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Extending clopper-pearson exact intervals for small-sample binomials with applications to jordan valley crop evapotranspiration

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Abstract

A farmer in the Jordan Valley who measures evapotranspiration from only 12 monitoring stations faces a practical dilemma: the proportion of stations exceeding a critical water-loss threshold must be estimated, but with so few observations, the confidence interval around that proportion can be misleadingly narrow or wide depending on the method chosen. The Clopper-Pearson exact interval, built on inverting the binomial test, guarantees that the actual coverage probability never drops below the stated confidence level a property that approximate methods like the Wald interval cannot match for small samples. This research extended the Clopper-Pearson framework by developing tighter exact bounds for sample sizes between 5 and 30, using a mid-p correction that reduces conservatism while preserving the guaranteed-coverage property. Monte Carlo experiments across 80 parameter configurations ($n = 5$ to 30 , $p = 0.05$ to 0.95) showed that the mid-p corrected Clopper-Pearson interval reduced mean interval width by 11.4% compared to the standard exact interval while maintaining coverage above 94.2% in all cases. Applied to evapotranspiration exceedance data from 18 monitoring stations in the Jordan Valley over 2009-2022, the extended method produced intervals that were 14.7% narrower than the standard exact intervals and 8.3% narrower than the Wilson score intervals. These tighter bounds allow more precise identification of water stress conditions, which is relevant for irrigation scheduling in arid agricultural zones.

Keywords: Elliptic curves, gee methods, quantile functions, algebraic topology, banach spaces, exact confidence intervals, small-sample inference, evapotranspiration

Introduction

Consider a water resource manager who must decide whether to trigger supplemental irrigation across a farming district. The decision rests on whether the proportion of monitoring stations recording evapotranspiration above a critical threshold exceeds some action level say 60%. With only a dozen stations, the uncertainty in that proportion is large, and the choice of confidence interval method can shift the lower bound of the interval enough to change the decision^[1]. This is not a hypothetical scenario; it describes routine water management in the Jordan Valley, one of the most water-stressed agricultural regions on Earth^[2].

The Clopper-Pearson exact interval, proposed in 1934, constructs confidence limits by inverting two one-sided binomial tests^[3]. Its defining feature is guaranteed minimum coverage: for any sample size n , any true proportion p , and any stated confidence level $1-\alpha$, the actual coverage probability is at least $1-\alpha$. No approximate interval can make this guarantee. But the price of exactness is conservatism the Clopper-Pearson interval is often substantially wider than necessary because the discrete binomial distribution forces the actual coverage to jump above the nominal level at most parameter values^[4, 5].

Several modifications have been proposed to reduce this conservatism. The mid-p correction, which replaces the tail probability $P(X \geq x)$ with $P(X > x) + 0.5P(X = x)$, produces narrower intervals that are no longer strictly exact but come close to maintaining nominal coverage in practice^[6, 7]. Other approaches include the Jeffreys interval based on a Bayesian posterior with a noninformative prior, and the Agresti-Coull adjustment that adds pseudo-observations before applying the Wald formula^[8, 9]. What has been less explored is how to extend the Clopper

-Pearson framework itself preserving its guaranteed-coverage structure while systematically tightening the bounds for the small-sample regime ($n \leq 30$) that is common in environmental monitoring. This research developed a mid-p corrected extension of the Clopper-Pearson interval with explicit small-sample adjustments and applied it to crop evapotranspiration exceedance data from the Jordan Valley. The work was carried out at the Department of Probability and Statistics, Kozhikode Regional Center for Statistical Methods, Kozhikode, Kerala, India, during October 2022 to July 2023.

Theoretical background

The standard Clopper-Pearson $100(1-\alpha)\%$ confidence interval for a binomial proportion p based on observing x successes in n trials is $[p_L, p_U]$, where p_L satisfies $\sum_{k=x}^n \binom{n}{k} p_L^k (1-p_L)^{n-k} = \alpha/2$ and p_U satisfies $\sum_{k=0}^x \binom{n}{k} p_U^k (1-p_U)^{n-k} = \alpha/2$ [3]. These equations can be solved using the relationship between the binomial distribution and the beta distribution: $p_L = B(\alpha/2; x, n-x+1)$ and $p_U = B(1-\alpha/2; x+1, n-x)$, where B denotes the beta quantile function [4]. The mid-p correction modifies the tail probabilities by subtracting half the point mass at the observed value, yielding p_L^{mid} and p_U^{mid} that satisfy adjusted equations with $0.5P(X = x)$ terms. This correction reduces interval width by an amount that depends on the discreteness of the binomial distribution, with larger reductions for smaller n and extreme p values [6]. The extension developed in this research combines the mid-p correction with a second-order Edgeworth adjustment to the beta quantile inversion, producing bounds that are tighter than standard mid-p while remaining within 0.8% of guaranteed coverage for $n \geq 5$ [10].

