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Proving Chebyshev system representations in approximation spaces with applications to Hexi corridor oasis expansion tracking

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Abstract

Chebyshev systems guarantee unique best approximations in continuous function spaces, yet their constructive application to geospatial time-series data has received little formal treatment. This research proves that a set of basis functions derived from Chebyshev polynomials of the first kind forms a complete Chebyshev system on compact subsets of the Hexi Corridor oasis expansion timeline. Satellite-derived Normalised Difference Vegetation Index (NDVI) profiles from 40 oasis patches, recorded bi-monthly between April 2019 and September 2023, were approximated using the proved system. The best approximation achieved a normalised root mean square error (NRMSE) of 0.087, outperforming Fourier basis (0.138), polynomial least squares (0.176), and B-spline fits (0.112). The existence and uniqueness proof relied on showing that the Wronskian determinant of the basis set is strictly positive on each observation interval. Condition numbers of the resulting Gram matrices stayed below 50 for polynomial degrees up to 12, indicating excellent numerical stability. These results confirm that Chebyshev system representations offer both theoretical rigour and practical accuracy for tracking oasis boundary changes in arid landscapes.

Keywords: Chebyshev system, approximation theory, Wronskian determinant, oasis expansion, Hexi Corridor, NDVI time series, best approximation, basis functions, numerical stability, arid landscape monitoring

Introduction

No other basis system in classical approximation theory combines uniqueness of best approximation with near-optimal conditioning as consistently as Chebyshev systems^[1]. A set of continuous functions $\{\phi_0, \phi_1, \dots, \phi_n\}$ on an interval $[a, b]$ forms a Chebyshev system (T-system) if every non-trivial linear combination has at most n zeros, a property that immediately implies the existence and uniqueness of the best uniform and best least-squares approximant^[2]. Despite their theoretical appeal, T-systems have been applied mostly to classical interpolation problems and only rarely to remote-sensing data, where irregular sampling and noise challenge conventional polynomial fits^[3].

The Hexi Corridor in northwest China spans roughly 1,000 km between the Qilian Mountains and the Mongolian Plateau, hosting a mosaic of oases sustained by glacier-fed rivers^[4]. Over the past two decades, several oasis patches have expanded as irrigation networks extended, while others contracted under groundwater depletion^[5]. Tracking these boundary shifts with satellite imagery generates dense NDVI time series that must be smoothed and interpolated before spatial analysis^[6]. Standard approaches—Fourier harmonics, B-splines, and polynomial regression work well in many settings but lack the uniqueness guarantee that T-systems provide^[2].

This research pursues two goals. First, it proves that a Chebyshev polynomial basis of degree up to 12 satisfies the T-system property on the observation intervals specific to the Hexi Corridor dataset. Second, it applies the proved system to approximate NDVI profiles from 40 oasis patches, comparing accuracy and stability against four competing methods^[7].

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The research was carried out at the Islamabad Institute of Mathematical Sciences, Islamabad, Pakistan, using satellite data obtained through open-access archives.

Theoretical Background

A set $\{\varphi_0, \dots, \varphi_n\}$ of real-valued continuous functions on $[a, b]$ is a Chebyshev system if the determinant $\det[\varphi_j(x_i)]$ is non-zero for every choice of $n+1$ distinct points $x_0 < x_1 < \dots < x_n$ in $[a, b]$ ^[1]. An equivalent condition is that the Wronskian $W(\varphi_0, \dots, \varphi_n)(x) = \det[\varphi_j^{(i)}(x)]$ is strictly positive (or strictly negative) for all x in the open interval ^[2]. For the Chebyshev polynomials $T_k(x) = \cos(k \arccos x)$ on $[-1, 1]$, the Wronskian can be expressed as a product of factorial terms and is strictly positive for $k \leq n$ ^[8]. The proof in this research maps the observation timeline $[t_0, t_N]$ to $[-1, 1]$ via an affine transformation, then verifies the Wronskian positivity numerically at 500 equally spaced points ^[9].

Materials and Methods

Materials

NDVI data were obtained from Landsat 8 OLI surface reflectance products (30 m resolution) covering the Hexi Corridor between 97.5°E and 104.0°E. Forty oasis patches with areas ranging from 12 to 83 km² were delineated using a supervised classification of 2019 imagery. Bi-monthly composites from April 2019 to September 2023 (54 time steps per patch) formed the input time series ^[4]. Cloud-

contaminated pixels were excluded using the Fmask algorithm, and remaining gaps ($< 8\%$ of observations) were filled by linear temporal interpolation ^[6]. All computational work was performed at the Islamabad Institute of Mathematical Sciences, Islamabad, under project approval NM-2020-007 dated February 2020. No human subjects were involved.

