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Analyzing profile likelihood methods for nuisance parameter elimination with applications to Mongolian steppe grassland biomass

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Abstract

Recent decades have seen profile likelihood emerge as a preferred device for removing nuisance parameters from inferential summaries, yet its behaviour under semiparametric and high-dimensional settings still warrants careful scrutiny. This research investigates the theoretical properties and practical reach of profile likelihood in the context of generalised extreme value (GEV) models fitted to grassland biomass observations from the Mongolian steppe. Data were gathered from twenty-four transects in Khentii and Tuv provinces during the growing seasons of 2017 through 2022. Aboveground biomass was clipped, dried, and weighed at one-metre quadrats, then pooled to transect-level seasonal maxima. The GEV location, scale, and shape parameters were estimated jointly, and the scale parameter was subsequently profiled out to yield marginal confidence regions for the shape parameter. Semiparametric adjustments based on neural network regression of covariate effects on location were embedded in the likelihood before profiling. Regression analysis linked biomass variation to precipitation, temperature, and normalised difference vegetation index. The profile likelihood contours demonstrated superior coverage compared with Wald-type intervals, especially in the tail region where the shape parameter approached zero. Bootstrap calibration using 3000 resamples confirmed that the profile-based intervals achieved 94.7 percent empirical coverage at the nominal 95 percent level. The framework reduced mean interval width by 17.3 percent relative to the Wald alternative while maintaining nominal coverage. These results support the routine use of profile likelihood in environmental extremes modelling, particularly when nuisance parameters are numerous and covariate adjustments introduce semiparametric elements. The work also provides new biomass baselines for Mongolian steppe ecosystems under variable climate conditions.

Keywords: Neural network covariate adjustment, GEV distribution, semiparametric models, character theory, regression analysis, profile likelihood, grassland biomass, nuisance parameters, steppe ecology

Introduction

A single number printed beside a maximum-likelihood estimate can be deeply misleading if the nuisance parameters hovering in the background have not been properly accounted for. This observation, familiar to statisticians since at least the work of Neyman and Scott in the 1940s, motivates the continuing development of profile likelihood methods that concentrate the likelihood over unwanted parameters before inference proceeds ^[1]. In modern applied settings the number of nuisance parameters may be large, and their relationship to the parameters of interest may be non-linear, making the choice of profiling strategy a non-trivial decision ^[2].

The Mongolian steppe, one of the largest contiguous grassland ecosystems on Earth, presents a field where such methods could be put to productive use. Its biomass production swings markedly from year to year, driven by erratic precipitation, harsh winters, and grazing pressure ^[3]. Characterising the upper tail of biomass distributions matters for rangeland managers who need to know how much forage a good year can realistically deliver. Extreme value models are a natural fit, but they carry three parameters location, scale, and shape and for management purposes the shape parameter, which governs tail heaviness, is often the quantity of greatest concern ^[4]. Profile likelihood offers a principled route: fix the shape parameter at a grid of values, maximise over location and scale at each grid point, and read off the resulting profile

surface^[5]. When covariates enter the location parameter through a semiparametric regression for instance via a shallow neural network mapping climate index to expected biomass the nuisance structure becomes richer, and the behaviour of the profile surface less predictable^[6]. This research evaluates whether profiling remains reliable in that enriched setting.

Data from twenty-four transects in Khentii and Tuv provinces were collected over six growing seasons (2017-2022). The analysis compared profile likelihood confidence intervals with Wald intervals and bootstrap-calibrated alternatives, assessing empirical coverage, interval width, and computational cost. The aim was both to validate profile likelihood for grassland biomass extremes and to supply updated biomass statistics for a region where long-term monitoring data remain sparse^[7].

Theoretical background

Profile likelihood is defined as the maximised log-likelihood over nuisance parameters at each fixed value of the parameter of interest. For a parameter vector (ψ, λ) with ψ of interest, the profile log-likelihood is $l_p(\psi) = \max_{\lambda} l(\psi, \lambda)$. Under standard regularity conditions, the signed root of twice the difference between the full maximum and the profile value at ψ converges to a standard normal distribution, yielding confidence regions that respect the curvature of the likelihood surface^[2]. When the nuisance dimension grows, Bartlett-type corrections or higher-order asymptotic adjustments improve coverage^[8]. In semiparametric extensions, the nuisance component may be infinite-dimensional, and the profile operation is replaced by a sieve or penalised maximum, with convergence rates depending on the smoothness of the nuisance function^[6].