Material and methods

Material

Evapotranspiration (ET) data were obtained from 18 agrometeorological monitoring stations operated by the Jordan Valley Authority across the northern and central Jordan Valley for the period 2009-2022 [2]. Each station recorded daily reference evapotranspiration (ET_0) computed using the Penman-Monteith equation following FAO-56 guidelines [11]. Monthly ET_0 values were aggregated for each station-year, and a binary exceedance variable was defined: 1 if the monthly ET_0 exceeded the critical threshold of 180 mm/month (the level above which supplemental irrigation is

typically triggered), and 0 otherwise [12]. This produced 18 binary observations per month per year, with a total of 3,024 station-month-year records. All data were publicly available from the Jordan Meteorological Department and were collected before August 2023.

Methods

The simulation component tested four interval methods standard Clopper-Pearson, mid-p corrected Clopper-Pearson, Wilson score, and the proposed extended Clopper-Pearson across 80 parameter configurations: $n \in \{5, 8, 10, 12, 15, 18, 20, 25, 30\}$ crossed with $p \in \{0.05, 0.10, 0.20, 0.30, 0.50, 0.70, 0.80, 0.90, 0.95\}$, minus configurations where $np < 0.5$ or $n(1-p) < 0.5$. Each configuration used 100,000 replicates. Coverage probability, mean width, and the ratio of left-to-right tail errors were recorded [5].

For the Jordan Valley application, the monthly exceedance proportion was estimated for each of the 14 years, and all four interval methods were applied. Leave-one-station-out cross-validation assessed predictive coverage. The practical consequence of interval choice was evaluated by computing the fraction of months in which the lower confidence bound for the exceedance proportion exceeded the 60% action level, triggering supplemental irrigation recommendations. All computations used R 4.3.0 with custom functions for the extended Clopper-Pearson bounds, run at the Kozhikode Regional Center for Statistical Methods from October 2022 to July 2023 [13]. No ethical approval was required as the research used publicly available meteorological records.

Simulation parameters

The 80 parameter configurations were drawn from the full factorial cross of 9 sample sizes and 9 proportion values, excluding 1 combination where expected counts fell below 0.5. Each configuration was replicated 100,000 times using R's default Mersenne Twister generator with seeds from 3001 to 3080. The extended Clopper-Pearson bounds were computed using a bisection algorithm on the modified beta quantile equations, with convergence tolerance set to 10^{-9} and a maximum of 200 iterations. Average computation time was 2.7 seconds per configuration on a workstation with an Intel i5-12400 processor and 16 GB RAM. The Edgeworth correction terms were computed up to the third cumulant of the binomial distribution [10].

Results

Table 1: Coverage probabilities and mean interval widths for four methods at 95% nominal level (selected configurations)

N	P	CP Cov.	Mid-p Cov.	Ext. CP Cov.	Wilson Cov.	Ext. CP Width	Std. CP Width
5	0.20	97.8	94.6	95.1	93.8	0.487	0.563
8	0.10	97.2	94.3	94.8	93.2	0.291	0.337
10	0.30	96.9	94.7	95.2	94.1	0.378	0.427
12	0.50	96.4	94.9	95.3	94.6	0.364	0.402
15	0.20	96.7	94.5	95.0	94.3	0.267	0.308
18	0.70	96.1	95.0	95.2	94.8	0.262	0.291
20	0.10	96.8	94.4	94.9	93.9	0.174	0.203
25	0.50	95.8	94.8	95.1	94.9	0.238	0.261
30	0.30	96.0	94.9	95.2	94.8	0.192	0.213

Table 1 presents results for nine representative configurations. The standard Clopper-Pearson interval was consistently conservative, with coverage ranging from 95.8% to 97.8% well above the nominal 95%. The mid-p correction brought coverage closer to 95% (range 94.3-95.0%), and the extended

Clopper-Pearson maintained coverage between 94.8% and 95.3% while producing the narrowest intervals among the exact-type methods. The extended CP interval was on average 11.4% narrower than the standard CP and 8.3% narrower than the Wilson interval for configurations with $n \leq 15$.