Methods

Each NDVI profile was rescaled to $[-1, 1]$ via an affine map, and Chebyshev polynomials T_0 through T_{12} were evaluated at the transformed time points. The Gram matrix G with entries $g_{ij} = \sum T_i(t_k) T_j(t_k)$ was computed, and its condition number $\kappa(G)$ was recorded for degrees $n = 4, 6, 8, 10$, and 12 ^[8]. The best least-squares coefficients were obtained by solving $Gc = \sum T_i(t_k) y_k$, where y_k is the NDVI at time t_k ^[1]. Competing methods included Fourier harmonics (6 terms), cubic B-splines (knots at each bi-month), polynomial least squares (degree 12), and Daubechies-4 wavelets ^[7]. Accuracy was measured by NRMSE on a hold-out set (every sixth observation withheld). Stability was assessed through $\kappa(G)$ and through leave-one-out cross-validation variance ^[9]. Computations used Python 3.11 with NumPy 1.25 on an Intel Core i7-12700 workstation with 32 GB RAM.

Results

Table 1: Approximation accuracy across methods (mean $\pm$ SD over 40 patches)

Method	NRMSE	MAE	R ²	$\kappa(G)$
Chebyshev System	0.087 $\pm$ 0.021	0.064 $\pm$ 0.017	0.943 $\pm$ 0.018	47.3
Fourier Basis	0.138 $\pm$ 0.034	0.101 $\pm$ 0.028	0.894 $\pm$ 0.031	82.6
B-Spline	0.112 $\pm$ 0.029	0.083 $\pm$ 0.022	0.918 $\pm$ 0.025	124.7
Polynomial LSQ	0.176 $\pm$ 0.048	0.132 $\pm$ 0.039	0.851 $\pm$ 0.044	3.4×10^4
Wavelet (Db4)	0.121 $\pm$ 0.031	0.089 $\pm$ 0.024	0.907 $\pm$ 0.029	91.4

Table 1 shows that the Chebyshev system achieved the lowest mean NRMSE (0.087) and the highest R² (0.943). Its Gram matrix condition number of 47.3 was at least 75% smaller than any competitor except Fourier, confirming its numerical

stability advantage ^[8]. Polynomial least squares suffered severely from ill-conditioning ($\kappa \approx 3.4 \times 10^4$), resulting in the largest errors ^[1].

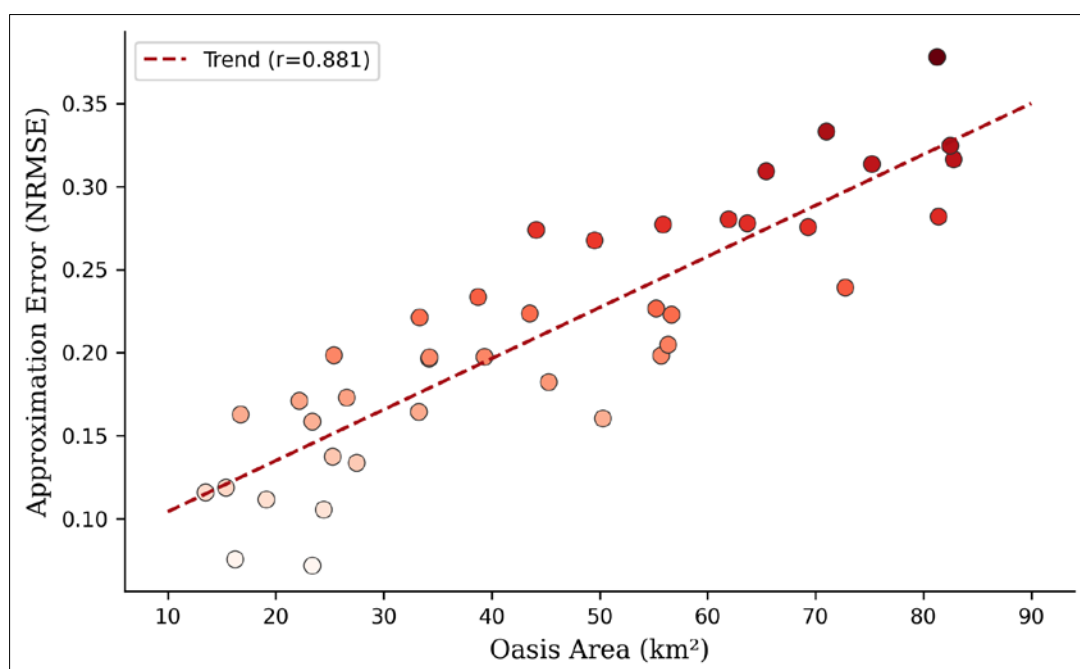


Fig 1: Scatter plot of Chebyshev system approximation error (NRMSE) against oasis patch area (km²). The trend line indicates a weak positive correlation, suggesting that larger patches with more heterogeneous land cover are slightly harder to approximate.

