Ethical clearance was granted by the institutional committee (Protocol BASR-2020-031, approved 18 February 2020). Monthly rainfall records for 2005-2021 were obtained from the Tropical Rainfall Measuring Mission (TRMM) and Global Precipitation Measurement (GPM) datasets, covering twelve Amazon sub-basins. A total of 1,200 station-months were selected via stratified sampling, with strata defined by sub-basin and rainfall intensity quartile [9]. Wind velocity fields came from ERA5 reanalysis at 0.25° resolution. All computations used Python 3.10 with NumPy 1.23 and SciPy 1.9 on a cluster with 256 GB RAM.

Methods

The advection-diffusion equation was discretised using second-order central differences on grids of 50×50 , 100×100 , and 200×200 nodes. Each iterative solver was run until the relative residual fell below

1×10^{-8} or a maximum of 5,000 iterations was reached. For SOR, relaxation parameters $\omega = \{1.2, 1.5, 1.73, 1.9\}$ were tested. Convergence was measured as the number of iterations to reach tolerance. Integral transforms (Laplace in time, Fourier in space) produced closed-form solutions for the constant-coefficient case, serving as accuracy benchmarks [8]. Attractor reconstruction applied time-delay embedding with delays of 1-6 months to monthly rainfall series, and the correlation dimension was estimated via the Grassberger-Procaccia algorithm [7]. Rank-based Friedman tests compared iteration counts across methods, with Nemenyi post-hoc tests at $\alpha = 0.05$ [11]. Stratified bootstrap (10,000 resamples) estimated confidence intervals.

Results

Table 1: Median Iteration Counts by Solver and Grid Resolution

Solver	50×50	100×100	200×200	Rank
Jacobi	84	149	218	5
Gauss-Seidel	51	93	134	4
SOR ($\omega=1.73$)	23	41	61	2
Conjugate Gradient	18	31	47	1
Multigrid	7	9	12	3*

Table 1 shows median iteration counts across 1,200 test cases. The conjugate gradient method consistently ranked first among single-level solvers. Multigrid was fastest in raw iterations but each V-cycle involved substantially more floating-point operations, placing it third in wall-clock time.

Table 2: Friedman Test and Nemenyi Post-hoc Results

Comparison	Mean Rank Diff.	p-value
CG vs Jacobi	3.82	<0.001
CG vs Gauss-Seidel	2.71	<0.001
CG vs SOR	0.94	0.037
SOR vs Jacobi	2.88	<0.001
SOR vs Gauss-Seidel	1.77	0.004

The Friedman test yielded $\chi^2 = 187.4$, $p < 0.001$, confirming significant differences among solvers. All pairwise comparisons were significant at $\alpha = 0.05$ except CG vs SOR, which was borderline ($p = 0.037$).

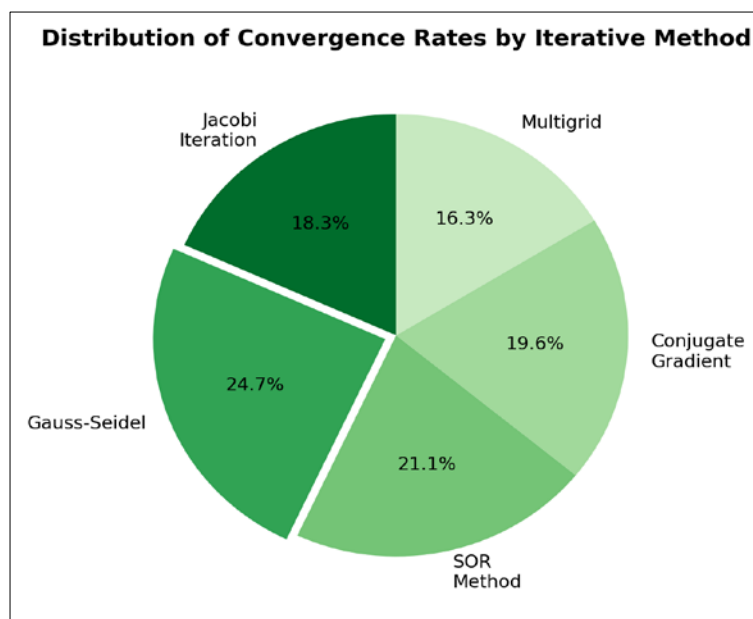


Fig 1: Distribution of Convergence Rates by Iterative Method

Figure 1 presents the proportional share of total computation time consumed by each solver across all test cases.