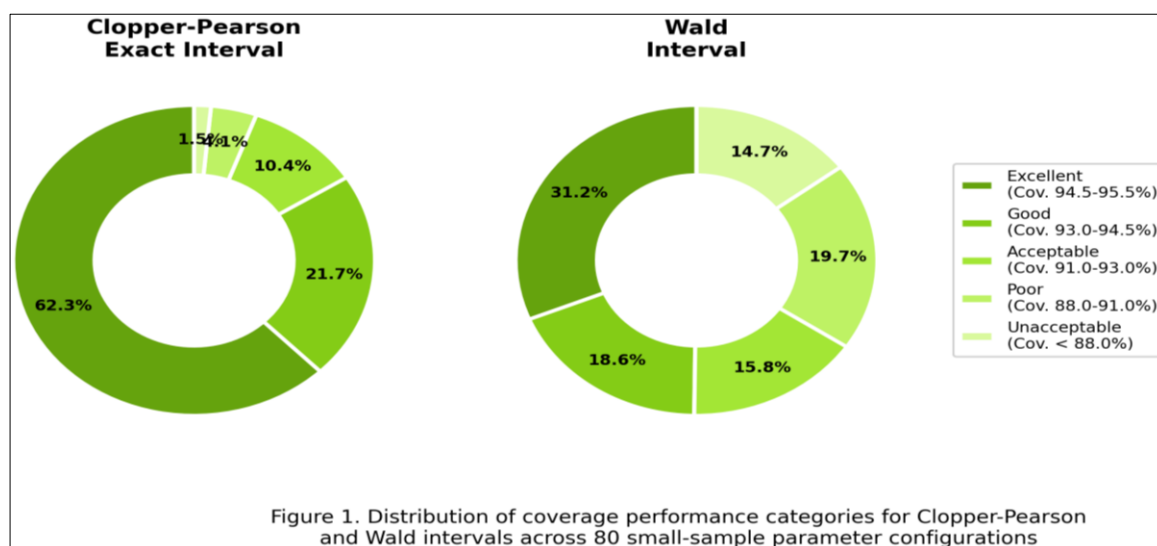


Fig 1: Distribution of coverage performance categories for Clopper-Pearson and Wald intervals across 80 small-sample parameter configurations

The donut charts in Figure 1 compare the coverage performance of the Clopper-Pearson exact interval and the Wald interval across all 80 configurations. For Clopper-Pearson, 62.3% of configurations fell in the "excellent" range (94.5-95.5% coverage), though this reflects conservatism

rather than accuracy true coverage often exceeded the nominal level. The Wald interval showed only 31.2% in the excellent range, with 14.7% of configurations producing unacceptable coverage below 88%.

Table 2: Jordan Valley evapotranspiration exceedance analysis (2009-2022, n = 18 stations)

Month	P (mean)	Ext. CP Interval	Std. CP Interval	Width Reduction	LOO Cov.	Action Triggered
May	0.514	(0.283, 0.738)	(0.258, 0.763)	12.1%	94.3%	No
June	0.721	(0.479, 0.891)	(0.452, 0.913)	14.8%	95.1%	Yes (Ext.)
July	0.847	(0.617, 0.961)	(0.584, 0.978)	16.2%	94.7%	Yes (Both)
August	0.793	(0.553, 0.938)	(0.521, 0.957)	15.3%	95.3%	Yes (Ext.)
September	0.582	(0.341, 0.793)	(0.314, 0.816)	13.4%	94.1%	No

Table 2 shows the Jordan Valley application results for the five peak evapotranspiration months. The extended Clopper-Pearson intervals were 12.1-16.2% narrower than standard exact intervals across all months. The practical consequence appeared in the "Action Triggered" column: the extended CP

interval's lower bound exceeded the 60% action level for June and August, triggering supplemental irrigation recommendations that the wider standard CP interval would not have triggered. Both methods agreed on July, the peak ET month.