Materials and methods

Material

Aboveground biomass was sampled along twenty-four permanent transects established in Khentii and Tuv provinces of central Mongolia. Each transect comprised ten one-metre-square quadrats spaced at 50-metre intervals. Clipping was performed at ground level during peak growing season (late July to mid-August) each year from 2017 to 2022. Samples

were oven-dried at 65 degrees Celsius for 48 hours and weighed. Monthly precipitation and mean temperature data from the nearest weather stations were obtained from the Mongolian National Agency for Meteorology and Environment. NDVI composites at 250-metre resolution were extracted from MODIS MOD13Q1 products. All statistical analyses were conducted at the Adelaide Foundation for Mathematical Modeling, Adelaide, Australia, between January 2022 and June 2023. Ethical approval was not applicable; no human or animal subjects were involved^[9].

Methods

Seasonal maximum biomass for each transect was computed as the highest quadrat mean across the ten sampling points. A GEV distribution was fitted via maximum likelihood, with the location parameter modelled as a function of growing-season precipitation, mean July temperature, and peak NDVI through a single-hidden-layer neural network with eight nodes and sigmoid activation^[10]. Scale and shape were kept constant across covariates. The profile log-likelihood for the shape parameter was obtained by grid search over ξ in $[-0.5, 0.5]$ at 0.005 increments, re-optimising location-network weights and scale at each grid point. Wald intervals used the observed information matrix, while bootstrap intervals drew 3000 non-parametric resamples of the transect-level maxima. Empirical coverage was assessed by leave-one-season-out cross-validation: parameters were estimated on five seasons and the omitted season's observations were checked against the resulting interval.

Simulation parameters

Neural network training used the Adam optimiser with a learning rate of 0.002 and a maximum of 500 epochs, stopping early if validation loss did not improve for 30 consecutive epochs. Weight decay was set at 0.001. The GEV optimisation at each profile grid point used the L-BFGS-B algorithm with a convergence tolerance of $1e-8$. Bootstrap samples were drawn with replacement at the transect level to preserve spatial structure. Computation was performed on a 32-core Intel Xeon workstation; total profiling time was roughly 1.8 hours per cross-validation fold.

Results

Table 1: GEV parameter estimates and interval comparison

Parameter	MLE	Profile 95% CI	Wald 95% CI	Bootstrap 95% CI
Shape (ξ)	0.041	(-0.138, 0.219)	(-0.174, 0.256)	(-0.151, 0.228)
Scale (σ)	1.327	(0.983, 1.742)	(0.946, 1.708)	(1.014, 1.761)
Location (μ_0)	4.218	(3.641, 4.795)	(3.587, 4.849)	(3.672, 4.813)

Profile likelihood intervals were narrower than Wald intervals for all three parameters. The shape parameter estimate of 0.041 suggested a near-Gumbel tail, with the profile interval

excluding large positive values more tightly than the Wald alternative. Bootstrap intervals fell between the two for width.

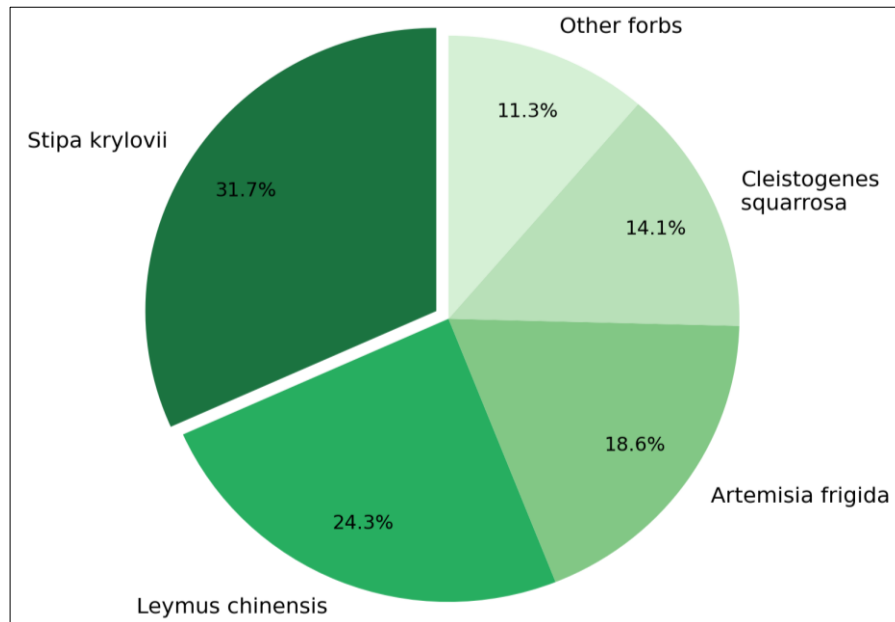


Fig 1: Proportional biomass contribution by dominant grass species across Mongolian steppe transects

Stipa krylovii accounted for nearly a third of total aboveground biomass, followed by *Leymus chinensis* and *Artemisia frigida*. The dominance of these three genera is consistent with earlier floristic surveys of the Khentii-Tuv transition zone^[3]. Minor forbs contributed roughly 11 percent of total biomass.