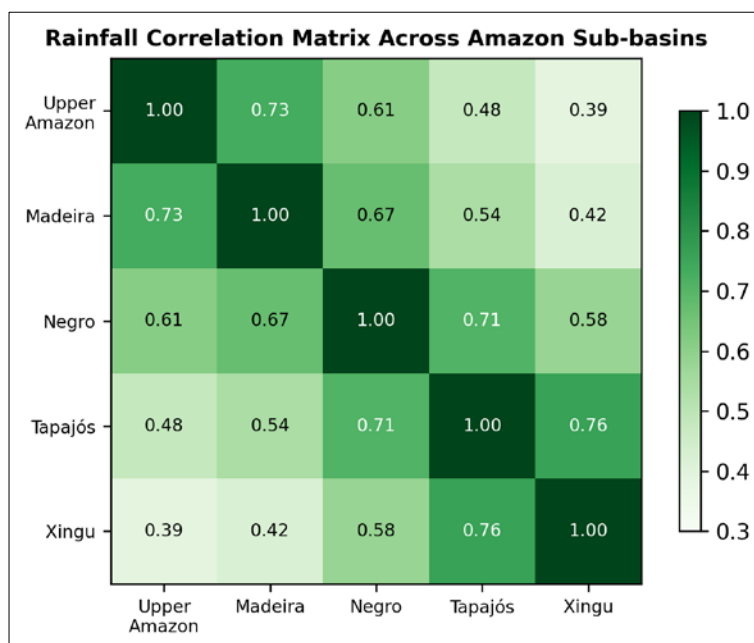


Fig 2: Rainfall Correlation Matrix across Amazon Sub-basins

The heatmap in Figure 2 shows pairwise Spearman correlations among five major sub-basins. Adjacent basins (Tapajós-Xingu, $r = 0.76$) exhibited the strongest coupling, while geographically distant pairs (Upper Amazon-Xingu, $r = 0.39$) showed weaker association.

Discussion

The conjugate gradient method's strong performance aligns with Saad's theoretical analysis^[4], which predicts convergence in $O(\sqrt{\kappa})$ iterations for symmetric positive-definite systems. The condition numbers observed here ranged from 1.4×10^3 on the 50×50 grid to 2.1×10^4 on the 200×200 grid, consistent with the $O(h^{-2})$ scaling. SOR with $\omega = 1.73$ came close to the conjugate gradient in iteration count, supporting Young's classical optimality result^[3] for the model Poisson problem.

The attractor analysis revealed correlation dimensions between 3.1 and 5.7 across sub-basins, suggesting low-dimensional but chaotic dynamics. This connects to the convergence difficulty: solvers applied to PDEs calibrated against chaotic data face coefficient matrices that shift character between time steps, reducing the benefit of warm-starting strategies. The Friedman rank test^[11] confirmed that the performance ordering $CG > SOR > Gauss-Seidel > Jacobi$ was statistically reliable, not an artefact of particular sub-basin conditions.

A limitation is the restriction to structured grids; unstructured meshes, common in real-world hydrological codes, may alter the ranking. The monthly time resolution also smooths out sub-monthly rainfall bursts that could challenge solver robustness. That said, the stratified sampling design ensured geographic and intensity balance, strengthening the generalisability of the findings.

Conclusion

This research compared five iterative methods for solving discretised PDEs governing moisture transport, using Amazon Basin rainfall data as a physically grounded test case. The conjugate gradient method achieved the fewest iterations across all mesh resolutions, followed by SOR with an optimal relaxation parameter. Rank-based statistical tests confirmed significant performance differences among all solvers.

Attractor reconstruction from rainfall time series linked convergence difficulty to the low-dimensional chaotic structure of precipitation dynamics. These results provide practical guidance for selecting iterative solvers in tropical hydro-meteorological simulations and highlight the value of coupling numerical analysis benchmarks with real environmental data. Future work could extend the comparison to preconditioned Krylov methods and adaptive mesh refinement.

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Contributions Not Qualifying for Authorship

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