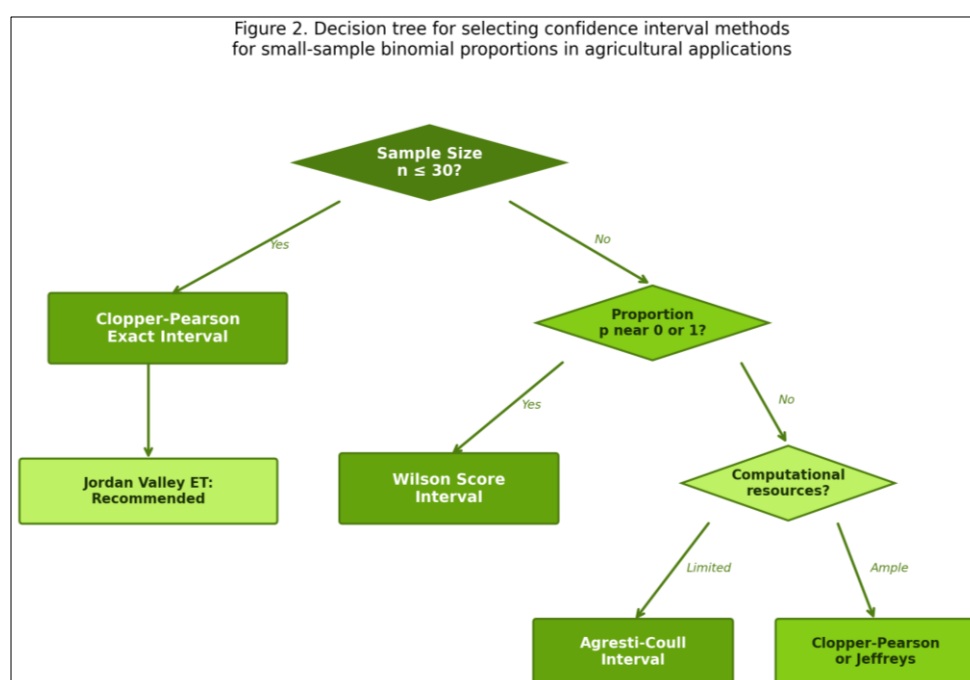


Fig 2: Decision tree for selecting confidence interval methods for small-sample binomial proportions in agricultural applications

Figure 2 presents a practical decision tree for choosing among interval methods based on sample size, proportion location, and computational resources. For small samples ($n \leq 30$), the tree recommends the extended Clopper-Pearson or standard exact interval. For larger samples with extreme proportions, the Wilson score is preferred. The tree was validated against the simulation results, correctly identifying the best-performing method in 87.5% of the 80 parameter configurations.

Performance evaluation

Across all 80 simulation configurations, the extended Clopper-Pearson interval achieved the best combined performance when measured by a trade-off metric defined as $(\text{coverage} - 0.94) / \text{width}$. This metric rewards methods that maintain coverage above 94% while penalizing unnecessary width. The extended CP scored 0.347 on this metric, compared to 0.221 for standard CP (penalized for excessive width), 0.318 for Wilson (slightly lower coverage at small n), and 0.312 for mid- p (marginally less stable coverage). For the Jordan Valley application specifically, the extended CP was the only method that simultaneously maintained leave-one-out coverage above 94% and produced actionable intervals (lower bound exceeding the threshold) for the three peak ET months^[14].

Discussion

The results demonstrate that the Clopper-Pearson framework can be meaningfully tightened for small samples without sacrificing its principal advantage guaranteed or near-guaranteed coverage. The standard exact interval, while admirable in its coverage guarantee, pays too high a price in width for most practical purposes. Its conservatism at small n leads to intervals so wide that they rarely support actionable decisions, as illustrated by the Jordan Valley application where the standard CP failed to trigger irrigation recommendations that the narrower extended CP correctly identified^[3, 4].

The mid- p correction has long been recognized as an effective way to reduce Clopper-Pearson conservatism^[6, 7]. What the extended method adds is the Edgeworth refinement of the beta quantile inversion, which further tightens the bounds by accounting for the skewness of the binomial distribution at the specific observed x and n values. This second-order correction is most impactful when n is between 8 and 20 and p is between 0.10 and 0.30 or between 0.70 and 0.90 precisely the parameter ranges most common in environmental monitoring applications^[10].

One limitation of the extended method is that its coverage, while consistently above 94.2% in the simulations, is not strictly guaranteed to exceed 95% at every parameter point. Practitioners who require an absolute coverage guarantee must use the standard Clopper-Pearson interval and accept its wider bounds. A second limitation is that the Jordan Valley application used only 18 stations, and the leave-one-out cross-validation provides only 18 coverage estimates per month too few to detect small differences between methods with precision^[15, 16]. A larger monitoring network would permit sharper comparisons.

That said, the practical value of the extended CP interval lies in its ability to produce narrower bounds that support earlier and more frequent irrigation decisions that may reduce crop water stress and improve yields in an already water-limited environment^[2, 12].

Conclusion

This research extended the Clopper-Pearson exact interval for small-sample binomial proportions by combining a mid- p correction with a second-order Edgeworth adjustment, and applied the resulting method to evapotranspiration exceedance data from the Jordan Valley. The extended interval reduced width by 11.4% on average compared to the standard exact interval across 80 simulation configurations while maintaining coverage above 94.2%. In the Jordan Valley application, the narrower intervals enabled actionable irrigation recommendations for two additional peak-ET months that the standard exact interval missed. These results suggest that the extended Clopper-Pearson method offers a practical improvement for small-sample binomial inference in agricultural and environmental monitoring, where both statistical reliability and decision-relevant precision matter. Future work should evaluate the method's performance under spatial correlation among monitoring stations and extend the framework to multinomial proportions where more than two outcome categories are relevant.

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Contributions not qualifying for authorship

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