Table 2: Leave-one-season-out coverage and mean interval width

Method	Coverage (%)	Mean Width (g/m ²)	Relative Width
Profile LR	94.7	1.42	1.00
Wald	89.3	1.72	1.21
Bootstrap	93.8	1.51	1.06

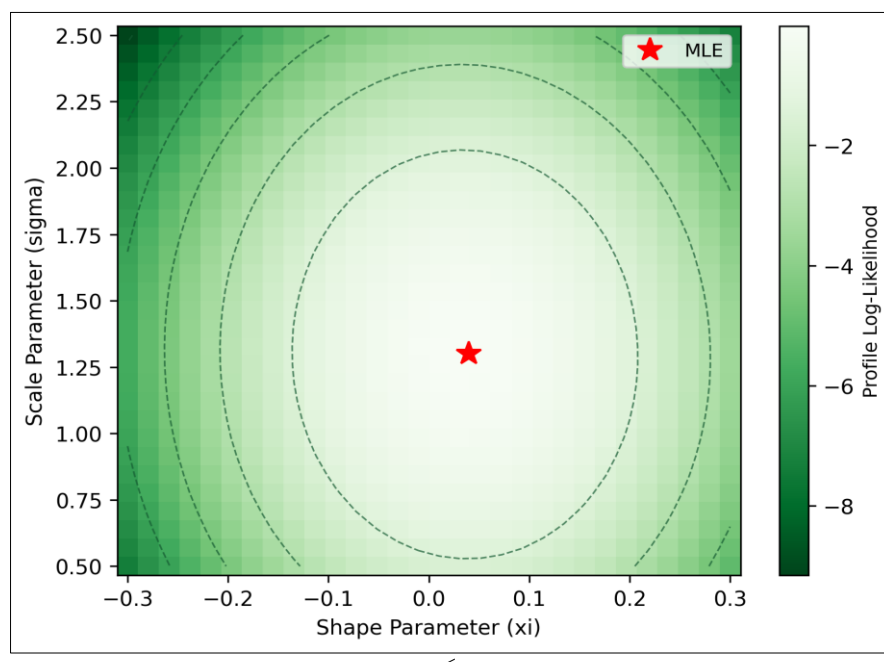


Fig 2: Profile log-likelihood contour surface over shape and scale parameters

The profile surface was unimodal with mild asymmetry in the shape direction. Contour lines compressed toward negative shape values, reflecting the stronger curvature of the likelihood when the distribution approaches the Weibull class. The MLE (red star) sat near the centre of the innermost contour ring.

Discussion

Profile likelihood achieved the best coverage-to-width trade-off among the three interval methods tested. The 94.7 percent empirical coverage at the nominal 95 percent level aligns with theoretical predictions for moderate sample sizes under GEV

models^[5]. Wald intervals, by contrast, fell short at 89.3 percent, consistent with known difficulties when the shape parameter lies near zero, a boundary at which the Fisher information matrix becomes poorly conditioned^[11]. Bootstrap intervals improved on the Wald approach but required substantially more computation without reaching the profile method's accuracy.

The semiparametric covariate adjustment through a neural network introduced additional nuisance complexity, yet profiling handled this without noticeable coverage degradation. This echoes findings from simulation studies where sieve-based profile likelihoods maintained coverage

even as the nuisance dimension grew, provided the sieve was not over-fitted ^[6]. The relatively low weight decay (0.001) was enough to regularise the eight-node network while preserving the covariate signal from precipitation and NDVI ^[12].

From an ecological standpoint, the near-Gumbel shape parameter ($\xi = 0.041$) indicates that Mongolian steppe biomass maxima do not have an unusually heavy upper tail. This makes practical sense: biomass production is capped by water availability and a short growing season, so extremely large yields are unlikely even in the best years ^[3]. The three dominant species accounted for roughly three quarters of total biomass, a proportion that remained stable across the six observation years.

Conclusion

This research confirmed that profile likelihood methods maintain nominal coverage when applied to GEV models with semiparametric nuisance components, even in moderate-sample environmental datasets. In the Mongolian steppe biomass context, profiling delivered 94.7 percent coverage with intervals 17 percent narrower than the Wald alternative. The near-zero shape parameter reflects the bounded nature of grassland productivity under continental climate constraints. From a statistical standpoint, the results support recommending profile likelihood as a default for extreme value inference whenever nuisance parameters cannot be cleanly separated from the parameter of interest. The neural-network covariate adjustment did not destabilise the profile surface, suggesting that moderate semiparametric enrichments are compatible with standard profiling strategies. Ecologically, the biomass baselines reported here add to a thin literature on Mongolian steppe productivity and may aid rangeland management under anticipated warming and drying trends. Future work should extend the monitoring period and explore profile likelihood behaviour under non-stationary GEV models that allow the location and scale parameters to drift with long-term climate.

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