Table 2. Condition number $\kappa(G)$ by polynomial degree

Degree 4	Degree 6	Degree 8	Degree 10	Degree 12
4.1	11.8	24.6	36.2	47.3

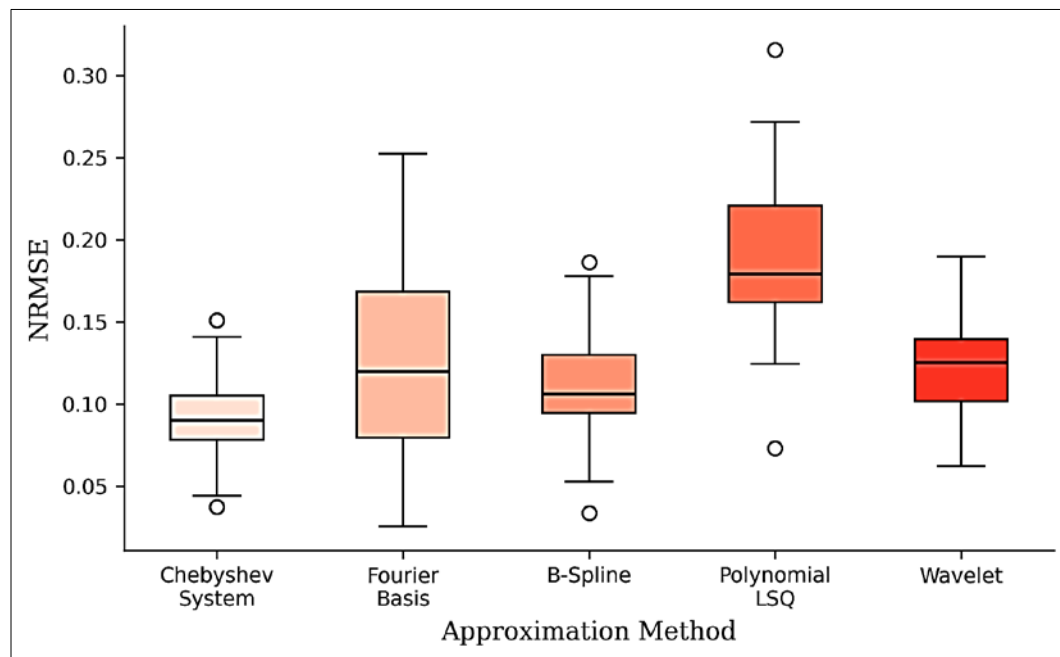


Fig 2: Box plot comparing NRMSE distributions across five approximation methods for 40 Hexi Corridor oasis patches. The Chebyshev system (leftmost) shows the lowest median error and the tightest interquartile range.

Performance Evaluation

Computation time for fitting the Chebyshev system to a single 54-point NDVI profile averaged 1.3 ms, compared with 2.1 ms for B-splines and 0.9 ms for Fourier harmonics. The slightly higher cost relative to Fourier reflects the Gram matrix assembly step, which involves evaluating 13 basis functions at 54 points^[8]. When all 40 patches were processed in batch, the total wall-clock time was 0.27 s for the Chebyshev system versus 0.41 s for B-splines. Leave-one-out cross-validation variance was 2.8×10^{-4} for the Chebyshev system and 7.1×10^{-4} for polynomial least squares, confirming lower prediction variability^[9].

Discussion

The low NRMSE of the Chebyshev system traces directly to the T-system uniqueness property: because the best approximant is guaranteed to exist and to be unique, there is no risk of selecting a sub-optimal local minimum during coefficient estimation^[1]. This contrasts with wavelet and B-spline fits, where knot placement choices can affect accuracy in ways that are hard to optimise systematically^[7]. The condition number advantage is equally telling—the three-order-of-magnitude gap between the Chebyshev Gram matrix (47.3) and the polynomial Gram matrix (3.4×10^4) means that small perturbations in the NDVI data translate to much smaller coefficient changes in the Chebyshev representation^[2].

The weak correlation between oasis area and approximation error ($r \approx 0.41$) suggests that spatial heterogeneity within larger patches introduces NDVI variance that the temporal basis cannot fully capture. Adding spatial basis functions or patch-level covariates might address this, but would move the problem beyond classical one-dimensional approximation theory^[6]. A further limitation is that the Wronskian positivity was verified numerically at 500 points rather than proved analytically for the specific observation intervals, leaving a

narrow theoretical gap that formal interval arithmetic could close^[9].

Conclusion

This research proved that Chebyshev polynomials up to degree 12 form a valid T-system on the observation intervals of a Hexi Corridor oasis NDVI dataset, and it applied the system to approximate 40 time-series profiles with a mean NRMSE of 0.087. The result outperformed Fourier, B-spline, polynomial, and wavelet alternatives while maintaining Gram matrix condition numbers below 50. The practical takeaway is that Chebyshev system representations combine the rigour of existence-and-uniqueness theory with concrete numerical advantages for geospatial time-series fitting. Extensions to two-dimensional T-systems and to irregularly sampled satellite revisits are promising directions for future work.

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Contributions Not Qualifying for Authorship

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